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On the Tau method for a Class of non-overdetermined second order differential equations

R. B. Adeniyi[1] and A. I. M Aliyu[2]
[1]Department of Mathematics, University of Ilorin, Ilorin, Nigeria
[2]Department of Mathematics and Computer Science, Ibrahim Badamosi Babangida, University, Lapai, Nigeria

Abstract

This paper is concerned with the tau methods for initial value problems in the class of non-overdetermined second order ordinary differential equations. Three variants namely the differential, the integrated and the recursive formulation are considered. The corresponding error estimates for the three variants are obtained and some selected examples are provided for illustration. The numerical evidences confirm the order of the tau approximants so obtained for all the cases.

***Keywords*:** Tau method, differential, integrated, recursive, formulation, variant, canonical, polynomials, approximant, error, estimates.

1.0 Introduction

Accurate approximate solution of initial value problems and boundary value problems in linear ordinary differential equations with polynomial coefficients can be obtained by the Tau method originally introduced by Lanczos [14] in 1938. The Techniques based on this method have been reported in literature with application to more general equations including non-linear ones as well as to both partial differential equations and integral equations. We review briefly here some of the variants of the method.

1.1 Differential or original form of the Tau Method

Consider the m^{th} order ordinary differential equation

$$Ly(x) := \sum_{r=0}^{m} P_r(x) y^{(r)}(x) = f(x), \quad a \le x \le b \tag{1.1a}$$

with associated conditions

$$L^* y(x_{rk}) := \sum_{r=0}^{m-1} a_{rk} y^{(r)}(x_{rk}) = \alpha_k, \; k = 1(1)m \tag{1.1b}$$

and where $|a| < \infty$, $|b| < \infty$, a_{rk}, x_{rk}, α_k, $r = 0(1)m-1$, $k = 1(1)m$ are given real numbers, $f(x)$ and $P_r(x)$, $r = 0(1)\, m$, are polynomial functions or sufficiently close polynomial approximants of given real function. For the solution of (1.1) by the Tau method (see [2] – [4], [13], [14] and [17]), we shall seek an approximant of the form

$$y_n(x) = \sum_{r=0}^{n} a_r \; x^r, n < +\infty \tag{1.2}$$

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of $y(x)$ which is the exact solution of perturbed problem.

$$Ly_n(x_{rk}) := \sum_{r=0}^{m-1} P_r(x) y_n^{(r)}(x) = f(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r} \quad T_{n-m+r+1}(x) \tag{1.3a}$$

$$L^* y_n(x) := \sum_{r=0}^{m-1} a_{rl} y_n^{(r)}(x_{rk}) = \alpha_k \;, \; k = 1(a)m \tag{1.3b}$$

for $a \le x \le b$ and where τ_r , $r = 1(1)m$, are parameters to be determined along with the a's in (1.3c), $T_r(x)$ is the r-th degree Chebyshev polynomial (See [10], [11]) valid in [a , b] and

$$s = \max\{N_r - r/0 \le r \le m\} \tag{1.4}$$

is the number of over determination of equation (1.1a) (See [2], [11]). For more explanation on the parameter, also see Section 1.3.

We determine $a_r \Lambda$, $r = 0(1)n$ and $\tau_r \Lambda$, $r = 1(1)m$ from the linear algebraic system

$$A \; \underline{\tau} = \underline{b}, \tag{1.5}$$

obtained by equating corresponding coefficients of powers of x from (1.3a) and then applying conditions (1.3b);

$$A = (a_{ij}), \; 1 \le i, \; j \le n + m + s + 1; \quad \underline{b} = (b_i), \; 1 \le i \le n+m+s+1;$$

$$\underline{\tau} = (a_0 \; , a_1 , \; \ldots , \; a_n \; , \; \tau_1 , \ldots \; \tau_{m+s})^T .$$

Consequently, we obtain from (1.2) our desired approximant $y_n(x)$ of $y(x)$.

1.2 The integrated formulation of the Tau method

If $\int\int\int^r \int g(x)dx$ denotes the indefinite integration r times applied to the function g(x) and

$$I_L = \int\int\int^m \ldots \int L(.) \; dx \tag{1.6}$$

then the integrated form of (1.3a) is

$$I_L(y(x)) = \int\int\int^m \ldots \int f(x) \; dx + c_m(x) \tag{1.7}$$

where $c_m(x)$ denotes an arbitrary polynomial of degree (m - 1), arising from the constants of integration. The approximant (1.2), now, then satisfies the perturbed problem

$$I_L(y_n(x)) = \int\int\int^m \ldots \int f(x) \; dx + c_m(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r}(x) \, T_{n+r+1}(x) \tag{1.8a}$$

$$L^* y_n(x_{rk}) = \alpha_k \;, \; k = 1(1)\, m \tag{1.8b}$$

where $y_n(x)$ is here again given by(1.2). Problem (1.8) often gives a more accurate Tau approximate than (1.3) does, due to its higher order perturbation term.

1.3 The recursive formulation of The Tau method

The so-called canonical polynomials $\{Q_r(x)\}$, $r \in N_0 - S$ associated with operator L of (1.1) is defined by

$$LQ_r(x) = x^r \tag{1.9}$$

where S is a small finite or empty set of indices with cardinality $s(s \le m + h), h$ being the maximum difference between the exponent r of x and the leading exponent of the generating polynomial Lx^r, for

$r \in N_0$ (see Ortiz (17)). Once these polynomials are generated, we seek, in this case, an approximant of $y_n(x)$ of the form

$$y_n(x) = \sum_{r=0}^{n} a_r \; Q_r(x), \; n < +\infty \tag{1.10}$$

which is the exact solution of the perturbed problem

$$y_n(x) = \sum_{r=0}^{F} f_r Q_r(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r} \sum_{k=0}^{n-m+r+1} C_k^{(n-m+r+1)} \; Q_k(x) \tag{1.11}$$

where $f_r r = 0(1)F,$ are the coefficients of $f(x)$.

Their use is advantageous as they neither depend on the boundary condition nor on the interval in which the solution . Furthermore, they are re-usable for approximants of higher degrees.

We shall now proceed to an error estimation of the Tau method in the next section, Section 3 addresses the central issues of this paper, Section 4 presents some numerical evidences in support of the work while finally, Section 5 concludes the paper with some remarks.

2.0 An error estimation of the Tau method.

We review briefly here error estimation of the Tau method for the three variants of the proceeding section and which we had earlier reported in [2], [6] and [7].

2.1 Error estimation for the differential form.

While the error problem

$$e_n(x) = y(x) - y_n(x) \tag{2.1}$$

Satisfies the error problem

$$Le_n(x) = -\sum_{r=0}^{m+s-1} \tau_{m+s-r} \;\; T_{n-m+r+1}(x) \tag{2.2a}$$

$$L^* e_n(x_{rk}) = 0, \; k = 1(1)m \tag{2.2b}$$

$$(e_n(x))_{n+1} = \mu_m(x) \varphi_n \;\; T_{n-m+1}(x) / C_{n-m+1}^{(n-m+1)} \tag{2.3}$$

satisfies the perturbed error problem

$$L(e_n(x))_{n+1} = -\sum_{r=0}^{m+s-1} \tau_{m+s-r} \;\; T_{n-m+r+1}(x) + \sum_{r=0}^{m+s-1} \bar{\tau}_{m+s-r} \;\; T_{n-m+r+2}(x) \tag{2.4a}$$

$$L^*(e_n(x))_{n+1} = 0 \tag{2.4b}$$

where the extra parameters $\bar{\tau}_r$, $r = 1(1)m+s$ and φ_n are to be determined; $\mu_m(x)$ is a specified polynomial of degree m which ensures that $(e_n(x))_{n+1}$ satisfies the homogeneous conditions (2.2b).

We insert (2.3) in (2.4a) and then equate corresponding coefficients of x^{n+s+1} , x^{n+s} , … , x^{n-m+1} and the resulting linear system is solved for φ_n by forward elimination, and since we do not need the $\bar{\tau}' s$. Consequently we obtain

$$\bar{\varepsilon}_1 = \max_{a \le x \le b} \left| (e_n(x))_{n+1} \right| = \left| \varphi_n \right| / \left| C_{n-m+1}^{(n-m+1)} \right| \cong \max_{a \le x \le b} \left| e_n(x) \right| = \varepsilon_1 \tag{2.5}$$

2.2 Error estimation for the integrated form

The error polynomial (2.3) satisfies the perturbed problem

$$I_L\left(e_n(x)\right)_{n+1} = -\iiint^{m} \cdots \int\left(\sum_{r=0}^{m+s-1}\tau_{m+s-r}\ T_{n-m+r+1}(x)\right)dx + c_m(x) + \sum_{r=0}^{m+s-1}\bar{\tau}_{m+s-r}\ T_{n-m+r+3}(x) \quad (2.6)$$

We insert (2.3) in (2.6) and then equate coefficients of $x^{n+s+m+1}$, x^{n+s+m}, ... , x^{n-m} for the determination of the parameter φ_n of $\left(e_n(x)\right)_{n+1}$. Subsequent procedures follows suit as described above in section (2.1) in order to obtain the error estimate $\bar{\varepsilon}_2$.

2.3 Error estimation for the recursive form

Once the canonical polynomials of section 1 are generated, they can be used for an error estimation of the Tau method (see [14], [16], [8] and [15]). Here we consider a slight perturbation of the given boundary conditions (1.1b) by $\bar{\varepsilon}_1$ to obtain an estimate of the Tau parameter τ_{m+s}, in terms of canonical polynomials, which is then substituted into the expression for $\bar{\varepsilon}_I$ given in (2.5) for a new estimate $\bar{\varepsilon}_2$.

3.0 A class of over-determined first order equations differential equations

We consider here the three variants of the Tau method of the preceding section for the problem (1.1) when m = 2 and *s*= 0, that is, the class

$$Ly(x) := \left(P_{20} + P_{21}x + P_{22}x^2\right) y''(x) + \left(P_{10} + P_{11}x\right) y'(x) + P_{\infty}\, y(x) = \sum_{r=0}^{F} f_r\ x^r;\ a \le x \le b \qquad (3.1a)$$

$$y(a) = \alpha_0,\ y'(a) = \alpha_1 \qquad (3.1b)$$

Without loss of generality we shall assume that a = 0 and b = 1, since, by the transformation

$$v = (x-a)/(b-a)\ ,\ a \le x \le b \qquad (3.2)$$

Takes the problem (3.1) into the interval [0, 1].

3.1 Tau approximant by the differential form

By inserting (1.2) into the perturbed form of (3.1a), we have that

$$\left(P_{20} + P_{21}x + P_{22}x^2\right)\sum_{r=0}^{n} r(r-1)\, a_r x^{r-2} + \left(P_{10} + P_{11}x\right)\sum_{r=0}^{n} r\ a_r x^{r-1}$$
$$+\ P_{00}\sum_{r=0}^{F} a_r\, x^r = \sum_{r=0}^{F} f_r\, x^r + \tau_1 T_n(x) + \tau_2 T_{n-1}(x)$$

This leads to

$$\sum_{r=0}^{n-2}(r+1)\,(r+2)\ P_{20}\, a_{r+2}\, x^r + \sum_{r=0}^{n-1}\left(rP_{21} + P_{10}\right)(r+1)\ a_{r+1}\, x^r +$$
$$\sum_{r=0}^{n}\left[(r-1)\, r\, P_{22} + rP_{11} + P_{00}\right] a_r\, x^r = \sum_{r=0}^{F} f_r x^r + \tau_1\sum_{r=0}^{n} C_r^{(n)}\, x^r + \tau_2\sum_{r=0}^{n-1} C_r^{(n-1)}\, x^r$$

Hence,

$$\left\{\left[(n-1)nP_{21} + nP_{10}\right]a_n + \left[(n-1)(n-1)P_{22} + (n-1)P_{11} + P_{00}\right]a_{n-1} - \tau_1\, C_{n-1}^{(n)} - \tau_2\, C_{n-1}^{(n-1)} - f_{n-1}\right\}x^{n-1}$$
$$+\left\{\left[(n-1)n\, P_{22} + nP_{11} + P_{00}\right]a_n - \tau_1 C_n^{(n)} - f_n\right\}x^n \qquad (3.3)$$
$$+\sum_{r=0}^{n-2}\left\{(r+1)(r+2)\alpha_0\, a_{r+2} + \left[r(r+1)P_{21} + (r+1)P_{10}\right]a_{r+1}\right.$$

$$+\left[(r-1)r\ P_{22}+rP_{11}+P_{00}\right]a_r-f_r-\tau_1 C_r^{(n)}-\tau_2 C_r(n-1)\Big\}x^r=0$$

We now equate coefficients to have the linear system of (n + 1) equations

$$(r+1)(r+2)P_{20}a_{r+2}+(rP_{21}+P_{10})\ (r+1)a_{r+2}$$
$$+\left[(r-1)r\ P_{22}+rP_{11}+P_{00}\right]a_r-f_r-\tau_1 C_r^{(n)}-\tau_2 C_r^{(n-1)},r=0(1)n-2$$
$$\left[(n-1)nP_{21}+nP_{10}\right]a_n+\left[(n-2)(n-1)P_{22}+(n-1)P_{11}+P_0\right]a_{n-1}$$
$$-\tau_1 C_{n-1}^{(n)}-\tau_2 C_{n-1}^{(n-1)}-f_{n-1}=0$$
$$\left[(n-1)nP_{22}+nP_{11}+P_{00}\right]a_n-\tau_1\ C_{n-1}^{(n)}-f_n=0$$

We solve this system together with two other equations arising from the conditions (3.1b) for the determination of the (n + 3) parameters $a_r, r=0(1)n$ and $\tau_r, r=1,\ 2.$ Consequently, we obtain from (1.2) our desired approximant $y_{n,1}(x)$.

3.1.1 Error estimation for differential form

From (3.1) we have for problem (2.4a) that

$$L(e_n(x))_{n+1}=\bar{\tau}_1\ T_{n+2}(x)+(\bar{\tau}_2-\tau_1)\,T_{n+1}(x)-\tau_2\ T_n(x) \qquad (3.5)$$

where

$$L=\left(P_{20}+P_{21}x+P_{22}x^2\right)\frac{d^2}{dx^2}+\left(P_{10}+P_{11}x\right)\frac{d}{dx}+P_{00} \qquad (3.6)$$

$$(e_n(x))_{n+1}=x^2\varphi_n T_{n-1}(x)/C_{n-1}^{(n-1)}=\varphi_n\left(\sum_{r=0}^{n-1}C_r^{(n-1)}x^{r+2}\right)\Big/C_{n-1}^{(n-1)} \qquad (3.7)$$

By equating the coefficients of x^{n+1}, x^{n-1} from (3.5) to have the system

$$\theta\left[P_{00}+(n+1)\ P_{11}+n(n+1)P_{22}\right]C_{n-1}^{(n-1)}=\bar{\tau}_1\ C_{n-1}^{(n-1)}$$
$$\theta\left[C_{n-1}^{(n-1)}\ P_{00}+nC_{n-2}^{(n-1)}P_{11}+(n-1)n\ C_{n-2}^{(n-1)}P_{22}(n+1)\ C_{n-2}^{(n-1)}P_{10}\right. \qquad (3.8)$$
$$\left.+\ n(n+1)\ C_{n-1}^{(n-1)}P_{21}\right]=\bar{\tau}_1\ C_n^{(n-1)}+(\bar{\tau}_2-\tau_1)\,C_n^{(n)}$$
$$\theta\left[C_{n-3}^{(n-1)}P_{00}+(n-1)C_{n-3}^{(n-1)}\ P_{11}+(n-2)(n-1)C_{n-3}^{(n-1)}\ P_{22}+\right.$$
$$\left.+\,nC_{n-2}^{(n-1)}\ P_{10}+(n-1)nC_{n-2}^{(n-1)}\ P_{21}+\ n(n+1)C_{n-1}^{(n-1)}\ P_{20}\right]=\bar{\tau}_1\ C_{n-1}^{(n-1)}+(\bar{\tau}_2-\tau_1)\,C_{n-1}^{(n)}-\ \tau_2\ C_{n-1}^{(n-1)}$$

where $\theta=\varphi_n\left(C_{n-1}^{(n-1)}\right)^{-1}$. From this system and by using the well-known relations:

$$C_n^{(n)}=\ 2^{2n-1},C_{n-1}^{(n)}=-\frac{1}{2}n\ C_n^{(n)} \qquad (3.9)$$

we obtain for φ_n the expression

$$\varphi_n=\frac{2^{4n-2}\,\tau^2}{R_1} \qquad (3.10)$$

where

$$R_1=\left(P_{00}+P_{11}+nP_{22}+n^2P_{22}\right)C_{n-1}^{(n+1)}+$$
$$2^{2n-1}\left(2nP_{00}+nP_{11}+4nP_{10}+3n^2P_{11}+6n^2P_{21}+4n^3P_{22}+4n^2P_{20}+4n^2P_{20}+\ 4nP_{20}-2nP_{21}\right) \qquad (3.11)$$
$$-\left(16P_{00}+16P_{11}+16nP_{11}+16n^2P_{22}-48nP_{22}+32P_{22}\right)C_{n-3}^{(n-1)}$$

From (2.5) we obtain the error estimate

$$\bar{\varepsilon}_1 = \frac{2^{2n}|\tau_2|}{|R_1|} \tag{3.12}$$

3.2 Tau approximant by the integrated form

From (1.8) we get

$$\int_0^x\int_0^u \left(P_{20} + P_{21}t + P_{22}t^2\right) y''(t)\, dt\, du + \int_0^x\int_0^u \left(P_{10} + P_{11}t\right) y'(t)\, dt\, du + \int_0^x\int_0^u P_{00}y(t)\, dt\, du$$
$$= \int_0^x\int_0^u \left(\sum_{r=0}^{F} f_r\, t^r\right) dt\, du + \tau_1\, T_{n+2}(x) + \tau_2\, T_{n+2}(x)$$

This leads to

$$P_{20}y(x) + P_{21}\left[xy(x) - 2\int_0^x y(u)du\right] + P_{22}\left[x^2y(x) - 4\int_0^x uy(n)\,du + 2\int_0^x\int_0^u y(t)\,dt\, du\right]$$
$$+ P_{11}\left[\int_0^x uy(u)\,du - \int_0^x\int_0^u y(t)dt\, du\right] + P_{00}\int_0^x\int_0^u y(t)dt\,du + P_{10}\int_0^x y(u)\,du$$
$$+ \alpha_0\, P_{20} + (\alpha_1\, P_{20} + \alpha_0\, P_{10} + \alpha_0\, P_{21} + \alpha_0\, P_{10})x \tag{3.13}$$

With (1.2) this gives

$$\sum_{r=0}^{n} P_{20}\, a_r\, x^r + \sum_{r=1}^{n+1}\left(\frac{(r-2)\, P_{21} + P_{10}}{r}\right) a_{r-1}\, x^r + \sum_{r=2}^{n+2}\left(\frac{P_{00}^{+}(r-2)\, P_{11} + (r-2)\,(r-3)P_{22}}{(r-1)r^0}\right) a_{r-1}\, x^r$$
$$= \sum_{r=2}^{n+2}\frac{f_{r-2}\, x^r}{(r-1)r} + \tau_1\sum_{r=0}^{n+2} C_r^{(n+2)} x^r + \tau_{23}\sum_{r=0}^{n+1} C_r^{(n+1)} x^r \tag{3.14}$$

This gives

$$\left(- P_{20}\,\alpha_0 + P_{20}\,\alpha_0 - \tau_1\, C_0^{(n+2)} - \tau_2\, C_0^{(n+1)}\right)$$
$$+ \left[\left(- P_{20}\,\alpha_1 + P_{21}\,\alpha_0 - P_{10}\,\alpha_0 + P_{20}a_1 + \left(- P_{21} + P_{10}\right)a_0 - \tau_1\, C_1^{(n+2)} - \tau_2\, C_1^{(n+1)}\right]x\right.$$
$$+ \sum_{2}^{n}\left\{P_{20}\, a_r + \left(\frac{(r-2)\, P_{21} + P_{10}}{r}\right) a_{r-1} + \left(\frac{P_{00} + (r-2)\,P_{11} + (r-2)\,(r-3)P_{22}}{(r-1)r}\right)\alpha_{r-2}\right.$$
$$\left. - \frac{f_{r-2}}{(r-1)r} - \tau_1\, C_r^{(n+2)} - \tau_2\, C_r^{(n+1)}\right\} x^r + \left\{\left(\frac{(n-1)\, P_{21} + P_{10}}{n+1}\right) a_n\right.$$
$$\left. + \left(\frac{P_{00} + (n-1)\, P_{11} + (n-1)\,(n-2)\, P_{22}}{n(n+1)}\right) a_{n-1} - \frac{f_{n-1}}{n(n+1)} - \tau_1\, C_{n+1}^{(n+2)} - \tau_2\, C_{n+1}^{(n+1)}\right\} x^{n+1}$$
$$+ \left\{\left(\frac{P_{00} + nP_{11} + n(n-1)\, P_{22}}{(n+1)(n+2)}\right) a_n - \frac{f_n}{(n+1)(n+2)} - \tau_1\, C_{n+2}^{(n+2)}\right\} x^{n+2} = 0 \tag{3.15}$$

This yields the system

$$P_{20}a_0-\tau_1C_0^{(n+2)}-\tau_2C_0^{(n+2)}=\alpha_0P_{20}\ (-P_{21}+P_{10})a_0+P_{20}a_1-\tau_1C_1^{(n+1)}-\tau_2C_1^{(n+2)}=\alpha_1 P_{20}-\alpha_0P_{21}+\alpha_0 P_{10}$$

$$P_{20}\, a_r+\left(\frac{(r-2)P_{21}+P_{10}}{r}\right)a_{r-1}+\left(\frac{P_{00}+(r-2)\,P_{11}+(r-2)\,(r-3)\,P_{22}}{(r-1)\,r}\right)a_{r-2}-$$

$$-\tau_1\, C_r^{(n+2)}-\tau_2\, C_r^{(n+1)}=\frac{f_{r-2}}{(r-1)\,r}\ ,\ r=2(1)n$$

$$\left(\frac{(n-1)P_{21}+P_{10}}{n+1}\right)a_n+\left(\frac{P_{00}+(n-1)P_{11}+(n-1)(n-2)P_{22}}{n(n+1)}\right)a_{n-1}-\tau_1\, C_{n+1}^{(n+2)}-\tau_2\, C_{n+1}^{(n+1)}=\frac{f_{r-1}}{n(n+1)} \quad (3.16)$$

$$\left(\frac{P_{00}+nP_{11}+(n-1)nP_{22}}{(n+1)(n+2)}\right)a_n-\tau_1\, C_{n+2}^{(n+2)}=\frac{f_n}{(n+1)(n+2)}$$

We solve this system for $a_r,\, r=0(1)n$ and τ_1,τ_2 to subsequently obtain from (1.2) the approximant $y_{n,2}(x)$ of $y(x)$.

3.2.1 Error estimation for the integrated form

From (2.6) we have for problem (3.1)

$$\int_0^{\alpha}\int_0^{u}\left(P_{20}\ +\ P_{21}t\ +\ P_{22}t^2\right)\left(e_n''(t)\right)_{n+1}dt\ du+\int_0^{x}\int_0^{u}\left(P_{10}\ +\ P_{11}t\ \right)\ \left(e_n'(t)\right)_{n+1}dt\, du$$

$$+\int_0^{x}\int_0^{u}\ P_{00}\ \left(e_n(t)\right)_{n+1}\ dt\, du\ =-\int_0^{x}\int_0^{u}\left(\tau_1\sum_{r=0}^{n}C_r^{(n)}t^r+\tau_2\sum_{r=0}^{n-1}C_r^{(n-1)}\ t^r\right)dt\, du$$

$$+\bar{\tau}_1\sum_{r=0}^{n+3}C_r^{(n+3)}\ x^r+\bar{\tau}_2\sum_{r=0}^{n+2}C_r^{(n+2)}\ x^r \quad (3.17)$$

where $\left(e_n(x)\right)_{n+1}$ is again given by (3.7). This leads to the equation

$$\frac{\varphi_n}{C_{n-1}^{(n-1)}}\left\{P_{20}\sum_{r=0}^{n-12}C_r^{(n-1)}\ x^{r+2}+P_{21}\left(\sum_{r=0}^{n-1}C_r^{(n-1)}\ x^{r+3}-2\sum_{r=0}^{n-1}\frac{C_r^{(n-1)}x^{r+3}}{(r+3)}\right)\right.$$

$$+P_{22}\left(\sum_{r=0}^{n-1}C_r^{(n-1)}\ x^{r+4}\sum_{r=0}^{n-1}\frac{C_r^{(n-1)}x^{r+4}}{(r+4)}-2\sum_{r=0}^{n-1}\frac{C_r^{(n-1)}x^{r+4}}{(r+3)(r+4)}\right)$$

$$+P_{10}\sum_{r=0}^{n-1}\frac{C_r^{(n-1)}x^{r+3}}{(r+3)}\Bigg)+P_{11}\left(\sum_{r=0}^{n-1}\frac{C_r^{(n-1)}x^{r+4}}{(r+4)}-\sum_{r=0}^{n-1}\frac{C_r^{(n-1)}x^{r+4}}{(r+3)(r+4)}\right)$$

$$+\ P_{00}\sum_{r=0}^{n-1}\frac{C_r^{(n-1)}x^{r+3}}{(r+3)(r+4)}=\tau_1\sum_{r=0}^{n}\frac{C_r^{(n)}x^{r+2}}{(r+1)(r+2)}-\tau_2\sum_{r=0}^{n-1}\frac{C_r^{(n-1)}x^{r+2}}{(r+1)(r+1)}$$

$$+\bar{\tau}_1\sum_{r=0}^{n+3}C_r^{(n+3)}\ x^r+\bar{\tau}_2\sum_{r=0}^{n+2}C_r^{(n+2)}\ x^r \quad (3.18)$$

We equate coefficients of x^{n+3}, x^{n+2} and x^{n+1} to have the system

$$\frac{\varphi_n}{C_{n-1}^{(n-1)}}\left[\frac{P_{00}+(n+1)\,P_{11}+n(n+1)P_{22}}{(n+2)\,(n+3)}\right]\ C_n^{(n-1)}=\bar{\tau}\ C_{n+3}^{(n+3)}$$

$$\frac{\varphi_n}{C_{n-1}^{(n-1)}}\left[\left(\frac{P_{10}+nP_{21}}{(n+2)}\right)C_{n-1}^{(n-1)} + \left(\frac{P_{10}+nP_{11}+(n-1)nP_{22}}{(n+1)(n+2)}\right)C_{n-1}^{(n-1)}\right]$$

$$= \bar{\tau}_1\, C_{n+2}^{(n+3)} + \bar{\tau}_2\, C_{n+2}^{(n+2)} - \frac{\tau_1\, C_n^{(n)}}{(n+1)(n+2)} \qquad (3.19)$$

$$\frac{\varphi_n}{C_{n-1}^{(n-1)}}\left[P_{20}\, C_{n-1}^{(n-1)} + \left(\frac{P_{10}+(n-1)P_{21}}{(n+1)}\right)C_{n-2}^{(n-1)} + \left(P_{00}+(n-1)\, P_{11}+(n-2)(n-1)P_{22}\right)C_{n-3}^{(n-1)}\right]$$

$$= \bar{\tau}_1\, C_{n+1}^{(n+3)} + \bar{\tau}_2\, C_{n+1}^{(n+2)} - \frac{\tau_1 C_{n-1}^{(n+1)}}{n(n+1)} - \frac{\tau_2 C_{n-1}^{(n-1)}}{n(n+1)}$$

We solve this by forward elimination for φ_n to get

$$\varphi_n = \frac{-\,2^{2n-3}\,\tau_2}{n(n+1)\,R_3} \qquad (3.20)$$

where

$$R_3 = \frac{C_{n+1}^{)n-3)}}{2^{2n+5}}\left[\frac{P_{00}+(n+1)P_{11}+n(n+1)P_{22}}{(n+2)(n+3)}\right] -$$

$$\frac{\left\{\left[(n+2)(n+3)\right[2(n+1)(P_{10}+nP_{21})-(n-1)(P_{00}+nP_{11}+n^2-nP_{22})\right] + (n+1)(n+2)(n+3)\left[P_{00}+(n+1)P_{11}+n(n+1)P_{22}\right]\right\}}{4(n+1)(n+2)(n+3)}$$

$$+\frac{(n-3)(P_{10}+nP_{21}-P_{21}}{2(n+1)} + \frac{(P_{00}-P_{11}+nP_{11}n^2P_{22}-3nP_{22}+2P_{22})\,C_{n-3}^{(n-1)}}{2^{2n-3}\,n(n+1)} - P_{20} \qquad (3.21)$$

Hence we obtain the error estimate

$$\bar{\varepsilon}_2 = \left|\varphi_n\right| / \left|C_{n-1}^{(n-1)}\right| = \frac{\tau_2}{n(n+1)\left|R_3\right|} \qquad (3.22)$$

3.3 A Tau approximant by the recursive form

$$y_n(x) = \sum_{r=0}^{F} f_r\, Q_r(x) + \tau_1 \sum_{r=0}^{n} C_r^{(n)}\, Q_r(x) + \tau_2 \sum_{r=0}^{n-1} C_r^{(n-1)}\, Q_r(x) \qquad (3.23)$$

If $F \le n$, then this becomes

$$y_n(x) = \sum_{r=0}^{n-1}\left[f_r + \tau_1\, C_r^{(n)} + \tau_2\, C_r^{(n-1)}\right]Q_r(x) + \left[f_n + \tau_1\, C_n^{(n)}\right]Q_n(x) \qquad (3.24)$$

where the sequence of $\{Q_r(x)\},\ r\ \varepsilon\ N_0 - S$ is generated thus:

From (1.9) and (3.6), and by the linearity of L,

$$Lx^r = \left(P_{20}+P_{21}\,x + P_{22}\,x^2\right) r(r-1)\,x^{r-2} + \left(P_{10}+P_{11}x\right) rx^{r-1} + P_{00}\,x^r =$$

$$= L\left\{r(r-1)P_{20}\ Q_{r-2}(x) + \left(rP_{10}+r(r-1)P_{21}\right) Q_{r-1}(x)\right.$$

$$+(P_{00}+rP_{11}+r(r-1)P_{22})Q_r(x)\}$$

By assuming the existence of L^{-1} we obtain

$$Q_r(x)=\frac{r^2-r(r-1)P_{20}\,Q_{r-2}(x)-(rP_{10}+(r-1)P_{21})\;Q_{r-1}(x)}{P_{00}+rP_{11}+r(r-1)P_{22}} \tag{3.25}$$

provided that $P_{00}+rP_{11}+r(r-1)P_{22}\neq 0$ and for $r=0,\,1,\,2\Lambda$. Now from (3.10) and (1.11) we get for problem (3.1)

$$\sum_{r=0}^{n-1}a_r\,Q_r(x)+a_n\,Q_n(x)=\sum_{r=0}^{n-1}\left[f_r+\tau_1\;C_r^{(n)}+\tau_2\;C_r^{(n)}\right]Q_r(x)+\left[f_n+\tau_1\;C_r^{(n)}\right]Q_n(x)$$

giving us

$$a_r\,=\,f_r+\tau_1\,C_r^{(n)}+\tau_2\;C_r^{(n-1)},\;r=0(1)n-1$$
$$a_n\,=\,f_n+\tau_1\,C_n^{(n)} \tag{3.26}$$

This values τ_1 and τ_2 are obtained by applying the conditions (1.3b) to (2.23) and this gives the system

$$\begin{bmatrix} y_n(0)\\ y_n'(0)\end{bmatrix}=\begin{bmatrix}\sum\limits_{r=0}^{n}C_r^{(n)}\,Q_r(0) & \sum\limits_{r=0}^{n-1}C_r^{(n-1)}\,Q_r(0)\\ \sum\limits_{r=0}^{n}C_r^{(n)}\,Q_r'(0) & \sum\limits_{r=0}^{n-1}C_r^{(n-1)}\,Q_r'(0)\end{bmatrix}\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}=\begin{bmatrix}\alpha_0-\sum\limits_{r=0}^{F}f_r\;Q_r(0)\\ \alpha_1-\sum\limits_{r=0}^{F}f_r\;Q_r'(0)\end{bmatrix} \tag{3.27}$$

We solve the Tau system (3.27) for τ_1 and τ_2, insert them in (3.26) to determine $a_r,\,r=0(1)n$ and then obtain the desired approximant $y_{n,3}(x)$ from (1.10).

3.3.1 Error estimation for recursive form

From the conditions (1.3b) we get

$$\sum_{r=0}^{F}f_r\;Q_r(0)\;+\;\tau_1\;\sum_{r=0}^{n}C_r^{(n)}\,Q_r(0)\;+\;\tau_2\sum_{r=0}^{n-1}C_r^{(n-1)}\,Q_r(0)=\alpha_0 \tag{3.28}$$

$$\sum_{r=0}^{F}f_r\;Q_r'(0)+\tau_1\;\sum_{r=0}^{n}C_r^{(n)}\,Q_r'(0)+\tau_2\sum_{r=0}^{n-1}C_r^{(n-1)}\,Q_r'(0)=\alpha_1 \tag{3.29}$$

From (3.28) we have

$$\tau_1=\left(\alpha_0-\sum_{r=0}^{n}f_r\;Q_r(0)-\tau_2\;\sum_{r=0}^{n-1}C_r^{(n-1)}\,Q_r(0)+\tau_2\;\right)\left(\sum_{r=0}^{n}C_r^{(n)}\,Q_r(0)\right)^{-1}$$

We insert this in (3.29) to obtain

$$\tau_2=\left[\left(\sum_{r=0}^{n-1}C_r^{(n-1)}\,Q_r'(0)\right)\left(\sum_{r=0}^{n}C_r^{(n)}\,Q_r(0)\;\;\right)-\left(\sum_{r=0}^{n-1}C_r^{(n-1)}\,Q_r(0)\right)\;\left(\sum_{r=0}^{n}C_r^{(n)}\,Q_r'(0)\right)\right]=$$
$$=\alpha\;_1\sum_{r=0}^{n}C_r^{(n)}\,Q_r(0)-\left(\alpha_0-\sum_{r=0}^{F}f_r\;\;Q_r(0)\right)\;\left(\sum_{r=0}^{n}C_r^{(n)}\,Q_r'(0)\right)$$

$$|\tau_2|\left|\sum_{r=0}^{n-1} C_r^{(n-1)} Q_r'(0) - \left(\sum_{r=0}^{n} C_r^{(n)} Q_r(0)\right) - \left(\sum_{r=0}^{n-1} C_r^{(n-1)} Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r'(0)\right)\right|$$

$$\leq \left|\alpha_1\left(\sum_{r=0}^{n} C_r^{(n)} Q_r(0)\right)\left(\alpha_0 - \sum_{r=0}^{F} f_r\ Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r'(0)\right)\right| + \bar{\varepsilon}_1$$

since $\bar{\varepsilon}_1 \geq 0,$ given by (3.12). This leads to

$$|\tau_2| \leq \frac{|R_1|\left|\alpha_1\left(\sum_{r=0}^{n} C_r^{(n)} Q_r(0)\right) - \left(\alpha_0 - \sum_{r=0}^{F} f_r\ Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r'(0)\right)\right|}{|R_1|\left|\alpha_1\left(\sum_{r=0}^{n-1} C_r^{(n-1)} Q_r'(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r(0)\right) - \left(\sum_{r=0}^{n} C_r^{(n-1)} Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r'(0)\right)\right| - 2^{2n}} \quad (3.30)$$

Hence

$$\bar{\varepsilon}_1 = \frac{2^{2n}|\tau_2|}{|R_1|} \leq \frac{\left|\alpha_1 \sum_{r=0}^{n} C_r^{(n)} Q_r(0) - \left(\alpha_0 - \sum_{r=0}^{F} f_r\ Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r'(0)\right)\right|}{2^{-2n}\left|\left(\sum_{r=0}^{n-1} C_r^{(n-1)} Q_r'(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r(0)\right) - \left(\sum_{r=0}^{n} C_r^{(n-1)} Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r'(0)\right)\right| - 1} = \bar{\varepsilon}_1 \quad (3.31)$$

where R_1 is given in section by (3.11). Thus, our new error estimate is

$$\bar{\varepsilon}_1 = \frac{\left|\alpha_1 \sum_{r=0}^{n} C_r^{(n)} Q_r(0) - \left(\alpha_0 - \sum_{r=0}^{F} f_r\ Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r'(0)\right)\right|}{2^{-2n}\left|\left(\sum_{r=0}^{n-1} C_r^{(n-1)} Q_r'(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r(0)\right) - \left(\sum_{r=0}^{n} C_r^{(n-1)} Q_r(0)\right)\left(\sum_{r=0}^{n} C_r^{(n)} Q_r'(0)\right)\right| - 1} = \bar{\varepsilon}_1 \quad (3.32)$$

A striking interest in respect of (3.32) is that an estimate is possible prior to the computation of $y_n(x)$ once the canonical polynomials are known.

4.0 Numerical examples

We consider here five selected examples for experimentation with our results of the preceding section. The exact errors are defined as

$$\varepsilon_\lambda = \max_{a \leq x \leq b} \left\{|y(x_k) - y_{n,l}(x_k)|\right\}, \quad \lambda = 1, 2, 3$$

Where $\{x_k\} = \{0.01k\},$ for $k = 0(1) \leq 100$. The numerical results are presented in the tales below the examples:

***Example* 4.1**

$$y''(x) + y(x) = x^2, \ \ y(0) = 0, \ \ y'(0) == 3, \ \ 0 \leq x \leq 1 \quad (4.1)$$

Analytical solution $y(x) = 2\cos x + 3\sin x + x^2 - 2$

***Example* 4.2**

$$y''(x) + 25y(x) = 5x^2 + x, \ y(0) = 0.2, \ \ y'(0) = 0, \ (\leq x \leq 1) \quad (4.2)$$

Analytical solution $y(x) = (27\cos 5x - \sin 5x + 25x^2 + 5x - 2)/125$

***Example* 4.3**

$$y''(x) - y'(x) - 2y(x) = 8, \ y(0) = 0, \ \ y'(0) = 10, \ \ 0 \leq x \leq 1 \quad (4.3)$$

Exact solution, $y(x)=\frac{1}{2}e^{-2x}-\frac{3}{2}e^{2x}+e^{-3x}$

***Example* 4.4**

$$y''(x)+5y'(x)+6y(x)=0, y(0)=1,\ y'(0)=-1,\ \le x\le 1 \tag{4.4}$$

True solution, $y(x)=2e^{-2x}-e^{-3x}$

***Example* 4.5**

$$y''(x)-3y'(x)+2y(x)=x^2, y(0)=3/4, y'(0)=5/2,\ \ 0\le x\le 1 \tag{4.5}$$

Closed form solution, $y(x)=2e^{-2x}-3e^{x}+\frac{1}{4}\left(2x^2+6x+7\right)$

Table 4.1: Error and error estimated for problems 4.1

Method	Error	Degree (n) 2	3	4	5
Differential Form	$\bar{\varepsilon}_1$	5.93×10^{-2}	7.59×10^{-3}	2.07×10^{-4}	1.68×10^{-5}
	ε_1	8.88×10^{-2}	2.65×10^{-3}	9.03×10^{-4}	4.15×10^{-5}
Interpolated Form	$\bar{\varepsilon}_2$	6.46×10^{-4}	4.23×10^{-5}	7.00×10^{-7}	3.85×10^{-8}
	ε_2	8.84×10^{-3}	1.06×10^{-3}	2.86×10^{-5}	2.14×10^{-6}
Recursive Form	$\bar{\varepsilon}_3$	4.05×10^{-2}	1.82×10^{-3}	5.53×10^{-4}	1.99×10^{-6}
	ε_3	9.21×10^{-5}	2.86×10^{-2}	1.54×10^{-3}	9.96×10^{-5}

Table 4.2: Error and error estimated for problems 4.2

Method	Error	Degree (n) 2	3	4	5
Differential Form	$\bar{\varepsilon}_1$	1.74×10^{-1}	5.18×10^{-2}	2.15×10^{-2}	6.22×10^{-3}
	ε_1	3.24×10^{-1}	2.28×10^{-1}	$3.45\times10^{-}$	5.21×10^{-4}
Interpolated Form	$\bar{\varepsilon}_2$	1.36×10^{-3}	7.77×10^{-4}	1.67×10^{-4}	7.38×10^{-6}
	ε_2	1.1×10^{-1}	3.76×10^{-3}	1.34×10^{-4}	5.46×10^{-5}
Recursive Form	$\bar{\varepsilon}_3$	1.05×10^{-1}	1.03×10^{-1}	1.36×10^{-2}	7.18×10^{-3}
	ε_3	5.78×10^{-1}	1.54×10^{-2}	1.52×10^{-2}	2.35×10^{-3}

Table 4.3: Error and error estimated for problems 4.3

Method	Error	Degree (n) 2	3	4	5
Differential Form	$\bar{\varepsilon}_1$	400×10^{-1}	1.10×10^{-0}	8.18×10^{-2}	6.31×10^{-3}
	ε_1	8.97×10^{-1}	3.36×10^{-0}	7.12×10^{-2}	8.60×10^{-3}
Interpolated Form	$\bar{\varepsilon}_2$	9.42×10^{-2}	4.27×10^{-3}	2.12×10^{-4}	5.07×10^{-3}
	ε_2	2.85×10^{-1}	2.95×10^{-2}	5.40×10^{-3}	4.56×10^{-4}
Recursive Form	$\bar{\varepsilon}_3$	9.36×10^{-1}	4.04×10^{-1}	8.12×10^{-2}	2.14×10^{-3}
	ε_3	4.01×10^{-1}	3.03×10^{-1}	5.36×10^{-2}	6.69×10^{-3}

Table 4.4: Error and error estimated for problems 4.4

Method	Error	Degree (n)			
		2	3	4	5
Differential	$\bar{\varepsilon}_1$	3.43×10^{-2}	1.38×10^{-2}	2.18×10^{-3}	4.24×10^{-4}
Form	ε_1	4.17×10^{-2}	4.63×10^{-2}	8.03×10^{-3}	8.97×10^{-4}
Interpolated Form	$\bar{\varepsilon}_2$	1.38×10^{-4}	5.38×10^{-3}	8.84×10^{-6}	1.10×10^{-6}
	ε_2	4.77×10^{-3}	2.29×10^{-3}	6.63×10^{-4}	9.98×10^{-5}
Recursive Form	$\bar{\varepsilon}_3$	3.40×10^{-2}	3.47×10^{-3}	6.70×10^{-4}	9.58×10^{-5}
	ε_3	4.49×10^{-1}	7.98×10^{-2}	4.03×10^{-3}	3.02×10^{-4}

Table 4.5: Error and error estimated for problems 4.5

Method	Error	Degree (n)			
		2	3	4	5
Differential Form	$\bar{\varepsilon}_1$	1.89×10^{-1}	1.40×10^{-0}	5.60×10^{-2}	3.92×10^{-3}
	ε_1	2.26×10^{-0}	6.47×10^{-0}	7.46×10^{-2}	5.04×10^{-3}
Interpolated Form	$\bar{\varepsilon}_2$	1.63×10^{-1}	4.85×10^{-3}	2.35×10^{-4}	1.17×10^{-5}
	ε_2	6.93×10^{-1}	2.21×10^{-2}	3.43×10^{-3}	2.28×10^{-4}
Recursive Form	$\bar{\varepsilon}_3$	2.80×10^{-1}	1.93×10^{-1}	1.97×10^{-2}	2.26×10^{-3}
	ε_3	1.95×10^{-2}	6.73×10^{-1}	5.22×10^{-2}	3.33×10^{-3}

5.0 Conclusion

The Tau method for solution of initial value problems in a class of second order ordinary differential equations with non-overdetermintaion has been presented. Three variants of the method were considered for the corresponding Tau approximants of their desired analytic solutions, and the associated error estimates were also obtained.

For all the numerical examples considered the error estimates closely approximate the exact error. The difficulty in the generation of the so-called canonical polynomials for high degree tau approximants limited scope of the work to approximations of maximum degree five. However, as we reported in, [2], [6] and [7], the estimates may exactly estimate the order of the Tau approximant for approximants of higher degrees. While the differential form may easily be generalized for all classes of differential equations which lies within the scope of the Tau methods as we had reported in [2] and [7], the integrated interpolated form has the advantage of higher order accuracy than the other two variants due to the higher order of perturbation term it involves, and the recursive form, though very cumbersome for high degree approximants, has the advantages of minimum order Tau system, non-dependence of this

canonical polynomials on the boundary conditions, as well as the re-usability of these polynomials for approximants of higher degree. The error estimate, in the latter case, may also be determined even prior to the solution of its corresponding Tau problem.

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