Further implications of the quick evaluation of determinants

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Abstract

Ching-Tzong Su and Feng Cheng Chang introduced a method of evaluation of determinants by order reduction in their paper "Quick evaluation of determinants. Here we moved further to state that the determinant of any given matrix is equal to the determinant of the reduced matrix of order (2×2) . In addition we assert that the process leads to a huge saving in computing storage space as a matrix of order $(n \times n)$ (n > 2) is reduced to order (2×2) . This was demonstrated by examples.

Keyword: matrix, order and determinant.

1.0 Introduction

Ching – Tzongh Su and Fen Ching Chang in their paper titled "Quick evaluation of Determinants [1] derived a quick method of evaluating the determinant of a given matrix by order reduction method. They showed that the reduction order formula is valid for $r \neq 0$ in

$$\det\begin{bmatrix} w & v \\ u & r \end{bmatrix} = r \det\left[w - \frac{vu}{r} \right]$$
(1.1)

where *w*, *v*, *u* and *r* are respectively a square matrix a column matrix, a row matrix and a scalar. The determinant of any large order matrix can therefore be easily obtained by applying this formula successively. In the process if r = 0, the modification is made by exchanging rows or columns of the matrix to avoid r = 0.

The required multiplications are found to be $\frac{1}{3}n(2n^2-3n+4)$ which is less than n³

operations needed for the product of $2n \times n$ matrices [1, 2]. It should be noted that a FORTRAN program involving the calculation of determinant is shown below:

PROGRAM MTDET

C DETERMINANT OF A GIVE MATRIX C ONLY 6 STATEMENT LINES B Y F. C. CHANG 10/16/94 DIMENSION A(10, 10) READ (5, *) n,((A(I,J),J=1,N), I = 1, N) WRITE(6, *) GIVEN MATRIX A(I, J) WRITE(6, 12) ((A(I, J), J = I, N), I = 1, N) 12 FORMAT(6,F8.2) DET = 1.0 DO 20 K = N, 1, -1 DO 10 I = 1, K - 1 DO 10 J = 1, K - 1 10 A(I, J) = A(I, J) - A(I, K) * A(K, J)/A(K, K) 20 DET = DET * A(K, K) WRIT(6, *) 'DETERMINANT = ', DET STOP END

2.0 Reducing the determinant of the matrix of any order $(n \times n) n > 2$ to the determinant of the matrix of order (2×2) .

By repeating the above process the (arbitrary) order of a given matrix can be reduced to the order of (2×2) . So for a matrix of order $(n \times n)$ as given below we have

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & \mathrm{K} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \mathrm{K} & a_{2n} \\ \mathrm{M} & \mathrm{M} & \mathrm{M} & \mathrm{M} \\ a_{n1} & a_{n2} & a_{n3} & \mathrm{K} & a_{nn} \end{pmatrix} = a_{nn} \det \begin{pmatrix} a_{11} & a_{12} & \mathrm{K} & a_{1n-1} \\ a_{21} & a_{22} & \mathrm{K} & a_{2n-1} & -\frac{\nu u}{r} \\ \mathrm{M} & \mathrm{M} & \mathrm{M} \\ a_{n1} & a_{n2} & \mathrm{K} & a_{m+n-1} \end{pmatrix}$$
(2.1)

where

$$v = \begin{pmatrix} a_{1n} \\ \\ a_{n-1}a_n \end{pmatrix}, v = (a_{n1} a_{n2} K a_{nn-1}), r = a_{nm}$$
(2.2)

For n = 4, we have the

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = r \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} & -\frac{vu}{r} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$$
(2.3)

where $v = \begin{pmatrix} a_{14} \\ a_{24} \\ a_{34} \end{pmatrix}$, $u = (a_{41} \ a_{42} \ a_{43})$ and $r = a_{44}$

$$\frac{vu}{r} = \frac{1}{a_{11}} \begin{pmatrix} a_{14}a_{41} & a_{14}a_{42} & a_{14}a_{43} \\ a_{24}a_{41} & a_{24}a_{42} & a_{24}a_{43} \\ a_{34}a_{41} & a_{34}a_{42} & a_{34}a_{43} \end{pmatrix}$$
(2.4)

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$$\therefore r \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} - \frac{vu}{r} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{44} \det \begin{pmatrix} a_{11} - \frac{a_{14}a_{41}}{a_{44}} & a_{12} - \frac{a_{14}a_{42}}{a_{44}} & a_{13} - \frac{a_{14}a_{43}}{a_{44}} \\ a_{21} - \frac{a_{24}a_{41}}{a_{44}} & a_{22} - \frac{a_{24}a_{42}}{a_{44}} & a_{23} - \frac{a_{24}a_{43}}{a_{44}} \\ a_{31} - \frac{a_{34}a_{41}}{a_{44}} & a_{32} - \frac{a_{34}a_{42}}{a_{44}} & a_{33} - \frac{a_{34}a_{43}}{a_{44}} \end{pmatrix}$$
(2.5)

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$$a_{44} \det \begin{pmatrix} a_{11} - \frac{a_{14}a_{41}}{a_{44}} & a_{12} - \frac{a_{14}a_{42}}{a_{44}} & a_{13} - \frac{a_{14}a_{43}}{a_{44}} \\ a_{21} - \frac{a_{24}a_{41}}{a_{44}} & a_{22} - \frac{a_{24}a_{42}}{a_{44}} & a_{23} - \frac{a_{24}a_{43}}{a_{44}} \\ a_{31} - \frac{a_{34}a_{41}}{a_{44}} & a_{32} - \frac{a_{34}a_{42}}{a_{44}} & a_{33} - \frac{a_{34}a_{43}}{a_{44}} \end{pmatrix} = a_{44} \det \begin{pmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31}' & a_{32}' & a_{33}' \end{pmatrix}$$
(2.6)

where $a_{11}^{(1)} = a_{11} - \frac{a_{14}a_{41}}{a_{44}}$, $a_{12}^{(1)} = a_{12} - \frac{a_{14}a_{42}}{a_{44}}$, $a_{13}^{(1)} = a_{13} - \frac{a_{14}a_{43}}{a_{44}}$ and so. So symbolically, we have (1) (1) (1)

$$a_{44} \det \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} \end{pmatrix} = a_{44} \cdot a_{33}^{(1)} \det \begin{pmatrix} a_{11}^{(2)} & a_{12}^{(2)} \\ a_{21}^{(2)} & a_{22}^{(2)} \end{pmatrix}$$
(2.7)

Hence for a matrix of order $n \times n$

$$\det \begin{pmatrix} a_{11} & a_{12} & \mathrm{K} & a_{1n} \\ a_{21} & a_{22} & \mathrm{K} & a_{2n} \\ \mathrm{M} & & & \\ a_{n1} & a_{n2} & \mathrm{K} & a_{nn} \end{pmatrix} = a_{nn} a_{n-1n-1}^{(1)} a_{n-2n-2}^{(2)} \mathrm{K} \ a_{33}^{(n-3)} \det \begin{pmatrix} a_{11}^{(n-2)} & a_{12}^{(n-2)} \\ a_{21}^{(n-2)} & a_{22}^{(n-2)} \end{pmatrix}$$
(2.8)

So the determinant of a matrix of order $(n \times n)$ is reduced to the determinant of a matrix order (2×2) which leads to a great reduction in computing storage space and the reduction in calculation and the associated error.

Example 2.1

Find the determinant of the matrix.

$$\begin{pmatrix}
1 & 0 & 0 \\
3 & 2 & -4 \\
4 & 1 & 3
\end{pmatrix}$$

By the process above

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & -4 \\ 4 & 1 & 3 \end{pmatrix} = 3 \det \begin{pmatrix} 1 & 0 \\ 3 & 2 & -\frac{vu}{r} \end{pmatrix}$$
(2.9)

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where
$$v = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$
, $u = \begin{pmatrix} 4 & 1 \end{pmatrix}$ and $r = 3$, therefore
 $vu = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \begin{pmatrix} 4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -16 & -4 \end{pmatrix}$

$$u = \begin{pmatrix} -4 \end{pmatrix} \begin{pmatrix} 4 & 1 \end{pmatrix} = \begin{pmatrix} -16 & -4 \end{pmatrix}$$
(2.10)

and

$$\frac{vu}{r} = \frac{1}{3} \begin{pmatrix} 0 & 0 \\ -16 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ -\frac{16}{3} & -\frac{4}{3} \end{pmatrix}$$
(2.11)

$$\therefore \qquad 3\det\left(\begin{array}{cc} 1 & 0 \\ 3 & 2 \end{array} - \frac{vu}{r}\right) = 3\det\left(\begin{array}{cc} 1 & 0 \\ 3 + \frac{16}{3} & 2 + \frac{4}{3} \end{array}\right) \tag{2.12}$$

$$= 3 \det \begin{pmatrix} 1 & 0 \\ \frac{9+16}{3} & \frac{10}{3} \end{pmatrix} = 3 \times \frac{10}{3} = 10$$
 (2.13)

Example 2.2

$$det \begin{pmatrix} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \\ -1 & -6 & 4 & 3 \end{pmatrix} = 3 det \begin{pmatrix} 2 & 5 & -3 \\ -2 & -3 & 2 & -\frac{vu}{r} \\ 1 & 3 & -2 & -\frac{vu}{r} \end{pmatrix}$$
(2.14)
$$v = \begin{pmatrix} -2 \\ -5 \\ 2 \end{pmatrix}, \quad u = (-1 & -6 & 4)$$

$$vu = \begin{pmatrix} 2 & 12 & -8 \\ 3 & 30 & -20 \\ -2 & -12 & 8 \end{pmatrix}, \quad \frac{vu}{r} = \begin{pmatrix} \frac{2}{3} & 4 & -8 \\ \frac{5}{3} & 10 & -\frac{20}{3} \\ -\frac{2}{3} & -4 & \frac{8}{3} \end{pmatrix}$$
(2.15)
$$\left(\begin{pmatrix} 2 & 5 & -3 \\ 2 & 5 & -3 \end{pmatrix} \right) \qquad \left(\begin{pmatrix} 2 & 5 & -3 \\ 2 & 5 & -3 \end{pmatrix} \right) \begin{pmatrix} \frac{2}{3} & 4 & -\frac{8}{3} \end{pmatrix}$$

$$3 \det \begin{bmatrix} -2 & -3 & 2 & -\frac{vu}{r} \\ 1 & 3 & -2 \end{bmatrix} = 3 \det \begin{bmatrix} -2 & -3 & 2 \\ 1 & 3 & -2 \end{bmatrix} - \begin{bmatrix} \frac{5}{3} & 10 & -\frac{20}{3} \\ -\frac{2}{3} & -4 & \frac{8}{3} \end{bmatrix}$$
(2.16)
$$= 3 \det \begin{bmatrix} \frac{4}{3} & 1 & -\frac{1}{3} \\ -\frac{11}{3} & -13 & \frac{26}{3} \\ \frac{5}{3} & 7 & -\frac{14}{3} \end{bmatrix} = 3 \times \frac{(-14)}{3} \det \begin{bmatrix} \frac{51}{42} & \frac{1}{2} \\ -\frac{12}{21} & 0 \end{bmatrix} = -4$$
(2.17)

3.0 Conclusion

Journal of the Nigerian Association of Mathematical Physics Volume 12 (May, 2008), 413 - 416 Quick evaluation of determinants V. U. Aihie J of NAMP From the above, it is clear that the determinant of any given matrix of order $(n \times n)$ (n > 2) can be reduced to the determinant of a matrix of order (2×2) by the repeated use of order reduction formula. The leads to simple and straight forward calculation and a huge reduction in computing storage space. This coupled with already noted advantage is a much better and efficient way of calculating the determinant of any given matrix of order $(n \times n)$ where n > 2.

References

- [1] Ching Tzong Su and Feng Cheng Chang (1996). Quick Evaluation of Determinants. Appl. Mathematics and Computation 75 117 118.
- [2] Keryszig Erwin (1979). Advanced Engineering Mathematics, John Wiley & Sons. Inc., New York, USA.