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Predator-prey mathematical model using Van Der Pol's equation

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Abstract

In this paper we present a predator-prey Mathematical Model using Van der Pol's equation. We consider a preditor (A) and a prey (B) which interact in the following way: when B is large, A increases, eventually causing the collapse of B to a low level. At the low level, the prey increase in numbers and the preditors decrease. The prey population eventually explodes to a high level, and the process repeats itself. In the absence of the predation, A the prey population has a stable low level, perhaps controlled by competition, a stable high level, and an unstable intermediate equilibrium. A model for this is derived by considering the deviation of prey and predators from the unstable equilibrium, say x for prey and y for predator. The model is converted to polar co-ordinates by introducing the new variables r and θ. The right-hand side of the equation is averaged for r and θ by integrating over θ from 0 to 2π while keeping r fixed. It was then shown that the point (x(t), y(t)) approaches a periodic orbit that has amplitude 2 as t increases.

Keywords: Predator – Prey, Van der Pol's Equation, Periodic, Orbit, Amplitude, Polar Co-ordinates.

1.0 Introduction

When species interact, the population dynamics of each species is affected. In general there is a whole web of interacting species, called a *trophic web*, which makes for structurally complex communities. We consider here systems involving two or more species, concentrating particularly on 2-species systems. There are three main types of interaction.

(i) If the growth rate of one population is decreased and the other increased the populations are in a *predator-prey* situation.

(ii) If the growth rate of each population is decreased, then it is *competition*.

(iii) If each population's growth rate is enhanced, then it is called *mutualism* or *symbiosis* [5].

Some mathematical models have been developed in this area.

In 1926, Volterra [8] first proposed a simple model for the predation of one species by another to explain the oscillatory levels of certain fish catches in the Adriatic. This model was based on four assumptions.

First, the prey grows unboundedly in a Malthusian way in the absence of any predation.
Secondly, the effect of the predation is to reduce the prey's per capita growth rate by a term proportional to the prey and predator populations.
Thirdly, in the absence of any prey for sustenance the predator's death rate results in exponential decay.
Fourthly, the prey's contribution to the predator's growth is proportional to the available prey as well as the size of the predator population. The model, is

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$$\frac{dN}{dt} = N(a - bp) \text{ and } \frac{\mathrm{dP}}{\mathrm{dt}} = P(cN - d)$$

when *N* is the prey population and *P* is the predator population. This model also called Lotka –Volterra model was analyzed.

Murray [5] modified the Lotka-Volterra Model by changing of the assumptions made by Voltera. The model he obtained is: $\frac{dN}{dt} = N\left[r\left(1 - \frac{N}{K}\right) - \frac{kP}{N + D}\right]$, $\frac{dP}{dt} = P\left[s\left(1 - \frac{hP}{N}\right)\right]$

where *r*, *K*, *k*, *D*, *S* and *h* are positive constants. He established the conditions under which this model is stable.

Murray [6] further extended this model to what he called "Competition Models: Principle of Competitive Exclusion". He considered the simple 2-species Lotka-Volterra competition model with each species N_1 and N_2 having logistic growth in the absence of the other. He formulated the model as

$$\frac{dN_1}{dt} = r_1 N_1\left[1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_2}\right], \ \frac{dN_2}{dt} = r_2 N_2\left[1 - \frac{N_2}{K_2} - b_{21}\frac{N_1}{K_2}\right]$$

where r_1, k_1, r_2, k_2, b_{12}, b_{21} are all positive constants. This model was analysed and the conditions for its stability established.

Beddington et al [1] presented some results on the dynamic complexity of coupled predator-prey systems.

Dunbar [2,3] studied in detail a modified Lotka-Volterra system with logistic growth of the prey and with both predator and prey dispersing by diffusion.

"*Predator-Prey* models are arguably the building blocks of the bio and ecosystems as biomasses are *grown* out of their resource masses. Species compete, evolve and disperse simply for the purpose of seeking resources to sustain their struggle for their very existence. Depending on their specific settings of applications, they can take the forms of resource – consumer, plant – herbivore, parasite-host, tumor cells (virus)-immune system, susceptible-infectious interactions, etc. They deal with the general loss-win interactions and hence may have applications outside of ecosystems. When seemingly competitive interactions are carefully examined, they are often in fact some forms of predator-prey interaction in disguise" [4].

Another approach to modeling the interaction between prey and predators was developed to account as well for organisms (such as bacteria) taking up nutrients and this is called Jacob-Monod Model. This model was discovered independently in the several diverse applications. It is akin to the Haldane-Briggs Model and Michaelis-Menten Model in Biochemistry the Jacob-Monod Model in microbial ecology, and the Beverton-Holt model in fisheries. It serves as one of the important building blocks in studies of complex biochemical reactions and in ecology [7].

Figure 1.1 below shows the periodic activity generated by the predator-prey model.

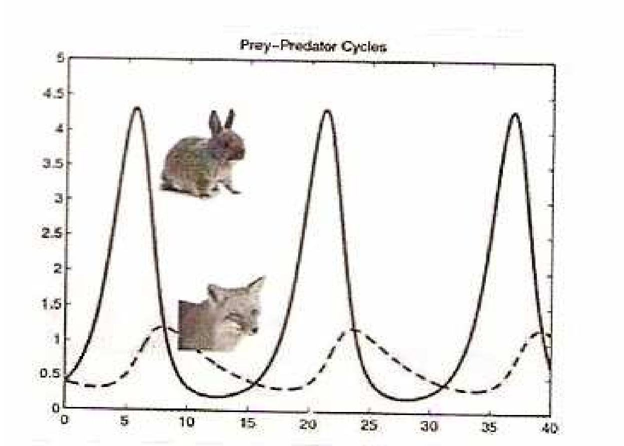

Figure 1.1: Periodic acitivity generated by the predator-prey model.

2.0 The model formulation

In this section we present the model for predator – prey using van der pol's equation. We consider predator (A) and prey (B) which interact in the following way: when B is large, A increases, eventually causing the collapse of B to a low level. At the low level, the prey increase in numbers and the predators decreases. The prey population eventually explodes to high level, and the process repeats itself. In the absence of the predator A, the prey population has a stable low level, perhaps controlled by competition, a stable high level, and an unstable intermediate equilibrium. The model for this can be derived by considering the deviation of prey and predators from the unstable equilibrium. There is need to study this model since many real life situations follow this pattern and hence this model can be applied to many areas of real life.

2.1 Parameters and symbols

$x(t) =$ deviation of prey from the unstable equilibrium at time t

$y(t) =$ deviation of predator from the unstable equilibrium at time t

μ = ratio of prey growth rate to predator response and $0 < \mu < 1$

2.2 The model

Using the above parameters, a simple model describing these interactions is

$$\frac{dx}{dt} = \mu\left(x - x^3/3\right) - y,\ \frac{dy}{dt} = x \tag{2.1}$$

3.0 Analysis of the model

The equilibrium curve $dx/dt = 0$ is a cubic as shown in Figure 3.1 below.

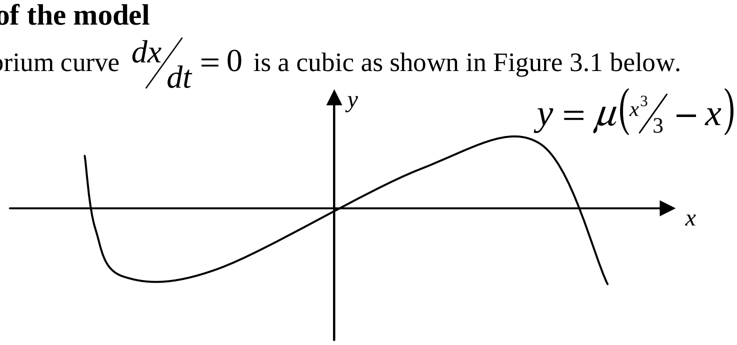

Figure 3.1

In this analysis we consider two cases: when $\mu = 0$ and when $\mu \neq 0$

Case I: when $\mu = 0$

If $\mu = 0$, from (2.1) we obtain $$\frac{d^2x}{dt^2} + x = 0 \qquad (3.1)$$

This gives $x(t) = A\cos t + B\sin t$, where *A* and *B* are constants and $y = A\sin t - B\cos t$, hence, when $\mu =$ 0

$$\left.\begin{aligned} x(t) &= A\cos t + \mathrm{B}\sin t \\ y(t) &= A\sin t - \mathrm{B}\cos t \end{aligned}\right\} \qquad (3.2)$$

This means that when the ratio of the prey growth rate to predator response (μ) is zero, the deviation of prey from the unstable equilibrium is periodic. This agrees with the condition we are discussing since the growth of the prey is periodic, (it fluctuates) as shown in the graph above.

Case II: when $\mu \neq 0$

We now convert the problem to polar form by taking $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$ to get.

$r\frac{dr}{dt} = \frac{xdx}{dt} + \frac{ydy}{dt} = \mu r\cos\theta\left(r\cos\theta - r^3\frac{\cos^3\theta}{3}\right)$, therefore $\frac{dr}{dt} = \mu r\left(\cos^2\theta - r^2\frac{\cos^4\theta}{3}\right)$

Since $\theta = \tan^{-1}\left(y/x\right)$ $\frac{d\theta}{dt} = 1 - \mu\sin\theta\cos\theta\left(1 - \frac{r^2\cos^2\theta}{3}\right)$

$$\frac{r^2 d\theta}{dt} = \frac{dy}{dt} - \frac{ydx}{dt} = r^2 - r\sin\theta\left(r\cos\theta - r^3\frac{\cos^3\theta}{3}\right)\mu$$

$$\therefore \qquad \frac{d\theta}{dt} = 1 - \mu\sin\theta\cos\theta\left(1 - \frac{r^2\cos^2\theta}{3}\right)$$

Hence the model in polar form is

$$\frac{dr}{dt} = \mu r\left(\cos^2\theta - r^2\frac{\cos^4\theta}{3}\right), \quad \frac{d\theta}{dt} = 1 - \mu\sin\theta\cos\theta\left(1 - r^2\frac{\cos^2\theta}{3}\right) \qquad (3.3)$$

If we average the right-hand side of (2.4) while keeping r fixed we have

$$\frac{d\bar{r}}{dt} = \frac{\mu\bar{r}}{2}\left(1 - \bar{r}\,{}^2\!/_4\right), \quad \frac{d\bar{\theta}}{dt} = 1$$

where $\bar{r}$ is the averaged value of *r* and $\bar{\theta}$ is the averaged value of θ.

Thus if $\bar{r}(0) \geq 0$, $\bar{r} \to 2$ as $t \to \infty$, this occurs on the slow time scale μt. Thus the point (*x*(*t*), *y*(*t*)) approaches a periodic orbit that has amplitude 2 as t increases.

3.0 Summary and conclusion

In this paper we have been able to formulate the predator-prey model by considering the deviation of prey (*x*) and preditors (*y*) from the unstable equilibrium. The formulated model is analysed. It was shown that the equilibrium curve $\left(dx/dt = 0\right)$ is a cubic and is shown in Figure 3.1. The model was further analyzed by considering the value of μ (the ratio of prey growth rate to predator response) It was shown that when μ = 0, x(t) (deviation of prey from the unstable equilibrium at time t) is found to be periodic (cyclical) $x(t) = A\cos t + B\sin t$. This means that x(t) fluctuates (rises and falls) which is in agreement with the situation we are modeling when $\mu \neq 0$, the model is converted to polar form by taking $x = r\cos\theta$, $y = r\sin\theta$ and $r^2 = x^2 + y^2$ and we obtained

$$\frac{dr}{dt} = \mu r\left(\cos^2\theta - r^2\frac{\cos^4\theta}{3}\right),\ \frac{d\theta}{dt} = 1 - \mu\sin\theta\cos\theta\left(1 - r^2\frac{\cos^2\theta}{3}\right).$$

We went further to average the right-hand side of the model by integrating over θ from 0 to 2π while keeping r fixed and we obtained $\frac{d\bar{r}}{dt} = \frac{\mu\bar{r}}{2}\left(1 - \frac{\bar{r}^2}{4}\right),\ \frac{d\bar{\theta}}{dt} = 1$. It was then shown that if $\bar{r}(0) \geq 0, \bar{r} \to 2$ as $t \to \infty$. This occurs on the slow time scale μt. This means that the point ($x(t)$, $y(t)$) approaches a periodic orbit that has amplitude 2 as *t* increases.

References

[1] Beddington, J.R., Free, C.A., Lawton, J.H. (1975). Dynamic Complexity in Predator-prey Models Framed in difference equations. Nature 255, 58-60.

[2] Dunbar, S.R. (1984). Traveling Wave Solutions of diffusive Lotka-Volterra equations. Journal of Mathematical Biology, 17, 11-32.

[3] Dunbar, S.R. (1984). Travelling Wave Solutions of diffusive Lotka-Volterra equations: A Heteroclinic connection in R^4. Trans. American Mathematical Society, 268, 557-594.

[4] Hoppensteadt, F. (2006). Predator-prey Model Scholarpedia (the free peer-reviewed encyclopedia) p. 5765-5772. Available on line (http://www.scholarpedia.org/article/predator-prey_model)

[5] Murray, J.D. (1989). Mathematical Biology, 2nd Edition. Springer-Verlag, New York 63, 72-78.

[6] Murray, J.D. (1989). Mathematical Biology, 2nd Edition. Springer-Verlag, New York78-83.

[7] Smith, H. and Waltman, P. (1997). The Mathematical Theory of Chemostats, Cambridge U. Press.

[8] Volterra, V. (1926). Variations and Fluctuations of a number of individuals in animal species living together. [In R.N. Chapman: Animal Ecology, New York, McGraw Hill, 1931, 409-448].