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A Chebyshev-collocation approach for a continuous formulation of hybrid methods for initial value problems in ordinary differential equations

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Abstract

In* [3] - [5] *we reported some classes of methods for the continuous solution of initial value problems in ordinary differential equations which was developed through multistep collocation and with Chebyshev basis functions. This paper extends the work to two classes of hybrid methods with equal success. Numerical evidences are provided in support of this work.

***Keywords*: Continuous, Scheme, Formulation, Approximant, Explict, Implicit, Predictor, Corrector, Method, Hybrid.

1.0 Introduction

Continuous formulations of linear multistep methods (LMMs) for the solution of the initial value problem in first order ordinary differential equation:

$$y'(x) = f\left(x, y(x)\right), \quad a \le x \le b < +\infty \qquad \ldots \qquad (1.1a)$$

$$y(a) = y_0 \qquad \ldots \qquad (1.1b)$$

have been reported in the literatures (see for example [2], [4], [12], [13], [15] [16]). Various types of basis functions have been employed for this purpose and some of these include the monomial x^r, the so-called canonical polynomials of Lanczos, $Q_r(x)$, $r \in N_0$, the Legendre polynomials $P_n(x)$, among others.

In our previous works [3] – [5], we have considered the choice of the Chebyshev polynomials with great success. This suggests the drive for the present work on the hybrid methods. The motivation for the choice stems from the desire to ensure equi-distribution of the error in our derived approximant Y(x) of y(x) through the entire range of integration. The Chybyshev polynomial $T_r(x)$, defined over the range [a , b] by

$$T_r(x) = \cos\left[r\cos^{-1}\left\{{(2x-a-b)}\big/{(b-a)} - 1\right\}\right] \equiv \sum_{k=0}^{r} c_k^{(r)}\ x^k \quad \ldots \qquad (1.2)$$

is most appropriate for this purpose due to its min-max approximation and equi-oscillation properties in its entire range of definition.

In Section 2 below we shall briefly review some antecedents of the continuous formulation of some schemes. Section 3 focuses on the central concern of this paper. Numerical examples will be provided in Section and while, in section 5, we shall finally conclude the paper with some remarks.

2.0 Some antecedents of continuous scheme for ODEs.

We review briefly here some of our recent works on the use of Chebyshev basis functions for

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continuous formulation of some linear multistep methods (LMMs) by collocation techniques.

Our earlier attempts towards continuous formulation of LMMs were reported in [2] – [5] and [12]. In [2], we employed the canonical polynomials $Q_r(x)$ defined by

$$LQ_r(x) = x^r, \quad \ldots \quad (2.1)$$

where L is the linear operator associated with the DE (1.1), as basis functions, in a perturbed form of (1.1a). So doing, we discretized the range of integration [a , b] such that

$$a = x_0 < x_1 < x_2 \pi \ldots < x_n = b \quad (2.2)$$

where

$$x_k = x_0 + kh \quad , \quad k = 0\ (1)n, \ h = (b-a)/n$$

and then sought the solution of the perturbed form of (1.1) over each of the range $[x_k\ ,\ x_{k+1}]$ to have

$$Y'(x) = f(x\ , Y) + \tau P_n(x), \qquad x_k \le x \le x_{k+1} \quad \ldots \quad (2.3a)$$

$$Y(x_k) = Y_k \quad \ldots \quad (2.3b)$$

where

$$Y(x) = \sum_{r=0}^{n} a_r\, Q_r(x) \cong y(x). \quad \ldots \quad (2.3c)$$

$P_n(x)$ is the n^{th} degree Legendre polynomial valid in $[x_k\ , x_{k+1}]$. To determine the value of τ in (2.3a) and the $(n + 1)$ co-efficients a_r , $r = 0$ (1) n, in (2.3c), we collocated at $(n + 1)$ selected points $\bar{x}_r$, $r = 1(1)\ n + 1$. The resulting equation together with (2.3b) were then solved. So doing the Trapezoidal rule, the Simpson's method and the Gragg and Stetter methods were recovered as discrete forms of their corresponding continuous formulations. In Onumanyi et al [12], we replaced (2.3c) by

$$Y(x) = \sum_{r=0}^{n} a_r\ x^r\ , n < +\infty \quad \ldots \quad (2.4)$$

in a non-perturbed collocation procedure to have the equivalent form of (2.3a) - (2.3c) as

$$Y'(x) = f(x, Y), \quad x_k \le x \le x_{k+1} \quad \ldots \quad (2.5a)$$

$$Y(x_k) = Y_k \quad \ldots \quad (2.5b)$$

$$Y(x) = \sum_{r=0}^{n} a_r\ x^r. \quad \ldots \quad (2.5c)$$

Various classes of LMMs were thus recovered depending on the choice of the collocation points. In [3] – [5] and [4] we replaced (2.3c) with

$$Y(x) = \sum_{0}^{m} a_r\ T_r(x)\ ,\ \mathrm{m} < +\infty \quad \ldots \quad (2.6)$$

in both perturbed and non-perturbed collocation techniques to have, respectively, the equivalent IVPs:

$$Y'(x) = f(x, Y(x)) + \tau\, p_n(x), Y(x_k) = Y_k \quad (2.7)$$

and

$$Y'(x) = f(x, Y(x)),\ Y(x_k) = Y_k \quad (2.8)$$

where Y(x) is given by (2.6). These led to the recovery of various classes of LMMs, differing as the choice of collocation methods. Amongst these were the Newton-Cotes related methods, the Adams – Moulton methods, the Adams – Bashforth methods and the Backward Differentiation methods

We proceed to the next section to derive some hybrid methods based on the choice of (2.6) , the reason for which had been earlier been stated in Section 1.

3.0 Derivation of hybrid methods

For clarity and sake of completeness we restate here the IVP:

$$y'(x) = f\left(x,\ y(x)\right), \quad a = x_0 \le x \le x_n = b \qquad \ldots \qquad (3.1a)$$

$$y(a) = y_0 \qquad (3.1b)$$

whose solution we shall seek over the sub-intervals arising from the subdivisions

$$a = x_0 < x_1 < x_2 < \ \ldots < x_k < x_{k+1} < x_{n-2} < x_{n-1} < x_n = b \qquad \ldots \qquad (3.2)$$

where $x_k = x_0 + kh$ and $h = {(b-a)}/{n}$. Without loss of generality, we shall assume that a = 0 in (1.1), as any problem in the general interval [a , b] may be transformed into [0 , b] by the substitution:

$$x = a + \left(1 - {a}/{b}\right)u, \ 0 \le u \le b. \qquad \ldots \qquad (3.3)$$

This makes for simplification of the arithmetic that later follows.
So then, we shall seek the solution of the problem

$$Y'(x) = f(x), Y(x), x_k \le x \le x_{k+p} \qquad (3.4a)$$

$$Y(x_k) = Y_k \qquad (3.4b)$$

where

$$Y(x) = \sum_{r=0}^{n} a_r\ T_r(x) = \sum_{r=0}^{n} a_r \sum_{k=0}^{r} c_k^{(r)}\ x^r$$

and p varies as the method to be derived.

In what now follows, we shall construct some explicit and implicit hybrid methods.

3.1 Explicit hybrid methods

3.1.1 A one-step explicit scheme

We consider here (3.4) with $n = 2$ and collocation points as $x_k, x_{k+\frac{2}{3}}$ while theinterpolation points is x_k . This leads to the equations

$$\frac{2}{h} a_1 - \frac{8}{h} a_2 = f_k \qquad (3.5)$$

$$\frac{2}{h} a_1 - \frac{8}{3h} a_2 = f_{k+\frac{2}{3}} \qquad (3.6)$$

$$a_0 - a_1 + a_2 = Y_k, \qquad (3.7)$$

since

$$Y(x) = a_0 + a_1 \left[\frac{2(x-x_k)}{h} - 1\right] + a_2 \left[2\left\{\frac{2(x-x_k)}{h} - 1\right\}^2 - 1\right], \text{ for } x_k \le x \le x_{k+1}. \quad (3.8)$$

We solve the system constituted by (3.5) – (3.7) for a_k , $k = 0(1)2$ to have

$$a_0 = y_k + \frac{9h}{32} f_{k+\frac{2}{3}} + \frac{7h}{32} f_k$$

$$a_1 = \frac{3h}{8} f_{k+\frac{2}{3}} + \frac{h}{8} f_k$$

$$a_2 = \frac{3h}{32} f_{k+\frac{2}{3}} - \frac{3h}{32} f_k.$$

We insert these in (3.8) to get the continuous scheme

$$Y(x) = Y_k + h\left[\beta_0(x)\ f_k + \beta_1(x)\ f_{k+2/3}\right], \tag{3.9}$$

where

$$\beta_0(x) = \frac{(x - x_k)}{h} - \frac{3(x - x_k)^2}{4h^2},\ \beta_1(x) = \frac{3(x - x_k)^2}{4h^2}.$$

At the grid point x_{k+1}, this yields the discrete form equivalent

$$Y_{k+1} = Y_k + \frac{h}{4}\left(f_k + 3f_{k+2/3}\right). \tag{3.10}$$

This is the one-step explicit hybrid scheme of order three with an error constant $c_4 = 23/108$.

For the determination of $f_{k+2/3}$ in (3.10) we consider an approximant of y(x):

$$Y(x) = a_0\ T_0(x) + ... + a_2\ T_2(x),\ \ x_k \le x \le x_{k+1}$$

$$= a_0 + a_1\left(2\frac{(x - x_k)}{h} - 1\right) + a_2\left[2\left(2\frac{(x - x_k)}{h} - 1\right)^2 - 1\right]. \tag{3.11}$$

By interpolating (3.11) at x_k as well as collocating the equation

$$Y'(x) = a_1\ T_1(x) + a_2\ T_2'(x) = f(x, Y(x)),\ x_k \le x \le x_{k+1} = \frac{2a_1}{h} + \frac{8a_2}{h}\left(\frac{2(x - x_k)}{h} - 1\right)$$

at x_k and $\mathbf{x}_{k+1}$ we obtain the linear system

$$\begin{pmatrix} 1 & -1 & 1 \\ - & \frac{2}{h} & -\frac{8}{h} \\ 0 & \frac{2}{h} & \frac{8}{h} \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} Y_k \\ f_k \\ f_{k+1} \end{pmatrix},$$

the solution of which yields the values

$$a_0 = Y_k + \frac{5}{16}h\ f_k + \frac{3}{18}h\ f_{k+1},\quad a_1 = \frac{1}{4}h\left(f_k + f_{k+1}\right)$$

$$a_2 = \frac{1}{16}h\left(f_{k+1} - f_k\right).$$

We insert these into (3.11) to get the continuous scheme

$$Y(x) = Y_k + h\left[\beta_0(x)f_k + \beta_1(x)f_{k+1}\right] \tag{3.12}$$

where

$$\beta_0(x) = \frac{(x - x_k)}{h} - \frac{(x - x_k)^2}{xh^2}$$

$$\beta_1(x) = \frac{(x - x_k)^2}{2h^2},$$

and which at the grid point $x_{k+2/3}$ gives

$$Y_{k+2/3} = Y_k + \frac{2h}{9}\left(2f_k + f_{k+1}\right), \tag{3.13}$$

a method of order two, with an error constant $c_3 = -\,{}^{5}\!/_{81}$.

Thus, the method (3.10) can be used as an accurate corrector formulae with the derived scheme (3.13). From (3.10) we obtain $f_{k+2/3}$ for our proposed continuous scheme (3.9).

3.1.2 Two–step explicit scheme

Now let us consider (3.4) with n = 3 to have

$$Y(x) = a_0\ T_0(x) + a_1\ T_1(x) + a_2\ T_2(x) + a_3\ T_3(x)$$

$$\begin{aligned} &= a_0 + a_1\left[\frac{2(x-x_k)}{h} - 3\right] + a_2\left[2\left\{\frac{2(x-x_k)}{h} - 3\right\}^2 - 1\right] \\ &\quad + a_3\left[4\left\{\frac{2(x-x_k)}{h} - 3\right\}^2 - 3\left\{\frac{2(x-x_k)}{h} - 3\right\}\right], \end{aligned} \tag{3.14}$$

for $x_k \le x \le x_{k+3}$

We collocate (3.4) with (3.14) at x_{k+1}, $x_{k+2.7}$ and x_{k+2} as well as interpolate to x_{k+2} so as to have the linear algebraic system:

$$\frac{2a_1}{h} + \frac{8}{h}a_2 + \frac{18}{h}a_3 = f_{k+1}$$

$$\frac{2a_1}{h} + \frac{96}{5h}a_2 + \frac{3306}{25h}a_3 = f_{k+2.7}$$

$$\frac{2a_1}{h} - \frac{8}{h}a_2 + \frac{18}{h}a_3 = f_{k+2}$$

$$a_0 + a_1 + a_2 + a_3 = Y_{k+2}$$

The solution of this system yields the values

$$a_0 = Y_{k+2} - \frac{113}{816}h\,f_{k+1} - \frac{145}{136}hf_{k+2} + \frac{25}{357}hf_{k+2.7}$$

$$a_1 = \frac{53}{272}h\,f_{k+1} + \frac{43}{112}hf_{k+2} - \frac{75}{952}hf_{k+2.7}$$

$$a_2 = \frac{-1}{16}h\,f_{k+1} + \frac{1}{16}hf_{k+2}$$

$$a_3 = \frac{5}{816}h\,f_{k+1} + \frac{25}{2856}hf_{k+2.7} - \frac{5}{336}hf_{k+2}\ .$$

These, when inserted in (3.14) yields the continuous approximant of $y(x)$ as

$$Y(x) = Y_k + h\left[\beta_1(x)\ f_{k+1} + \beta_2(x)f_{k+1} + \beta_{27/10}(x)f_{k+2.7}\right]$$

$$\beta_1(x) = -\frac{122}{51} + \frac{54}{17h}(x-x_k) - \frac{47}{34h^2}(x-x_k)^2 + \frac{40}{204h^3}(x-x_k)^3 \tag{3.15}$$

$$\beta_2(x) = {}^{20}\!/_{21} - \frac{27}{h}(x - x_k) + \frac{37}{14}\frac{(x - x_k)^2}{h^2} - \frac{40}{84h^3}(x - x_k)^3$$

$$\beta_{2.7}(x) = -\frac{200}{257} + \frac{200}{119h}(x - x_k) - \frac{900}{714h^2}(x - x_k)^2 + \frac{200}{714h^3}(x - x_k)^3$$

At the grid point x_{k+3}, this yields the explicit hybrid scheme of two-step:

$$Y_{k+3} = Y_{k+2} + \frac{h}{714}\left(221 f_{k+2} - 7 f_{k+1} + 500 f_{k+2.7}\right). \tag{3.16}$$

It is of order four with an error constant c_5 = 3.103049136. See Lambert [7]. We obtain f_{k+1}, f_{k+2} and f_{k+2}, from (3.16) for the proposed continuous scheme (3.15).

3.2 Implicit hybrid methods

3.2.2 A One-step implicit scheme

Suppose now, we consider (3.4) for n = 3 and p = 1 to have

$$Y(x) = a_0 T_0(x) + a_1 T_1(x) + a_3 T_3(x) + , \quad x_k \le x \le x_{k+1}$$

$$= a_0 + a_1\left[\frac{2(x - x_k)}{h} - 1\right] + a_2\left[2\left\{\frac{2(x - x_k)}{h} - 1\right\}^2 - 1\right]$$

$$+ a_3\left[4\left\{\frac{2(x - x_k)}{h} - 1\right\}^3 - 3\left\{\frac{2(x - x_k)}{h} - 1\right\}\right]. \tag{3.17}$$

We collocate (3.4) with (3.17) for $n = 3$ *and* $p = 1$ *at* $x_k, x_{k+\frac{1}{2}}$ *and* x_{k+1}

as well as interpolate at x_k to have the matrix equation:

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ a & 2/h & -8/h & 18/h \\ 0 & 2/h & 0 & -6/h \\ 0 & 2/h & 8/h & 18/h \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} Y_k \\ f_k \\ f_{k+1/2} \\ f_{k+1} \end{pmatrix}.$$

We solve this to obtain

$$a_0 = Y_k + \frac{h}{48} f_k + \frac{7h}{48} f_{k+1} + \frac{h}{3} f_{k+1/2}$$

$$a_1 = \frac{h}{16} f_k + \frac{3h}{8} f_{k+1/2} + \frac{h}{16} f_{k+1}$$

$$a_2 = \frac{h}{16} f_k - \frac{h}{16} f_{k+1}$$

$$a_2 = \frac{h}{48} f_k - \frac{h}{24} f_{k+1/2} + \frac{h}{48} f_{k+1}.$$

These, with (3.17) yield the continuous scheme

$$Y(x) = Y_k + h\left[\beta_0(x) f_k + \beta_{\frac{1}{2}}(x) f_{k+1/2} + \beta_1(x) f_{k+1}\right] \tag{3.18}$$

where

$$\beta_0(x) = -\frac{(x-x_k)^2}{2h^2} + \frac{2(x-x_k)^3}{3h^3}$$

$$\beta_{1/2}(x) = \frac{2(x-x_k)^3}{h^2} - \frac{4(x-x_k)^3}{3h^3}$$

$$\beta_1(x) = \frac{(x-x_k)}{h} - \frac{3(x-x_k)^2}{2h^2} + \frac{2(x-x_k)^3}{3h^3} .$$

At $x = x_{k+1}$, this gives the one-step implicit hybrid scheme

$$Y_{k+1} = Y_k + \frac{h}{6}\left(f_k + 4 f_{k+1/2} + f_{k+1} \right) \tag{3.19}$$

of order four and error constant $c_5 = 1/2880$. The scheme (3.19) is the Gragg and Stetter method (see [8] and [9]). It can be used as an accurate corrector formula with the scheme (predictor):

$$Y_{k+1/2} = \frac{1}{2}(Y_k + Y_{k+1}) - \frac{h}{8}(f_{k+1} - f_k) \tag{3.20}$$

(see [2]) in a predictor-corrector algorithm as discussed fully by Lambert [8]. From (3.19) and (3.20) we determine $f_{k+1/2}$ and f_{k+1} for the continuous scheme (3.18).

3.2.3 A two–step implicit scheme

We shall consider here (3.4) for $n = 4$ and $p = 2$ and thus have the approximant of $y(x)$ as

$$Y(x) = a_0T_0(x) + a_1T_1(x) + a_2T_2(x) + a_3T_3(x) + a_4T_4(x), \quad x_k \le x \le x_{k+2}$$

$$= a_0 + a_1\left[\frac{x-x_k}{h} - 1\right] + a_2\left[2\left(\frac{2(x-x_k)}{h} - 1\right)^2 - 1\right] +$$

$$a_3\left[4\left(\frac{x-x_k}{h} - 1\right)^3 - 3\left(\frac{x-x_k}{h} - 1\right)\right] + a_4\left[8\left(\frac{x-x_k}{h} - 1\right)^4 - 8\left(\frac{x-x_k}{h} - 1\right)^2 + 1\right] \tag{3.21}$$

We collocate (3.4) together with (3.18) for $n = 4$ and $p = 2$ at x_k, x_{k+1}, x_{k+2}, and interpolate at $x_{k+3/2}$ to get the linear system:

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 1 & 1/2 & -1/2 & -1 & -1/2 \\ 0 & 1 & -4 & 9 & -16 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 1 & 4 & 9 & 16 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} Y_{k+1} \\ Y_{k+3/2} \\ hf_k \\ hf_{k+1} \\ hf_{k+2} \end{pmatrix} .$$

This yields the values

$$a_0 = \frac{13}{35} Y_{k+1} + \frac{22}{35} Y_{k+3/2} + \frac{127}{840} hf_k - \frac{121}{420} hf_{k+1} - \frac{3}{56} hf_{k+2}$$
$$a_1 = \frac{-4}{5} Y_{k+1} + \frac{45}{5} Y_{k+3/2} + \frac{-11}{120} hf_k - \frac{23}{60} hf_{k+1} + \frac{31}{168} hf_{k+2}$$
$$a_2 = \frac{-12}{35} Y_{k+1} + \frac{12}{35} Y_{k+3/2} + \frac{39}{280} hf_k - \frac{11}{70} hf_{k+1} - \frac{5}{165} hf_{k+2}$$
$$a_3 = \frac{h}{24} f_k - \frac{h}{12} f_{k+1} + \frac{h}{24} f_{k+2}$$
$$a_4 = \frac{2}{7} Y_{k+1} - \frac{2}{7} Y_{k+3/2} - \frac{h}{84} f_k - \frac{11}{84} hf_{k+1} + \frac{h}{48} hf_{k+2}.$$

We substitute these into (3.18) to get the continuous scheme

$$Y(x) = \alpha_1(x) Y_{k+1} + \alpha_{3/2}(x) Y_{k+3/2} + h\left[\beta_0(x) Y_k + \beta_1(x) f_{k+1} + \beta_2(x) f_{k+2}\right]$$

where $\alpha_1(x) = \frac{39}{35} - \frac{4(x-x_k)}{h} + \frac{376}{35}\frac{(x-x_k)^2}{h^2} - \frac{64}{7h^3}(x-x_k)^3 + \frac{16}{7}\frac{(x-x_k)^4}{h^4}$

$$\alpha_{3/2}(x) = \frac{-4}{35} + \frac{4(x-x_k)}{h} - \frac{376}{35}\frac{(x-x_k)^2}{h^2} - \frac{64}{7h^3}(x-x_k)^3 - \frac{16}{7}\frac{(x-x_k)^4}{h^4}$$
$$\beta_0(x) = \frac{23}{70} - \frac{(x-x_k)}{12h} - \frac{293(x-x_k)^2}{420\ h^2} + \frac{23(x-x_k)^3}{42h^3} - \frac{8(x-x_k)^4}{h^4}$$
$$\beta_1(x) = \frac{-43}{70} - \frac{11(x-x_k)}{6h} + \frac{622}{105\ h^2}\frac{(x-x_k)^2}{h^2} - \frac{95(x-x_k)^3}{21h^3} + \frac{22}{21}\frac{(x-x_k)^4}{h^4}$$
$$\beta_2(x) = \frac{-2}{7} + \frac{25}{84}\frac{(x-x_k)}{h} + \frac{11(x-x_k)^2}{28h^2} - \frac{25(x-x_k)^3}{42h^3} + \frac{8(x-x_k)^4}{42h^3}$$
$$\beta_2(x) = \frac{-2}{7} + \frac{25}{84}\frac{(x-x_k)}{h} + \frac{11(x-x_k)^2}{28h^2} - \frac{25(x-x_k)^3}{42h^3} + \frac{8(x-x_k)^4}{42h^3}$$

This, at the grid point x_{k+2}, yield the two-step implicit hybrid scheme

$$Y_{k+2} + \frac{17}{32} Y_{k+1} - \frac{52}{35} Y_{k+3/2} = h\left[\frac{8}{35} f_k - \frac{1}{70} f_{k+1} + \frac{1}{6} f_{k+2}\right]$$

of order four with error constant $c_1 = -13/105$.

4.0 Numerical examples

From our earlier works (see [2] - [4]), it is evident that the continuous schemes perform much better in terms of efficiency and cost compared viz-a-viz their discrete form equivalents. For the sake of completeness, and emphasis, we provide again here some numerical evidences in support of this assertation.

***Example* 4.1**

We consider $y' - xy = 0,\ y(0) = 1,\ 0 \le x \le 1$, whose analytic solution is $y(x) = \exp\left(\frac{1}{2}x^2\right)$.

The numerical results obtained from the experimentation of the schemes (3.9), (3.10), (3.15) and (3.16) on this example are presented in Table.

***Example* 4.2**

We consider here the IVP in first order system of equations

$$L\underline{Y} = \underline{O}\text{, where } L = \begin{bmatrix} d/dx & -1 \\ -2 & d/dx + 1 \end{bmatrix}, \quad \underline{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \underline{Y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and whose exact solution is $\underline{Y}(x) = \left[\frac{1}{3}\left(e^x - e^{-2x}\right), \frac{1}{3}\left(e^x + 2e^{-2x}\right)\right]$, for $0 \le x \le 1$.

Numerical evidences for this example based on experimentation with the scheme (3.9) – (3.10) and (3.15) – (3.16) are presented in Table 4.2.

Table 4.1 Errors of methods for Example 4.1 with h = 0.1, $\delta x = h/10$.

X	Discrete method (3.10)	Continuous Scheme (3.9)	Discrete Method (3.16)	Continuous Method (3.15)
0.00	0.000	0.000	0.000000	0.000000
0.01		1.110E-7		9.044090E-4
0.02		4.270E-7		1.607698E-3
0.03		9.040E-7		2.109971E-3
0.04		1.467E-6		2.411305E-3
0.05		2.010E-6		2.511744E-3
0.06		2.399E-6		2.411304E-3
0.07		2.468E-6		2.109970E-3
0.08		2.022E-6		1.607697E-3
0.09		8.330E-7		9.04409E-4
0.10	1.354 E - 6	1.354E-6	0.000000	0.00000
0.11		7.840E-7		9.317440E-4
0.12		7.100E-7		1.656287E-3
0.13		2.780E-6		2.173737E-3
0.14		5.045E-6		2.484176E-3
0.15		7.096E-6		2.587649E-3
0.16		8.487E-6		2.484171E-3
0.17		8.747E-6		2.173734E-3
0.18		7.368E-6		1.656284E-3
0.19		3.811E-6		9.317430E-4
0.20	2.494E - 6	2.494E-6	0.000000	0.000000

Table 4.2: Errors of methods for example 4.2 with h = 0.1, $\delta x = h/10$.

X	**Discrete method (3.10)**	**Continuous Scheme (3.9)**	**Discrete Method (3.16)**	**Continuous Method (3.15)**
0.00	0.000	0.00000	0.000000	0.000000
0.01		5.890000E-7		6.268552E-3
0.02		3.800000E-7		3.977540E-4
0.03		3.550000E-6		8.87050043
0.04		1.408400E-5		1.565214E-3
0.05		3.405200E-5		2.425818E-3
0.06		6.623900E-5		3.464937E-3
0.07		1.133870E-4		4.678145E-3
0.08		1.781940E-4		6.061114E-3
0.09		2.633130E-4		7.609612-4
0.10	3.713590E-4	3.713590E-4	9,319504E-3	9.319504E-3
0.11		4.636890E-4		9.316058E-3

0.12		5.559650E-4		9.474634E-3
0.13		6.50682E-4		9.791371E-3
0.14		7.502950E-4		1.0262491E-3
0.15		8.572220E-4		1.0884303E-2
0.16		9.738410E-4		1.1653204E-2
0.17		1.102499E-3		1.2565669E-3
0.18		1.245507E-3		1.3618258E-2
0.19		1.40514E-3		1.4807609E-4
0.20	1.583641E-3	1.583641E-3	1.6130439E-2	1.6130439E-2

Table 4.3: Errors for example

x	**Discrete method (3.16)**	**Continuous scheme (3.15)**
0.00	0.0	0.0
0.01		9.044090 E-4
0.02		1.607698 E-3
0.03		2.109971 E-3
0.04		2.411305 E-3
0.05		2.511744 E-3
0.06		2.411304 E-3
0.07		2.109970 E-3
0.08		1.607697 E-3
0.09		9.044090 E-4
0.10	1.000000 E – 9	1.000000 E-9
0.11		9.317440 E-4
0.12		1.656287 E – 3
0.13		2.173737 E – 3
0.14		2.484176 E – 3
0.15		2.587649 E – 3
0.16		2.484171 E – 3
0.17		2.173734 E-3
0.18		1656284 E – 3
0.19		9.317430 E – 4
0.20	1.000000 E- 9	1.000000 E - 9
Max. Error in [0 , 1]	3.220 000 e – 7	7.479670 E - 3

5.0 Conclusion

A method for the derivation of continuous hybrid schemes for the solution of IVPs in ordinary differential equations has been presented. For this purpose the Chebyshev polynomial has been employed as the basis function and a collocation approach was adopted. The schemes reproduced their corresponding discrete finite difference equivalents at approximate chosen/selected points. The higher

the step number of the methods the larger the dimension of the matrix equation that is involved and consequently the higher the computational cost. The continuous schemes are desirable as they exhibit the features of efficiency since they do not require additional interpolation to yield as many results as desirable at the off-grid points and that, at no extra cost. Numerical evidences also demonstrate the accuracy of these schemes.

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