Journal of the Nigerian Association of Mathematical Physics
***Volume* 12 (May, 2008), 399 – 408**
© J. of NAMP

On the Tau method for certain over-determined first order differential equations

R. B Adeniyi[1] and E. O. Edungbola[2]
Department of Mathematics,
University of Ilorin, Ilorin, Nigeria

Abstract

In Adeniyi et al.* [7], *we reported a generalization of the original formulation of the Tau method of Lanczos [14] with its associated error estimation. The generalized Tau method is not however, possible for all three variants of the method, namely the differential (or original), the integrated and the recursive formulation, due to the difficulty in constructing the so-called canonical polynomial which is the basis function required for the recursive form. Yet, it is worthwhile to compare the three variants, as much as feasible. Consequently, in this paper, we present the three variants together with their error estimates for a class of first orderoverdetermined ordinary differential equations with unit overdetermination. Numerical evidences are provided in support of the accuracy of our results.

***Keywords*:** Tau method, variants, error, error estimate, recursive, differential, integrated, approximant, canonical polynomials, formulation.

1.0 Introduction

Accurate approximate solution of linear ordinary differential equations with polynomial coefficients may be obtained by the tau method of Lanczos (See [2] – [4], [13], [14], [17]) introduced in 1938. The method is related to the principle of economization of a differentiable function, implicitly defined by a linear differential equation (DE) with polynomial coefficients. Since then, variants of this method have emerged, some of which we now highlight below:

1.1 Differential or original form of the Tau method.

We describe briefly here the original formulation of the tau method by considering the boundary value problem (BVP) in the m-th order linear DE:

$$Ly(x):=\sum_{r=0}^{m} P_r(x)y^{(r)}(x) = f(x), \quad a\le x \le b \tag{1.1a}$$

$$L^* y(x_{rk}):=\sum_{r=0}^{m-1} a_{rk}y^{(r)}(x_{rk}) = \alpha_k \ , \ k = 1(1)m \tag{1.1b}$$

where
$$y_n(x) = \sum_{r=0}^{n} a_r \ x^r \ , \ 0 < n < +\infty \tag{1.2}$$

of $y(x)$ which is the exact solution of perturbed problem:

$$Ly_n(x_{rk}):=\sum_{r=0}^{m-1} P_r(x)y_n^{(r)}(x) = f(x) + \sum_{r=0}^{m+s-1}\tau_{m+s-r} \ T_{n-m+r+1}(x) \tag{1.3a}$$

[1]**e-mail: adeniyibr@yahoo.com.**
[1]Author thankfully acknowledges financial support from the University's Senate Research Grant.

$$L * y_n(x) := \sum_{r=0}^{m-1} a_{rl} y_n^{(r)}(x_{rk}) = \alpha_k \ , \ k = 1(a)m \tag{1.3b}$$

for a $\leq$ x $\leq$ b and where τ_r , $r = 1(1)m$, are parameters to be determined along with the a's in (1.3), $T_r(x)$ is the r-th degree Chebyshev polynomial (See [10], [11]) valid in [a , b] and

$$s = \max \{ N_r - r / 0 \leq r \leq m \} \tag{1.4}$$

is the number of overdetermination of equation (1.1a) (See [2], [11]). For more explanation on the parameter, also see Section 1.3.

We determine a_r , $r = 0(1)n, \quad and \ \ \tau_r$, $r = 1(1)m$ from the linear algebraic system

$$A \underline{\tau} = \underline{b}, \tag{1.5}$$

obtained by equating corresponding coefficients of powers of x from (1.3a) and then applying conditions (1.3b);

$$A = (a_{ij}), 1 \leq i, \ j \leq n + m + s + 1; \quad \underline{b} = (b_i), 1 \leq i \leq n + m + s + 1;$$

$$\underline{\tau} = (a_0 \ , a_1 \ , \ \ldots \ , a_n \ , \tau_1 \ , \ldots \ \tau_{m+s})^T.$$

Consequently, we obtain from (1.2) our desired approximant $y_n(x)$ of $y(x)$.

1.2 The integrated formulation of the Tau method

If $\int\int\int^{r} \cdot \int g(x)dx$ denotes the indefinite integration r times applied to the function g(x) and

$$I_L = \int\int\int^{m} \ldots \int L(.) \ dx \tag{1.6}$$

then the integrated form of (1.3a) is

$$I_L(y(x)) = \int\int\int^{m} \ldots \int f(x) \ dx + c_m(x) \tag{1.7}$$

where $c_m(x)$ denotes an arbitrary polynomial of degree (m - 1), arising from the constants of integration. The approximant (1.2), now, then satisfies the perturbed problem

$$I_L(y_n(x)) = \int\int\int^{m} \ldots \int f(x) \ dx + c_m(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r}(x) \, T_{n+r+1}(x) \tag{1.8a}$$

$$L * y_n(x_{rk}) = \alpha_k \ , \ k = 1(1) \, m \tag{1.8b}$$

The tau problem (1.8) often gives a more accurate approximant of y(x) than does (1.3) due to its higher order perturbation term. See ([2], [10], [11]).

1.3 The recursive formulation of the Tau method.

To give some flexibility in computation of tau solution, Lanczos [13] introduced a systematic use of the so-called canonical polynomial, $Q_r(x)$, defined by

$$LQ_r(x) = x^r \tag{1.9}$$

where L is given by (1.1a), $r \,\varepsilon\, N_0 - S$, S is a small finite or empty set of indices with cardinality s($s \leq m + h$); h is the maximum difference between the exponent r of x and the leading exponent of the generating polynomial L x^r, for $r \,\varepsilon\, N_0$.

Due to the difficulties in the construction of these polynomials, Ortiz [17] in 1969 proposed a recursive generation of the polynomials. Once these polynomials are generated, the tau approximant of $y(x)$ is then obtained as

$$y_n(x) = \sum_{r=0}^{F} f_r Q_r(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r} \sum_{k=0}^{n-m+r+1} C_k^{(n-m+r+1)} \ Q_k(x) \tag{1.10}$$

where f_r, $r = 0(1) \, F$, are the coefficients of $f(x)$, $C_k^{(r)}$ is the coefficient of x^k in the r-th degree Chebyshev polynomial, and the approximant $y_n(x)$ now takes the form

$$y_n(x) = \sum_{r=0}^{n} a_r \; Q_r(x) \qquad (1.11)$$

Their use offer some advantages as the Q'$_r$s neither depend on the boundary condition nor on the interval in which the solution is sought. They are also re-useable for approximants of higher degrees.

In what follows in the next section we shall review briefly an error estimation technique of the Tau method which we had reported in [6] and [17], section 3 focuses on the central concerns of this paper, some numerical examples will be considered in section 4 and we shall finally conclude the paper in section 5 with some concluding remarks.

2.0 An error estimation of the Tau method.

We review here error estimations for the three variants of the tau method discussed in the preceding section.

2.1 Error estimation for the differential form.

For an error estimation of the Tau method we constructed, based on the error of the Lanczos economization process, the error approximant.

$$(e_n(x))_{n+1} = \; \varphi_n \; \mu_m(x)\, T_{n-m+1}(x) / C_{n-m+1}^{(n-m+1)} \cong y(x) - y_n(x) = e_n(x) \qquad (2.1)$$

which satisfies, exactly, the perturbed error problem

$$L(e_n(x))_{n+1} = -\sum_{r=0}^{m+s-1} \tau_{m+s-r} \; T_{n-m+r+1}(x) + \sum_{r=0}^{m+s-1} \bar{\tau}_{m+s-r} \; T_{n-m+r+2}(x) \qquad (2.2a)$$

$$L^*(e_n(x_{rk}))_{n+1} = 0 \qquad (2.2b)$$

and where the extra parameters $\bar{\tau}_r$, $r = 1(1)\,m+s$ and φ_n are to be determined; $\mu_m(x)$ is a specified polynomial of degree m which ensures that $(e_n(x))_{n+1}$ satisfies the homogeneous conditions associated with $e_n(x)$.

We insert (2.1) in (2.2a) and then equate corresponding coefficients of x^{n+s+1} , x^{n+s} , ... , x^{n-m+1}. The resulting linear system is solved for φ_n by forward elimination, and consequently we obtain the apriori error estimate

$$\bar{\varepsilon}_1 = \max_{a \le x \le b} \left|(e_n(x))_{n+1}\right| = \; |\varphi_n| / \left|C_{n-m+1}^{(n-m+1)}\right| \cong \max_{a \le x \le b} |e_n(x)| = \varepsilon_1 \qquad (2.3)$$

2.2 Error Estimation for the integrated form

For an error estimation of the integrated form, we have from (2.2a)

$$I_L(e_n(x))_{n+1} = -\int\int\int^m \ldots \int \left(\sum_{r=0}^{m+s-1} \tau_{m+s-r} \; T_{n-m+r+1}(x) \right) dx + c_m(x) + \sum_{r=0}^{m+s-1} \bar{\tau}_{m+s-r} \; T_{n-m+r+3}(x) \qquad (3.4)$$

where $(e_n(x))_{n+1}$ is given by (2.1). Once (2.1) is inserted in (2.4) we equate corresponding coefficients of $x^{n+s+m+1}$, x^{n+s+m} … , x^{n+1}. Subsequent procedures follow as described in section 2.2 to obtain the error estimate ε_I .

2.3 Error estimation for the recursive form

Once the canonical polynomials of section 1 are generated, they can be used for an error estimation of the tau method (see [6], [8], [15]). Here we consider a slight perturbation of the given boundary conditions (1.1b) by $\bar{\varepsilon}_1$ to obtain an estimate of the Tau parameter τ_{m+s} , in terms of canonical polynomials, which is then substituted into the expression for $\bar{\varepsilon}_I$ given in (2.3) for a new estimate $\bar{\varepsilon}_2$. See section 3.3.1 below.

3.0 A class of over-determined first order equations

We consider here the three variants of the Tau method of the preceding section and their error estimation as applied to the class (1.1) when m = 1 and for s = 1, that is,

$$Ly(x) := \left(\alpha_0 + \alpha_1 x + \alpha_2 x^2\right) y'(x) + \left(\beta_0 + \beta_1 x\right) y(x) = \sum_{r=0}^{F} f_r\ x^r;\ a \le x \le b \qquad (3.1a)$$

where $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1$ are given constants and $F \le n + 1$.

Without loss of generality we shall assume that a = 0 and b = 1, since, by the transformation

$$u = (x-a)/(b-a)\ ,\ a \le x \le b$$

we may readily transform (3.1) into the interval [0 , 1]

3.1 Tau approximant by the differential form

If we insert (1.2) into the perturbed form of (3.1a), we have that

$$\left(\alpha_0 + \alpha_1 x + \alpha_2 x^2\right) \sum_{r=0}^{n} r\, a_r\ x^{r-1} + \left(\beta_0 + \beta_1 x\right) \sum_{r=0}^{n} a_r\ x^r = \sum_{r=0}^{F} f_r\ x^r + \tau_1 T_{n+1}(x) + \tau_2\ T_n(x)$$

That is $\left(\beta_0\ \alpha_0 + \alpha_0 a_1 - \tau_2\ C_0^{(n)} - \tau_1\ C_0^{(n+1)} - f_0\right)$

$$+ \sum_{r=1}^{n-1}\left[\left(\alpha_2(r-1) + \beta_1\right) a_{r-1} + \left(\alpha_1 r + \beta_0\right) a_r + (r+1)\alpha_0\ a_{r+1} - \tau_2\, C_r^{(n)} - \tau_1\, C_r^{(n+1)} - f_r\right] x^r$$

$$+ \left[\left(\alpha_2(n-1) + \beta_1\right) a_{n-1} + \left(\alpha_1 n + \beta_0\right) a_n - \tau_2\, C_n^{(n)} - \tau_1\ C_n^{(n+1)} - f_n\right] x^n$$

$$+ \left[\left(\alpha_2 n + \beta_1\right) a_n - \tau_1\ C_{n+1}^{(n+1)} - f_{n+1}\right] x^{n+1} = 0$$

We equate corresponding coefficients of x to zero to get the system:

$$\beta_0\ \alpha_0 + \alpha_0\, a_1 - \tau_1\ C_0^{(n)} - \tau_1\, C_0^{(n+1)} = f_0$$

$$\left(\alpha_2(r-1) + \beta_1\right) a_{r-1} + \left(\alpha_1 r + \beta_0\right) a_r + \alpha_0 (r+1) a_{r+1} - \tau_2\, C_r^{(n)} - \tau_1\, C_r^{(n+1)} = f_r\ ,\ r = 1(1)n-1$$

$$\left(\alpha_2(n-1) + \beta_1\right) a_{n-1} + \left(\alpha_1 n + \beta_0\right) a_n - \tau_2 C_n^{(n)} - \tau_1 C_n^{(n+1)}\ f_n \left(\alpha_2\, n + \beta_1\right) a_n - \tau_1 C_n^{(n+1)} = f_{n+1}$$

We solve this system together with $a_0 = A$, obtained from the condition (3.16), so as to determine a_r , r = 0(1)n in (1.2). Consequently we obtain the desired approximant $y_{n,1}(x)$ of $y(x)$.

3.1.1 Error estimation for differential form

From (2.2a), we have for problem (3.1)

$$L\left(e_n(x)\right)_{n+1} = \bar{\tau}_1\ T_{n+2}(x) + \left(\bar{\tau}_2 - \tau_1\right) T_{n+1}(x) - \tau_2\ T_n(x)$$

where

$$L = \left(\alpha_0 + \alpha_1 x + \alpha_2 x^2\right) \frac{d}{dx} + \left(\beta_0 + \beta_1 x\right) \qquad (3.4)$$

$$\left(e_n(x)\right)_{n+1} = \varphi_n\, x\, T_n(x) / C_n^{(n)} = \varphi_n \left(\sum_{r=0}^{n} C_r^{(n)}\ x^{r+1}\right) \Big/ C_r^{(n)} \qquad (3.5)$$

By equating the coefficients of x^{n+2}, x^{n+1} and x^n in (3.3), we have the system

$$\theta\left[\beta_1\, C_n^{(n)} + (n+1)\alpha_2\, C_n^{(n)}\right] = \bar{\tau}_1\ C_{n+2}^{(n+2)}$$

$$\theta\left[\alpha_1 (n+1) C_n^{(n)} + n\alpha_2\, C_{n-1}^{(n)} + \beta_0 C_n^{(n)} + \beta_1 C_{n-1}^{(n)}\right] = \bar{\tau}_1\ C_{n+1}^{(n+2)} + \left(\bar{\tau}_2 - \tau_1\right) C_{n+1}^{(n+1)}$$

$$\theta\left[\alpha_0 (n+1) C_n^{(n)} + n\alpha_2\, C_{n-1}^{(n)} + (n-1)\alpha_2 C_{n-2}^{(n)} + \beta_0 C_{n-1}^{(n)} + \beta_1 C_{n-2}^{(n)}\right] =$$

$$\bar{\tau}_1\ C_{n-1}^{(n+2)} + \left(\bar{\tau}_2 - \tau_1\right) C_n^{(n+1)} - \tau_2 C_n^{(n)},$$

where $\theta = \varphi_n \left(C_n^{(n)}\right)^{-1}$. We solve this system by forward elimination for φ_n using the well-known relations

$$C_n^{(n)} = 2^{2n-1} \quad , \; C_{n-1}^{(n)} = -\frac{1}{2} n \, C_n^{(n)}$$

to have

$$\varphi_n = -\, 2^{4n+2} \;\; \tau_2 / R_1 \tag{3.6}$$

where $R_1 = 16 \left[2^{2n-1}(n+1)\alpha_0 \; - 2^{2n-R} \; n^2 \, \alpha_1 + (n-1)\alpha_2 \, C_{n-2}^{(n)} \; - 2^{2n-2} n \, \beta_0 + \beta_1 C_{n-2}^{(n)} \right]$

$$-[\beta_1 + (n+1)\alpha_2]\, C_n^{(n+2)} + 2^{2n+1}(n+1) \left[2(n+1)\alpha_1 + (3n+2)\alpha_2 + 2\beta_0 + 2\beta_1\right] \tag{3.7}$$

From (2.3) we obtain the error estimate

$$\bar{\varepsilon}_1 = 2^{2n+3} \left|\tau_2\right| / \left|R_1\right| \tag{3.8}$$

3.2 Tau approximant by the integrated form

From (1.7) we get

$$\int_0^x \left(\alpha_0 + \alpha_1 u + \alpha_2 \, u^2\right) y'(u) du + \int_0^x (\beta_0 + \beta_1 u) y \;(u) du \;= \int_0^x \left(f_0 + f_1 u + \; f_2 u^2\right) du$$

or $\left(\alpha_0 + \alpha_1 x + \alpha_2 \; x^2\right) y(x) - \int_0^x (\alpha_1 + 2\alpha_2 u) y(u)\, du + (\beta_0 + \beta_1 u) y(u)\, du = \alpha_0 \, A + \sum_{r=0}^{F} \frac{f_x \; r^{r+1}}{r+1}$ and from (1.8a) together with (1.2) we have

$$\sum_{r=0}^{n} \left(\alpha_0 \; + \alpha_1 x + \alpha_2 \, x^2\right) a_r \, x^r - \sum_{r=0}^{n} \left(\frac{\alpha_1 \, a_r \, x^{r+1}}{r+1} + \frac{2\alpha_2 \;\, a_r \, x^{r+2}}{r+2} \right) +$$

$$\sum_{r=0}^{m} \left(\frac{\beta_0 \, a_r \, x^{r+1}}{r+1} + \frac{\beta_1 \;\, a_r \, x^{r+2}}{r+2} \right) a_r \; = \alpha_0 \;\; A + \sum_{r=0}^{F} \frac{f_r \;\; x^{r+1}}{r+1} + \tau_1 \, T_{n+2}(x) \;\; + \;\; \tau_2 \, T_{n+2}(x)$$

That is,

$$\left(\alpha_0 \;\, a_0 \; - \tau_2 \;\, C_0^{(n+1)} \; - \tau_2 C_0^{(n+1)} \; - \alpha_0 \; A\right)$$

$$+ \; \sum_{r=1}^{n} \left[\left(\alpha_2 \; - \frac{2\alpha_2}{r} + \frac{\beta_1}{r} \right) a_{r-2} \; + \left(\alpha_1 \; - \frac{\alpha_1}{r} + \frac{\beta_0}{r} \right) a_{r-1} + \alpha_0 \, a_r - \tau_2 \, C_r^{(n+1)} - \tau_1 \, C_r^{(n+2)} - \frac{f_{r-1}}{r} \right] x^r +$$

$$\left[\left(\alpha_2 - \frac{2\alpha_2}{n+1} + \frac{\beta_1}{n+1} \right) a_{n-1} \; + \left(\alpha_1 \; - \frac{\alpha_1}{n+1} + \frac{\beta_0}{n+1} \right) a_n - \tau_2 \, C_{n+1}^{(n+1)} - \tau_1 \, C_{n+1}^{(n+2)} \; - \frac{f_n}{n+1} \right] x^{n+1} +$$

$$\left[\left(\alpha_2 \; - \frac{2\alpha_2}{n+2} + \frac{\beta_1}{n+2} \right) a_n - \;\, \tau_1 \, C_{n+2}^{(n+2)} \; - \frac{f_{n+1}}{n+2} \right] x^{n+2} \; = \; 0$$

From this we obtain the linear system

$$\begin{aligned} &\alpha_0 \;\, a_0 \; - \tau_2 \, C_0^{(n+1)} \; - \; \tau_1 \, C_0^{(n+2)} \; = \alpha_0 \, A \\ &\frac{1}{r}(r\alpha_2 \; - \, 2\alpha_2 \; + \; \beta_1)\, a_{r-2} + \frac{1}{r}(r\alpha_1 \; - \; \alpha_1 \; + \; \beta_0)\, a_{r-1} + \alpha_0 \, a_r - \tau_2 \, C_r^{(n+1)} \; - \\ &\tau_1 \, C_r^{(n+2)} \; = \frac{f_{r-1}}{r} \; , \; r \; = 1 \; (1) \, n \end{aligned} \tag{3.9}$$

$$\left(\frac{n\alpha_2 - \alpha_2 + \beta_1}{n+1}\right) a_{n-1} + \left(\frac{n\alpha_1 - \alpha_1 + \beta_0}{n+1}\right) a_n - \tau_2 C_{n+1}^{(n+1)} - \tau_1 C_{n+1}^{(n+2)} = \frac{f_n}{n+1} x^{n+1}$$

$$\left(\frac{n\alpha_2 + \beta_0}{n+2}\right) a_n + \tau_1 C_{n+2}^{(n+2)} = \frac{f_n}{n+2}$$

We solve this for a_r , $r = 0(1)n$ *and* τ_1, τ_2. Subsequently, we obtain from (1.2) our approximant $y_{n,2}(x)$.

3.2.1 Error estimation for the integrated form

From (2.4) we have for problem (3.1)

$$\int_0^x \left(\alpha_0 + \alpha_1 u + \alpha_2 u^2\right)\left(e(u)_{n+1}\right)_{n+1} du + \int_0^x \left(\beta_0 + \beta_1 u\right)\left(e_n(u)\right)_{n+1} du$$

$$= -\int_0^x \left[\tau_1 T_{n+1}(u) + \tau_2 T_n(u)\right] du + \bar{\tau} T_{n+2}(x) + \bar{\tau}_2 T_{n+1}(x)$$

$$\frac{\phi_n}{C_n^{(n)}}\left[\sum_{r=0}^{n}\left(\alpha_0 + \alpha_1 x + \alpha_2 x^2\right) a_r x^{r+1} - \sum_{r=0}^{n}\left(\frac{\alpha_1 a_r x^{r+2}}{r+2} + \frac{2\alpha_2 a_r x^{r+3}}{r+3}\right) + \right.$$

$$\left.\sum_{r=0}^{n}\left(\frac{\beta_0 a_r x^{r+2}}{r+2} + \frac{\beta_1 a_r x^{r+3}}{r+3}\right)\right] = -\tau_1 \sum_{r=0}^{n+1}\frac{C_r^{(n+1)} x^{r+1}}{r+1} - \tau_2 \sum_{r=0}^{n}\frac{C_r^{(n+1)} x^{r+1}}{r+1} +$$

$$\bar{\tau}_1 \sum_{r=0}^{n+3} C_r^{(n+3)} x^r + \bar{\tau}_2 \sum_{r=0}^{n+2} C_r^{(n+2)} x^r$$

where $\left(e_n(x)\right)_{n+1}$ is again given by (3.5). This leads to

$$\frac{\varphi_n}{C_n^{(n)}}\left[\sum_{r=0}^{n}\left(\alpha_0 + \alpha_1 x + \alpha_2 x^2\right) a_r x^{r+1} - \sum_{r=0}^{n}\left(\frac{\alpha_1 a_r x^{r+2}}{r+2} + \frac{2\alpha_2 a_r x^{r+3}}{r+3}\right) + \right.$$

$$\left.\sum_{r=0}^{n}\left(\frac{\beta_0 a_r x^{r+2}}{r+2} + \frac{\beta_1 a_r x^{r+3}}{r+3}\right)\right] = -\tau_1 \sum_{r=0}^{n+1}\frac{C_r^{(n+1)} x^{r+1}}{r+1} - \tau_2 \sum_{r=0}^{n}\frac{C_r^{(n+1)} x^{r+1}}{r+1}$$

$$+ \bar{\tau}_1 \sum_{r=0}^{n+3} C_r^{(n+3)} x^r + \bar{\tau}_2 \sum_{r=0}^{n+2} C_r^{(n+2)} x^r$$

We equate corresponding coefficients of x^{n+3}, and x^{n+1} to have the system

$$\frac{\varphi_n}{C_n^{(n)}}\left[\alpha_2 C_n^{(n)} - \frac{2\alpha_2 C_n^{(n)}}{n+2} + \frac{\beta_1 C_n^{(n)}}{n+3}\right] = \bar{\tau}\, C_{n+3}^{(n+3)}$$

$$\frac{\varphi_n}{C_n^{(n)}}\left[\alpha_1 C_n^{(n)} + \alpha_2 C_{n-1}^{(n)} - \frac{\alpha_1 C_n^{(n)}}{n+2} - \frac{2\alpha_2 C_{n-1}^{(n)}}{n+2} + \frac{\beta_1 C_n^{(n)}}{n+2} + \frac{\beta_0 C_n^{(n)}}{n+2}\right] \tag{3.11}$$

$$= \bar{\tau}_1\, C_{n+2}^{(n+3)} + \bar{\tau}_2\, C_{n+2}^{(n+2)} - \frac{1}{2}\tau_1\, C_{n+1}^{(n+1)}$$

$$\frac{\varphi_n}{C_n^{(n)}}\left[\alpha_0 C_n^{(n)} + \alpha_1 C_{n-1}^{(n)} + \alpha_2 C_{n-2}^{(n)} - \frac{\alpha_1 C_{n-1}^{(n)}}{n+1} - \frac{2\alpha_2 C_{n-2}^{(n)}}{n+1} + \frac{\beta_1 C_{n-2}^{(n)}}{n+1} + \frac{\beta_0 C_{n-1}^{(n)}}{n+1}\right]$$

$$= \bar{\tau}_1\, C_{n+1}^{(n+3)} + \tau_2\, C_{n+1}^{(n+2)} - \frac{\tau_1 C_n^{(n+1)}}{n+1} - \frac{\tau_2 C_n^{(n)}}{n+1}$$

We solve this by forward elimination for φ_n to have $\varphi_n = \dfrac{-2^{4n+4}(n+3)\,\tau_2}{R_2}$

where

$$\begin{aligned} R_2 &= 2^{2n+5}(n+1)(n+2)\alpha_0 + 32\left[(n+1)(n+3)C_{n-2}^{(n)} - 2^{2n-1}n(n+2)(n+3) + 2^{2n-1}(n+1)^2(n+2)\right]\alpha_1 \\ &+ 2^{2n+4}(n+3)\beta_0 + (n+1)\left[32(n+3)C_{n-2}^{(n)} - C_{n-1}^{(n+3)} + 2^{2n+4}(n+3)\right]\beta_1 \\ &+ (n+1)\left[2^{2n+4}\, n(n+3) - 64(n+3)C_{n-2}^{(n)} - nC_{n-1}^{(n+3)}\right]\alpha_2 \\ &+ n\,(n+1)\left[2^{2n+3}(n+2)(n+3) - C_{n-1}^{(n+3)}\right] \end{aligned} \tag{3.12}$$

Hence the error estimate is $\bar{\varepsilon}_2 = 2^{2n+5}(n+3)\,{}^{|\tau_2|}\!/_{|R_2|}$

3.3 A Tau approximant by the recursive form

For the problem (3.1), from (1.10) we have that

$$y_n(x) = \sum_{r=0}^{F} f_r\, Q_r(x) + \tau_1 \sum_{r=0}^{n+1} C_r^{(n+1)} Q_r(x) + \tau_2 \sum_{r=0}^{n} C_r^{(n)} Q_r(n) \tag{3.13}$$

If $F \le n + 1$, then this becomes

$$y_n(x) = \sum_{r=0}^{n}\left[f_r + \tau_1 C_r^{(n+1)} + \tau_2 C_r^{(n)}\right]Q_r(x) + \left[f_{n+1} + \tau_1 C_{n+1}^{(n+1)}\right]Q_{n+1}(x) \tag{3.14}$$

where the sequence of canonical polynomials $\{Q_r(x)\}$, r ε N_0 – S, is generated thus:
From (1.9) and (3.4), and by the linearity of L,

$$\begin{aligned} Lx^r &= \left(\alpha_0 + \alpha_1 x + \alpha_2 x^2\right) x^{r-1} + \left(\beta_0 + \beta_1 x\right) x^r \\ &= L\left\{\alpha_0\, rQ_{r-1}(x) + \left(\alpha_1 r + \beta_0\right) Q_r(x) + \left(\alpha_2 r + \beta_1\right) Q_{r+1}(x)\right\} \end{aligned}$$

By assuming the existence of L^{-1} we obtain

$$Q_{r+1}(x) = \left[x^r - \alpha_0\, rQ_{r-1}(x) - \left(\alpha_1 r + \beta_0\, Q_r(x)\right)\right]\left(\alpha_2\, r + \beta_1\right)^{-1} \tag{3.15}$$

for r = 0 , 2 , … and provided that $\alpha_2\, r + \beta_1 \neq 0.$

From (3.15) we generate as many canonical polynomials as needed, depending on the degree of the approximant $y_n(x)$. We see that $Q_0(x)$ is undefined by (3.15) and hence

$S \equiv \{0\}$ with $s = 1$. The difficulty in generating the polynomials $Q_r(x)$ for large value of r will limit our discussion here to a typical approximant of degree 5. Consequently we shall let n = 5 in (3.14) to have.

$$y_5(x) = \sum_{r=0}^{5}\left[f_r + \tau_1\ C_r^{(6)} + \tau_2\ C_r^{(6)}\right]Q_r(x) + \left[f_6 + \tau_1\ C_6^{(6)}\right]Q_6(x) \tag{3.16}$$

where $Q_r(x)$, r = 1(1)6, are obtained from (3.15). Since $Q_0(x)$ cannot be determined we shall equate its coefficient to zero to have the equation

$$\lambda_1\ \tau_1 + \lambda_2\ \tau_2 = \eta_1 \tag{3.17}$$

Whatever now remains of (3.16) constitutes our new $y_5(x)$, call this $Y_5(x)$, say. Applying the condition (3.1b) to $Y_5(x)$ also yields a second equation

$$v_1\ \tau_1 + v_2\ \tau_2 = \eta_2 \tag{3.18}$$

The quantities $\lambda_1, \lambda_2, \eta_1,\ v_2, v_2,\ and\ \ \eta_1$ are defined as

$$\lambda_1 = C_1^{(6)}\rho_1 + C_2^{(6)}\rho_2 + C_3^{(6)}\rho_3 + C_4^{(6)}\rho_4 + C_5^{(6)}\rho_5 + C_6^{(6)}\rho_6 \tag{3.19}$$

$$\lambda_2 = C_1^{(5)}\rho_1 + C_2^{(5)}\rho_2 + C_3^{(5)}\rho_3 + C_4^{(5)}\rho_4 + C_5^{(5)}\rho_5 + C_6^{(5)}\rho_6 \tag{3.20}$$

$$v_1 = C_1^{(6)}\gamma_1 + C_2^{(6)}\gamma_2 + C_3^{(6)}\gamma_3 + C_4^{(6)}\gamma_4 + C_5^{(6)}\gamma_5 + C_6^{(6)}\gamma_6 \tag{3.22}$$

$$v_2 = C_1^{(5)}\gamma_1 + C_2^{(5)}\gamma_2 + C_3^{(5)}\gamma_3 + C_4^{(5)}\gamma_4 + C_5^{(5)}\gamma_5 \tag{3.23}$$

$$\eta_1 = f_0 + f_1\rho_1 + f_2\rho_2 + f_3\rho_3 + f_4\rho_4 + f_5\rho_5 + f_6\rho_6 \tag{3.21}$$

$$\eta_2 = A + f_1\gamma_1 + f_2\gamma_2 + f_3\gamma_3 + f_4\gamma_4 + f_5\gamma_5 + f_6\gamma_6 \tag{3.24}$$

where $\rho_1 = -\dfrac{\beta_0}{\beta_1},\ \rho_2 = -\dfrac{\beta_0(\alpha_1+\beta_0)}{\beta_1(\alpha_2+\beta_1)} - \dfrac{\alpha_0}{\alpha_2+\beta_1}$

$$\rho_3 = \frac{2\alpha_0\beta_0}{\beta_1(2\alpha_2+\beta_1)} + \frac{(2\alpha_2+\beta_0)}{(2\alpha_2+\beta_1)}\left[\frac{\alpha_0}{\alpha_2+\beta_1} - \frac{\beta_0(\alpha_1+\beta_0)}{\beta_1(\alpha_2+\beta_1)}\right]$$

$$\rho_4 = \frac{3\alpha_0}{3\alpha_2+\beta_1}\left[\frac{\alpha_0}{\alpha_2+\beta_1} - \frac{\beta_0(\alpha_1+\beta_0)}{\beta_1(\alpha_2+\beta_1)}\right] -$$
$$\frac{(3\alpha_1+\beta_0)}{(3\alpha_2+\beta_1)}\left[\frac{2\alpha_0\beta_0}{\beta_1(2\alpha_2+\beta_1)} + \frac{(2\alpha_1+\beta_0)}{(2\alpha_2+\beta_1)}\left(\frac{\alpha_0}{\alpha_2+\beta_1} - \frac{\beta_0(\alpha_1+\beta_0)}{\beta_1(\alpha_2+\beta_1)}\right)\right]$$

$$\rho_5 = \frac{-4\alpha_0}{(4\alpha_2+\beta_1)}\left[\frac{2\alpha_0\beta_0}{\beta_1(2\alpha_2+\beta_1)} + \frac{(2\alpha_1+\beta_0)}{(2\alpha_2+\beta_1)}\left(\frac{\alpha_0}{\alpha_2+\beta_1} - \frac{\beta_0(\alpha_1+\beta_0)}{\beta_1(\alpha_2+\beta_1)}\right)\right]$$

$$+\frac{(4\alpha_1+\beta_0)}{(4\alpha_2+\beta_1)}\left[\frac{3\alpha_0}{(3\alpha_2+\beta_1)}\left(\frac{\alpha_0}{\alpha_2+\beta_1} - \frac{\beta_0(\alpha_1+\beta_0)}{\beta_1(\alpha_2+\beta_1)}\right) - \frac{(3\alpha_1+\beta_0)}{(3\alpha_2+\beta_1)}\right.$$
$$\left.+\left(\frac{2\alpha_0\beta_0}{\beta_1(2\alpha_2+\beta_1)} + \frac{(2\alpha_1+\beta_0)}{(2\alpha_2+\beta_1)}\left(\frac{\alpha_0}{\alpha_2+\beta_1} - \frac{\beta_0(\alpha_1+\beta_0)}{\beta_1(\alpha_2+\beta_1)}\right)\right)\right]$$

$$\rho_6 = -\frac{5\alpha_0}{(5\alpha_2+\beta_1)}\left[\frac{3\alpha_0\ \beta_0}{(3\alpha_2+\beta_1)}\left(\frac{\alpha_0}{\alpha_2+\beta_1}-\frac{\beta_0(\alpha_1+\beta_0)}{\beta_1(\alpha_2+\beta_1)}\right)\right.$$

$$\left.-\frac{(3\alpha_1+\beta_0)}{(3\alpha_2+\beta_1)}\left(\frac{2\alpha_0\ \beta_0}{\beta_1(2\alpha_2+\beta_1)}+\frac{(2\alpha_1+\beta_0)}{(2\alpha_2+\beta_1)}\left(\frac{\alpha_0}{\alpha_2+\beta_1}-\frac{\beta_0(\alpha_1+\beta_0)}{\beta_1(\alpha_2+\beta_1)}\right)\right)\right]$$

$$-\frac{5\alpha_1+\beta_0}{5\alpha_2+\beta_1}\left[\frac{4\alpha_0}{(4\alpha_2+\beta_1)}\left(\frac{2\alpha_0\ \beta_0}{\beta_1(2\alpha_2+\beta_1)}+\frac{(2\alpha_1+\beta_0)}{(2\alpha_2+\beta_1)}\left(\frac{\alpha_0}{\alpha_2+\beta_1}-\frac{\beta_0(\alpha_1+\beta_0)}{\beta_1(\alpha_2+\beta_1)}\right)\right)\right.$$

$$+\frac{(4\alpha_1+\beta_0)}{(4\alpha_2+\beta_1)}\left(\frac{3\alpha_0}{(3\alpha_2+\beta_1)}\left(\frac{\alpha_0}{\alpha_2+\beta_1}-\frac{\beta_0(\alpha_1+\beta_0)}{\beta_1(\alpha_2+\beta_1)}\right)\right.$$

$$\left.\left.-\frac{(3\alpha_1+\beta_0)}{(3\alpha_2+\beta_1)}\left(\frac{2\alpha_0\ \beta_0}{\beta_1(2\alpha_2+\beta_1)}+\frac{(2\alpha_1+\beta_0)}{(2\alpha_2+\beta_1)}\left(\frac{\alpha_0}{\alpha_2+\beta_1}-\frac{\beta_0(\alpha_1+\beta_0)}{\beta_1(\alpha_2+\beta_1)}\right)\right)\right)\right]$$

$$\gamma_1 = -\frac{1}{\beta_1},\ \gamma_2 = -\frac{(\alpha_1+\beta_0)}{\beta_1(\alpha_2+\beta_1)},\ \gamma_3 = \frac{(2\alpha_1+\beta_0)(\alpha_1+\beta_0)}{\beta_1(2\alpha_2+\beta_1)(\alpha_2+\beta_1)}-\frac{2\alpha_0}{\beta_1(2\alpha_2+\beta_1)}$$

$$\gamma_4 = -\frac{3\alpha_0(\alpha_1+\beta_0)}{\beta_1(3\alpha_2+\beta_1)(\alpha_2+\beta_1)}+\frac{(3\alpha_1+\beta_0)}{(3\alpha_2+\beta_1)}\left[\frac{2\alpha_0}{\beta_1(2\alpha_2+\beta_1)}-\frac{(2\alpha_1+\beta_0)(\alpha_1+\beta_0)}{\beta_1(2\alpha_2+\beta_1)(\alpha_2+\beta_1)}\right]$$

$$\gamma_5 = \frac{4\alpha_0}{\beta_1(4\alpha_2+\beta_1)}\left[\frac{2\alpha_0}{\beta_1(2\alpha_2+\beta_1)}-\frac{(2\alpha_1+\beta_0)(\alpha_1+\beta_0)}{\beta_1(2\alpha_2+\beta_1)(\alpha_2+\beta_1)}\right]$$

$$-\frac{(4\alpha_1+\beta_0)}{(4\alpha_2+\beta_1)}\left[\frac{3\alpha_0(\alpha_1+\beta_0)}{\beta_1(3\alpha_2+\beta_1)(\alpha_2+\beta_1)}+\frac{(3\alpha_1+\beta_0)}{(3\alpha_2+\beta_1)}\left(\frac{2\alpha_0}{\beta_1(2\alpha_2+\beta_1)}-\frac{(2\alpha_1+\beta_0)(\alpha_1+\beta_0)}{\beta_1(2\alpha_2+\beta_1)(\alpha_2+\beta_1)}\right)\right]$$

$$\gamma_6 = -\frac{5\alpha_0}{(5\alpha_2+\beta_1)}\left[\frac{3\alpha_0(\alpha_1+\beta_0)}{\beta_1(3\alpha_2+\beta_1)(\alpha_2+\beta_1)}+\frac{(3\alpha_1+\beta_0)}{(3\alpha_2+\beta_1)}-\frac{(4\alpha_1+\beta_0)}{(4\alpha_2+\beta_1)}\right.$$

$$\left.\left(\frac{2\alpha_0}{\beta_1(2\alpha_2+\beta_1)}-\frac{(2\alpha_1+\beta_0)(\alpha_1+\beta_0)}{\beta_1(2\alpha_2+\beta_1)(\alpha_2+\beta_1)}\right)\right]$$

$$-\frac{(5\alpha_1+\beta_0)}{(5\alpha_2+\beta_1)}\left[\frac{4\alpha_0}{(4\alpha_2+\beta_1)}\left(\frac{2\alpha_0}{\beta_1(2\alpha_2+\beta_1)}-\frac{(2\alpha_1+\beta_0)(\alpha_1+\beta_0)}{\beta_1(2\alpha_2+\beta_1)(\alpha_2+\beta_1)}\right)\right.$$

$$\left.-\frac{(4\alpha_1+\beta_0)}{(4\alpha_2+\beta_1)}\left(\frac{3\alpha_0(\alpha_1+\beta_1)}{(3\alpha_2+\beta_1)(\alpha_2+\beta_1)}+\frac{(3\alpha_1+\beta_0)}{(3\alpha_2+\beta_1)}\left(\frac{2\alpha_0}{\beta_1(2\alpha_2+\beta_1)}-\frac{(2\alpha_1+\beta_0)(\alpha_1+\beta_0)}{\beta_1(2\alpha_2+\beta_1)(\alpha_2+\beta_1)}\right)\right)\right]$$

The system constituted by (3.17) and (3.18) is then solved for τ_1 *and* τ_2, whose values, are substituted back into $Y_5(x)$ so as to obtain the desired approximant $y_{n,3}(x)$ of $y(x)$

3.3.1 Error estimation for recursive form

From (3.14) we have $\tau_1 = \dfrac{\eta_1 - \lambda_2 \tau_2}{\lambda_2}$. Insert (3.22) in (3.15) to get

$$\frac{v_1}{\lambda_1}(\eta_1 - \lambda_2 \tau_2) + v_2 \tau_2 = \eta_2 \text{ or } (\lambda_1 v_2 - \lambda_2 v_1)\tau_2 = \lambda_1 \eta_2 - \eta_1 v_1$$

Hence $|\lambda_1 v_2 - \lambda_2 v_1|\,|\tau_2| \le |\lambda_1 \eta_2 - \eta_1 v_1| + \varepsilon_1$ since $\varepsilon_1 \ge 0$. That is

$$|\lambda_1 v_2 - \lambda_2 v_1|\,|\tau_2| \le |\lambda_1 \eta_2 - \eta_1 v_1| + \frac{2^{2n+3}|\tau_2|}{|R_1|}$$

which gives $|\tau_2| \le \left[\dfrac{|R_1|\,|\lambda_1 \eta_2 - \eta_1 v_1|}{|R_1|\,|\lambda_1 v_2 - \lambda_2 v_1| - 2^{2n+3}}\right]$. Hence, from (3.7)

$$\bar{\varepsilon}_1 \le \frac{2^{2n+3}}{|R_1|}\left[\frac{|R_1|\,|\lambda_1 \eta_2 - \eta_1 v_1|}{|R_1|\,|\lambda_1 v_2 - \lambda_2 v_1| - 2^{2n+3}}\right] = \frac{|\lambda_1 \eta_2 - \eta_1 v_1|}{2^{-(2n+3)}|R_1||\lambda_1 \eta_2 - \eta_1 v_1| - 1} = \bar{\varepsilon}_3$$

So then our error estimate is $\bar{\varepsilon}_3 = \dfrac{|\lambda_1 \eta_2 - \eta_1 v_1|}{2^{-(2n+3)}|R_1|\,|\lambda_1 v_2 - \lambda_2 v_1|^{-1}}$

where λ_1, λ_2, η_1, v_2 *and* η_2 are given by (3.19) - (3.24), and R_1 by (3.7)

4.0 Numerical examples

We consider here some selected examples for experimentation with the preceding discourse. For these examples, the exact errors are obtained as $\varepsilon^* = \max_{a \le x \le b}\{|y(x_k) - y_{n,l}(x_k)|\}$, $\lambda = 1(1)3$ where $\{x_k\} = \{0.01k\}$ for $k = 0(1)\ 100$. As the remark in section 3.3 relating to the use of canonical polynomials limits the scope of this work on the recursive formulation and for purpose of meaningful comparison, we shall consider an approximant of degree 5. The results are presented in Table 4.1 below:

***Example* 4.1:**

$$y'(x) + 2xy(x) = 0,\ y(0) = 1,\ y(x) = e^{-x},\ 0 \le x \le 1 \quad (4.1)$$

***Example* 4.2** $\quad y'(x) + xy(x) = 0,\ y(0) = 1,\ y(x) = \exp\left(\frac{1}{2}x^2\right),\ 0 \le x \le 1 \quad (4.2)$

***Example* 4.3** $\quad y'(x) - (1 + 2x)y(x) = 0,\ y(0) = 1,\ y(x) = \exp(x + x^2),\ 0 \le x \le 1 \quad (4.3)$

***Example* 4.4** $\quad y'(x) + 2xy(x) = 4x,\ y(0) = 3,\ y(x) = 2 + \exp(-x^2),\ 0 \le x \le 1 \quad (4.4)$

***Example* 4.5** $\quad (2 - x + x^2)y'(x) + (2x - 1)y(x) = 0,\ y(x) = 2/(2 - x + x^2),\ 0 \le x \le 1 \quad (4.5)$

Table 4.1: Error and Error Estimated For Fifth Degree Approximant

Method	**Error**	**Problem**				
		Problem 4.1	**Problem 4.2**	**Problem 4.3**	**Problem 4.4**	**Problem 4.5**
Differential Form	$\bar{\varepsilon}_1$	$4.08X10^{-5}$	$9.81X10^{-5}$	$1.34X10^{-2}$	$4.08X10^{-5}$	$2.080X10^{-4}$
	ε_1	$3.57X10^{-5}$	$6.12X10^{-5}$	$1.07X10^{-2}$	$3.57X10^{-5}$	$1.95X10^{-4}$
Interpreted Form	$\bar{\varepsilon}_2$	$1.85X10^{-6}$	$3.97X10^{-6}$	$5.15X10^{-6}$	$1.85X10^{-6}$	$6.72X10^{-6}$
	ε_2	$2.12X10^{-5}$	$2.38X10^{-5}$	$2.82X10^{-3}$	$2.12X10^{-5}$	$8.49x10^{-4}$
Recursive Form	$\bar{\varepsilon}_3$	$4.05x10^{-5}$	$9.81x10^{-5}$	$1.34X10^{-2}$	$4.08x10^{-5}$	$2.80x10^{-4}$
	ε_3	$3.57x10^{-5}$	$6.12x10^{-5}$	$1.07x10^{-2}$	$3.57x10^{-5}$	$1.95x10^{-4}$

5.0 Conclusion

Three variants of the tau method and their corresponding error estimates for a class of overdetermined first order ordinary differential equations with unit overdetermination have been presented. For meaningful comparison, the results obtained have been applied to some selected members of this class for a fifth degree approximant, since the recursive form becomes too cumbersome for approximants of higher degrees. The fifth degree approximant was also chosen, rather than lower degree approximants, because convergence will be better achieved for reasonably large n, the degree of the approximant. We note the effectiveness of the three variants as the error estimates compare favorably with the exact error in all the three cases.

References

[1] Adeniyi, R.B. (2007), "On a class of optimal order tau methods for initial value problems in ordinary differential equations", Kenya Journal of Sciences, Series A, Vol. 12, No. 1, pp 17 – 30.

[1] Adeniyi, R.B. (1991): On the tau method for numerical solution of ordinary differential equations, Doctoral Thesis, University of Ilorin, Nigeria.

[3] Adeniyi, R.B. (2000), "Optimality of an error estimate of a one-step numerical integration method for certain initial value problems, Intern. J. Computer Math., Vol. 75, pp 283 - 295

[4] Adeniyi, R.B. and Edungbola E.O. (2007), " On the recursive formulation of the tau method for a class of overdetermined first order equations", Abacus, Journal of the Mathematical Association of Nigeria, vol. 34., No. 2B, pp 249 – 261.

[5] Adeniyi, R.B. and Erebholo F.O., "An error estimation of a numerical integration scheme for certain initial boundary value problems in partial differential equations, J. Nig. Math. Soc. (To Appear)

[7] Adeniyi, R.B. and Onumanyi, P., (1991): Error estimation in the numerical solution of ordinary differential equations with the tau method, Comp. Maths. Applic, Vol. 21, No 9, pp. 19 – 27.

[7] Adeniyi, R.B., Onumanyi, P. and Taiwo, O.A., (1990): A computational Error estimate of the tau method for non-linear ordinary differential equations, J. Nig. Maths. Soc. Pp. 21 -32.

[8] Crisci M.R. and Ortiz E.L. (1981), "Existence and convergence results for the numerical solution of differential equation with the tau method, Imperial College Research Rep., pp 1 - 16

[9] Crisci, M.R. and Russo, E., (1983): An extension of Ortiz's recursive formulation of the tau method for certain linear system of ordinary differential equations, Math. Comput., Vol. 41 pp. 431 – 435.

[10] Fox, L., (1968): Numerical solution of ordinary and partial differential equations, Pergmen Press, Oxford.

[11] Fox, L. and Parker, I.B., (1968): Chebyshev polynomials in numerical analysis, Oxford University Press, Oxford.

[12] Freilich, J.G. and Ortiz, E.L., (1982): Numerical solution of systems of differential equations: an error analysis, Math. Comput, Vol. 39 pp. 467 – 479.

[13] Lanczos, C. (1956), Applied analysis, Prentice Hall, New Jersey

(14) Lanczos, C., (1938): Trigonometric interpolation of empirical and analytic functions, J. Math. And Physics, Vol. 17 pp. 123 - 199

[15] Namasivayam S and Ortiz, E.L. (1981), "Perturbation term and approximation error in the numerical solution of differential equation with the tau method, Imperial College Research Rep. NAS, pp 1 - 5

[16] Onumanyi, P. and Ortiz, E.L., (1982): Numerical solutions of higher order boundary value problems for ordinary differential equations with an estimation of the error, Intern. J. Numer. Mech. Engrg., Vol. 18 pp. 775 – 781.

[17] Ortiz, E.L., (1969): The Tau method, SIAM J. Numer. Anal. Vol. 6 pp 480 – 492

[18] Ortiz, E.L., (1974): Canonical polynomials in the Lanczos tau method, Studies in Numerical analysis (edited by scaife, B.K.P), Academic Press, New York.