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Strategies for deriving multi-derivative GMLM for the numerical solution of IVPs in ODEs

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Abstract

In the search for high accuracy, reliable and efficient numerical methods for initial value problems (IVPs) in ordinary differential equations (ODEs) the need arises to incorporate more analytic information of derivatives from the underlining ODEs into the design of a method. This presentation considers a strategy for deriving multi-derivative general multi-linear methods (GMLM) which is referred by Burrage and Butcher, the original inventors, as general linear methods (GLM), for the numerical solution of ordinary differential equations, see Hairer, Norsett and Wanner [31 pp385-401].The proposed approach is to use continuous polynomial interpolation and collocation of the solution of the initial value problem. The purpose is to show that interpolation and collocation can he used as a veritable tool in the design of efficient, high order and highly stable GLM compared to the other means of Taylor series expansion, integration and differentiation, amongst other methods of deriving computational methods for IVPs in ODEs.

1.0 Introduction

1.1 General multi-linear methods (GMLM)

In previous attempts at the numerical solution of the initial value problem the author has considered the use of rational interpolant to derive discrete variable methods for this class of problems. However, in [28] and [29] a different approach has been presented for the derivation of methods using collocation to introduce initial value methods that are capable of continuous output of the numerical solution of IVPs in ODEs.

In this presentation we wish to further consider strategies for the derivation of methods for dense out put. Already, methods in this regard are in [1] – [8], [16], [17], [25], [26], [27], [30], - [33]. Further theories of this class of continuous methods can be found in [9] – [15], [18] – [24] and [31] – [36].

The numerical solution of the initial value problem,

$$y^1 = f(x,y),\ y(a) = y_0 \qquad 1.1$$

in ordinary differential equations may employ the general linear methods (GLM) of Burrage and Butcher, considered in Hairer, Norsett and Wanner [31,pp385-401]

$$y_i^{(n+1)} = \sum_{j=1}^{k} a_{ij}^{[1]} y_j^{(n)} + h\sum_{j=1}^{s} b_{ij}^{[1]} f\left(x_n + c_j h, v_j^{(n)}\right),\ i = 1,2,\ldots,k \qquad (1.2a)$$

where

$$v_i^{(n)} = \sum_{j=1}^{k} a_{ij}^{[2]} y_j^{(n)} + h\sum_{j=1}^{s} b_{ij}^{[2]} f\left(x_n + c_j h, v_j^{(n)}\right),\ i = 1,2,\ldots,s \qquad (1.2b)$$

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the $v_i^{(n)}, i=1,2,...,s$ are the internal stages and $y_i^{(n+1)}, i=1,2...,k$ so called external stages are the generated numerical solution vectors we seek of the initial problem in (1.1). The statement in (1.2) presents a GLM as a composition of $k-LMM$ with s number of hybrid points. It is in this regard that the GLM (1.2) of Burrage and Butcher considered in Hairer, Norsett and Wanner, [31, pp385-401] shall be referred to as general Multi – linear methods (GMLM) herein. To present the picture differently, see the connection with RKM by writing the above (1.2) equivalently as

$$\begin{cases} y_i^{(n+1)} = \sum_{j=1}^{k} a_{ij}^{[1]} y_j^{(n)} + h\sum_{j=1}^{s} b_{ij}^{[1]} k_j^{(n)} & i=1,2...,k \\ k_i^{(n)} = f\left(x_n + c_i h, \sum_{j=1}^{k} a_{ij}^{[1]} y_j^{(n)} + h\sum_{j=1}^{s} b_{ij}^{[1]} k_j^{(n)} \right) & i=1,2,...,s \end{cases} \tag{1.3}$$

Really, the difficulty is more constructing the GMLM directly from this structural form, although a direct restructuring of (1.2) into (1.3) as in the above has been employed in Yakubu [30]. It thus becomes glaring that the GLM are indeed also linear multi-step RKM and even more. A further generalization of GMLM is the multi-derivative GMLM

$$\begin{cases} y_i^{(n+1)} = \sum_{j=1}^{k} a_{ij}^{[1]} y_j^{(n)} + \sum_{r=1}^{q} h^r \left(\sum_{j=1}^{k} a_{ij}^{[r,1]} f\left(x_n + c_j h, v_j^{(n)}\right) \right), \ i=1,2...,k; q\geq 1 \\ V_i^{(n+1)} = \sum_{j=1}^{k} a_{ij}^{[2]} y_j^{(n)} + \sum_{r=1}^{q} h^r \left(\sum_{j=1}^{s} a_{ij}^{[r,2]} f\left(x_n + c_j h, v_j^{(n)}\right) \right), \ i=1,2...,s \end{cases} \tag{1.4}$$

In a more traditional notational equivalent in the spirit we have found above is this one,

$$\begin{aligned} y_i^{(n+1)} &= \sum_{j=1}^{k} a_{ij}^{[1]} y_j^{(n)} + \sum_{r=1}^{q} h^r \left(\sum_{j=1}^{k} b_{ij}^{[r,1]} k_i^{(n)} \right), \ i=1,2...,k; q\geq 1 \\ k_i^{(n)} &= f\sum_{j=1}^{k} a_{ij}^{[2]} y_j^{(n)} + \sum_{r=1}^{q} h^r \left(\sum_{j=1}^{s} a_{ij}^{[r,2]} f\left(x_n + c_j h, k_j^{(n)}\right) \right), \ \ i=1,2...,s \end{aligned} \tag{1.5}$$

The method parameters $a_{ij}^{[1]}, a_{ij}^{[2]}, b_{ij}^{[1]}, b_{ij}^{[2]}, b_{ij}^{[r,1]}, b_{ij}^{[r,2]},$ are the determinable real constants that uniquely characterize a GMLM. In particular, with $q = 2$ is the second derivative GLM. In fact the methods may as well be referred to as second derivative linear multi-step RKM in a sense of the foregoing. Further methods in this regard can be found in [1]-[33].

2.0 The continuous multi-derivative GMLM

In this section interpolation and collocation is employed in the derivation of multi-derivative GMLM referred to as multi-derivative GLM in Hairer, Norsett and Wanner, [31, pp385-401], but first introduced by Barrage and Butcher. The demerit of obtaining methods this way is the fact that it dose not give a straight forward way of finding error constant of the resultant method, although this is readily obtained by a later means of Taylor's series expansion after deriving the method. However, its merit is of immense value, to mention just a few, this approach gives the benefit of obtaining discrete and continuous versions of multi-derivative LMM, RKM, composite and hybrid method since the process employs continuous interpolation of the solution vectors of the initial value problem (1.1). It is worthwhile to also remark that the continuous version of a given method provides the benefit of getting the solution output values at any desired point in the integration interval and what is more!, variable step-size, variable order variable method implementation of that family of methods which then is with continuous coefficients comes almost for free, although the process of achieving this is by no means trivial. This is what is exploited to get parallel block methods or RKM. In following Onumanyi et al [33],

the new methods will be highlighted as in the following, let Y_{n+j} for this purpose be the approximation to the numerical solution y_{n+j} (we can then put that $Y_{n+j} = y_{n+j}$ wherever appropriate in subsequent derivation) of the exact solution $y(x_{n+j})$ of (1.1). A q-step s-multi-derivative collocation method where $\{x_{n+j}, j = 0,1,2,...,k-1, k \geq 1\}$ interpolation points of the solution and $\{x_{n+j}^{[q]}; j = 0,1,2,...,m_q - 1, q = 1,2,...,s; s \geq 1\}$ collocation points of the s-order derivative involves finding a polynomial of degree $p = k + \sum_{q=1}^{s} m_q - 1; k > 0, m_q > 0, s \geq 1$ that satisfies the interpolation conditions

$$Y(x_{n+j}) = Y_{n+j};\ j = 0,1,2,...,k-1, k \geq 1 \qquad (2.1)$$

and the collocation conditions

$$\left\{\begin{array}{l} Y^{/}\left(x_{n+j}^{[1]}\right) = f\left(x_{n+j}^{[1]}, Y\left(x_{n+j}^{[1]}\right)\right);\ j = 0,1,2,...,m_1 - 1, m_1 \geq 1 \\ Y^{(q)}\left(x_{n+j}^{[q]}\right) = f^{(q-1)}\left(x_{n+j}^{[q]}, Y\left(x_{n+j}^{[q]}\right)\right);\ j = 0,1,2,...,m_q - 1; q = 2,3,...,s \end{array}\right. , m_q \geq 1 \qquad (2.2)$$

It is to be noted that some interpolation points may well be collocation points and vice-versa, the coincident is allowed. A solution method to the initial value problem (1.1) which employs the information of multi-derivatives like the stiffly stable second derivative methods of Enright [14] is therefore given by the continuous s-order derivative GMLM of the general structure

$$Y(x) = \sum_{j=0}^{k-1} \alpha_{n+1}^{[s]}(x) y_{n+j} + \sum_{r=1}^{s} h^r \left(\sum_{j=0}^{m_r - 1} \beta_j^{[r]}(x) f^{(r-1)}\left(x_{n+j}^{[r]}, Y\left(x_{n+j}^{[r]}\right)\right)\right); \qquad (2.3)$$

assuming of course that the conditions (2.2,2.2) above have been satisfied, where there is understanding that

$$\left\{\begin{array}{l} f^{(0)}\left(x_{n+j}^{[1]}, Y\left(X_{n+j}^{[1]}\right)\right) = f\left(x_{n+j}^{[1]}, Y\left(X_{n+j}^{[1]}\right)\right) \qquad f^{(1)}\left(x_{n+j}^{[2]}, Y\left(X_{n+j}^{[2]}\right)\right) = f^{/}\left(x_{n+j}^{[2]}, Y\left(X_{n+j}^{[2]}\right)\right) \\ f^{(q)}\left(x_{n+j}^{[q]}, Y\left(X_{n+j}^{[q]}\right)\right) = f^{(q-1)}\left(x_{n+j}^{[q1]}, Y\left(x_{n+j}^{[q]}\right)\right) \quad j = 0(1)m_r -,; q = 1,2,...,s; s \geq 1 \end{array}\right. \qquad (2.4)$$

and $\alpha_j^{[s]}(x)$ and $\{\beta_j^{[r]}(x)\}_{r=1}^{s}$ are the real continuous coefficients assumed to be polynomials given as

$$\alpha_j^{[s]}(x) = \sum_{i=0}^{k+\sum_{j=1}^{s} m_j - 1} \alpha_{j,i}^{[s]} \psi_i(x);$$

$$\beta_j^{[s]}(x) = \sum_{i=0}^{k+\sum_{j=1}^{s} m_j - 1} \alpha_{j,i}^{[s]} \psi_i(x); \qquad \psi_i(x) = x^i; 1,2,...,s \qquad (2.5)$$

The $\{\psi_i(x)\}_{i=0}$ are the monomial basis interpolating functions, however any other polynomial basis function $\{P_i(x)\}_{i=0}$ which are usually orthogonal in an interval on the real line will do as well. The expression in (2.3) gives rise to both continuous and discrete appropriate LMM, hybrid, parallel block, and

RKM methods. In fact, by appropriately evaluating $Y(x), Y^{(q)}(x),\ q = 1,2\dots,s$ at non interpolatory points

$$t_c \in \left(\bigcup_{j=0}^{k-1}\left\{w : w\in \left(x_{n+j}, x_{n+1}\right)\right\}\right)\bigcup\left\{\bigcup_{q-1}^{s}\left\{x_n^{[q]},\dots,x^{[q]}_{n+m_1+\dots+m_s-1}\right\}\right\}\setminus\left\{x_n, x_{n+1},\dots,x_{n+k-1}\right\} \qquad (2.6)$$

and also at non-collocatory points

$$t_1 \in \bigcup_{r=1s}^{s}\left(\bigcup_{j=0}^{m_r}\left\{w : w\in \left(x^{[q]}_{n+j}, x^{[q]}_{n+j+1}\right)\right\}\right)\setminus\bigcup_{q=1}^{s}\left\{x_n^{[q]},\dots,x^{[q]}_{n+m_s-1}\right\}$$

denoting the results as $Y_t^{(x)}, Y_t^{(q)}$ we get the discrete version

$$Y_{t_c} = \sum_{j=0}^{k-1}\alpha^{[s]}_{j^*}y_{n+j} + \sum_{r=1}^{s} h^r\left(\sum_{j=0}^{m_r-1}\beta^{[r]}_{j^*} f^{(r-1)}\left(x^{[r]}_{n+j}, Y^{[r]}_{n+j}\right)\right)$$

where now

$$Y^{[r]}_{n+j} = f\left(x^{[r]}_{n+i}, \sum_{j=0}^{k-1}\alpha^{[s]}_{j^{**}}y_{n+j} + \sum_{l=1}^{s}h^r\left(\sum_{j=0}^{m_r-1}\beta^{[r]}_{j^{**}} f^{(l-1)}\left(x^{[r]}_{n+j}, Y\left(x^{[r]}_{n+j}\right)\right)\right)\right);$$

is a particular method of interest which continuous form is the collocation method in (2.3). Thus evaluating (2.3), and its derivatives at several point s of reasonable $\{t_l, t_c\}$ leads to multiple LMM, RKM, composite methods or their hybrids whose solution to the IVP in (1.1) can be formed into a vector that is referred to as a block and thus can be implemented as a parallel block method. The motivation to introduce the derivatives of the solution in (2.3) is to have higher order, or rather to bypass stability and order barrier constrains of convectional LMM and RKM, another is for stiff stability reason and incidentally the arising methods may be known for smaller error constants when compared with the convectional LMM for the same k step. The overhead in computation however, is in the evaluation of the s-order derivatives, but this cost may not be so significant in cases where (1.1) is autonomous and indeed many real life applications arises from which leads to the need to solve numerically autonomous ordinary differential systems. An example to point to, is in the modeling of the spread of diseases. The problem now is to determine the constants in (2.4),. Invoking (2.4) in (2.3), the result is

$$Y(x) = \sum_{i=0}^{k+\sum_{j=1}^{s} m_j -1}\left(\sum_{j=0}^{k-1}\alpha^{[s]}_{ji}y_{n+j} + \sum_{r=1}^{s}h^r\left(\sum_{j=0}^{m_r-1}\beta^{[r]}_{ji} f^{(r-1)}\left(x^{[r]}_{n+j}, Y\left(x^{[r]}_{n+j}\right)\right)\right)\right)\psi_i(x) \qquad (2.7)$$

By writing this as

$$Y(x) = \sum_{i=0}^{k+\sum_{j=1}^{s} m_j -1}\alpha^{[s]}_i\psi_i(x)$$

$$\alpha^{[s]}_i = \sum_{j=0}^{k-1}\alpha^{[s]}_{ji}y_{n+j} + \sum_{r=1}^{s}h^r\left(\sum_{j=0}^{m_r-1}\beta^{[r]}_{ji} f^{(r-1)}\left(x^{[r]}_{n+j}, Y\left(x^{[r]}_{n+j}\right)\right)\right) \qquad (2.8)$$

it becomes apparent that the solution of the IVP (1.1) is been approximated by a polynomial of degree $k + m_1 + \dots + m_s - 1$. Lets have the definitions of the following vectors,

$$a^{[s]} = \left(a_0^{[s]}, a_1^{[s]}, \dots, a_{k+m_1+\dots+m_s-1}^{[s]}\right)^T$$

$$\psi(x) = \left(\psi_0(x), \psi_1(x), \dots, \psi_{k+m_1+\dots+m_s-1}(x)\right)^T$$

$$v^{[s]} = \Big(Y_n \cdots Y_{n+k-1},\ f\left(x_n^{[1]}, Y_n\right) \dddot{f}\ \left(x_{n+1}^{[1]}, Y\left(x_{n+1}^{[1]}\right)\right), \dots,\ f\left(x_{n+m_1-1}^{1}, Y\left(x_{n+m_1-1}^{[1]}\right)\right), \dots, \tag{2.9}$$

$$, \dots,\ f^{(s-l)}\left(x_n^{[s]}, Y\left(x_n^{[s]}\right)\right), f^{(s-l)}\left(x_{n+1}^{[s]}, Y\left(x_{n+1}^{[s]}\right)\right)^T$$

recalling the interpolatory collocatory conditions (2.1) and (2.8) we have

$$F\ a^{[s]} = v^{[s]}$$

where

$$
F=\left(\begin{array}{ccccccc}
1 & x_n & x_n^2 & x_n^3 & \cdots\; x_n^s & \cdots & x_n^{k-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
1 & x_{n+k-1} & x_{n+m-1}^2 & x_{n+m-1}^3 & x_{n+m-1}^s & \cdots & x_{n+m-1}^{k-1} \\
0 & 1 & 2x_n^{[1]} & 3x_n^{[1]2} & \cdots\; s x_n^{[1]s-1} & \cdots & (k-1)x_n^{[1]k-2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 1 & 2x_{n+m_1-1}^{[1]} & 3x_{n+m_1-1}^{[1]2} & \cdots\; s x_{n+m_1-1}^{[1]s-1} & \cdots & (k-1)x_{n+m_1-1}^{[1]k-2} \\
0 & 0 & 2 & 6x_n^{[2]} & \cdots\; s(s-1)x_n^{[2]s-2} & \cdots & (k-1)x_n^{[2]k-3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 2 & 6x_{n+m_2-1}^{[2]} & s(s-1)x_{n+m_2-1}^{[2]s-2} & \cdots & (k-1)\cdots(k-2)x_{n+m_2-1}^{[2]k-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & & \vdots \\
0 & 0 & 0 & & s(s-1)\cdots 2.1 & x_n^{[s]} \quad \cdots & (k-1)\cdots(k-s)x_n^{[s]k-s-1} \\
\vdots & 0 & 0 & & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & s(s-1)\cdots 2.1 & x_{n+m_1-1}^{[s]} \quad \cdots & (k-1)\cdots(k-s)x_{n+m_s-1}^{[s]k-s-1}
\end{array}\right.
$$

$$
\left.\begin{array}{cccc}
\cdots & x_n^{k+m_1-1} & \cdots & x_n^{k+m_2-1} \\
 & \vdots & & \vdots \\
 & x_{n+k-1}^{k+m_1-1} & \cdots & x_{n+k-1}^{k+m_2-1} \\
\cdots & & \cdots & \vdots \\
\cdots & (k+m_1-1)x_n^{[1]k+m_1-2} & \cdots & (k+m_1+m_2-1)x_n^{[1]k+m_1+m_2-2} \\
 & & \cdots & \vdots \\
\cdots & (k+m_1-1)x_{n+m_1-1}^{[1]k+m_1-2} & \cdots & (k+m_1+m_2-1)x_{n+m_1-1}^{[1]k+m_1+m_2-2} \\
\cdots & (k+m_1-1)(k+m_1-1)x_n^{[2]k+m_1-3} & \cdots & (k+m_1+m_2-1)(k+m_1+m_2-1)x_n^{[2]k+m_1+m_2-3} \\
 & \vdots & \cdots & \vdots \\
\cdots & (k+m_1-1)(k+m_1-2)x_{n+m_2-1}^{[2]k+m_1-3} & \cdots & (k+m_1+m_2-1)(k+m_1+m_2-2)x_{n+m_2-1}^{[2]k+m_1+m_2-3} \\
\cdots & \vdots & \cdots & \vdots \\
\cdots & (k+m_1-1)\cdots(k+m_1-s)x_n^{[2]k+m_1-s-1} & \cdots & (k+m_1+m_2-1)\cdots(k+m_1+m_2-s)x_n^{[2]k+m_1+m_2-s-1} \\
 & \vdots & \cdots & \vdots \\
\cdots & (k+m_1-1)\cdots(k+m_1-s)x_{n+m_s-1}^{[2]k+m_1-s-1} & \cdots & (k+m_1+m_2-1)\cdots(k+m_1+m_2-s)x_{n+m_s-1}^{[2]k+m_1+m_2-s-1}
\end{array}\right)
$$

$$
\left.\begin{array}{ll}
\mathrm{K} & x_n^{k+m_1+\ldots+m_{s-1}-1} \\
 & \mathrm{M} \\
\mathrm{K} & x_{n+k-1}^{k+m_1+\ldots m_{s-1}-1} \\
\mathrm{K} & (k+m_1+\ldots+m_{s-1}-1)x_n^{[1]k+m_1+\ldots+m_{s-1}-2} \\
 & \mathrm{M} \\
\mathrm{K} & (k+m_1+\ldots+m_{s-1}-1)x_{n+m_1-1}^{[1]k+m_1+\ldots+m_{s-1}-2} \\
\mathrm{K} & (k+m_1+\ldots+m_{s-1}-1)(k+m_1+\ldots+m_{s-1}-2)x_n^{[2]k+m_1+\ldots+m_{s-1}-3} \\
 & \mathrm{M} \\
\mathrm{K} & (k+m_1+\ldots+m_{s-1}-1)(k+m_1+\ldots+m_{s-1}-2)x_{n+m_2-1}^{[2]k+m_1+\ldots+m_{s-1}-3} \\
\mathrm{K} & \mathrm{M} \\
\mathrm{K} & (k+m_1+\ldots+m_{s-1}-1)(k+m_1+\ldots+m_{s-1}-s)x_n^{[s]k+m_1+\ldots+m_{s-1}-s-1} \\
 & \mathrm{M} \\
\mathrm{K} & (k+m_1+\ldots+m_{s-1}-1)(k+m_1+\ldots+m_{s-1}-s)x_{n+m_s-1}^{[s]k+m_1+\ldots+m_{s-1}-s-1}
\end{array}\right.
$$

$$
\left.\begin{array}{ll}
\mathrm{K} & x_n^{k+m_1+\ldots+m_s-1} \\
 & \mathrm{M} \\
\mathrm{K} & x_{n+k-1}^{k+m_1+\ldots m_s-1} \\
\mathrm{K} & (k+m_1+\ldots+m_s-1)x_n^{[1]k+m_1+\ldots+m_s-2} \\
 & \mathrm{M} \\
\mathrm{K} & (k+m_1+\ldots+m_s-1)x_{n+m_1-1}^{[1]k+m_1+\ldots+m_s-2} \\
\mathrm{K} & (k+m_1+\ldots+m_s-1)(k+m_1+\ldots+m_s-2)x_n^{[2]k+m_1+\ldots+m_s-3} \\
 & \mathrm{M} \\
\mathrm{K} & (k+m_1+\ldots+m_s-1)(k+m_1+\ldots+m_s-2)x_{n+m_2-1}^{[2]k+m_1+\ldots+m_s-3} \\
\mathrm{K} & \mathrm{M} \\
\mathrm{K} & (k+m_1+\ldots+m_s-1)(k+m_1+\ldots+m_s-s)x_n^{[s]k+m_1+\ldots+m_s-s-1} \\
 & \mathrm{M} \\
\mathrm{K} & (k+m_1+\ldots+m_s-1)(k+m_1+\ldots+m_s-s)x_{n+m_s-1}^{[s]k+m_1+\ldots+m_s-s-1}
\end{array}\right) \qquad (2.10)
$$

The F is of dimension $(k+m_1+\ldots+m_s)(k+m_1+\ldots+m_s)$. the s-order derivative collocation method (2.3) is now

$$Y(x)=\left(v^{[s]}\right)^T\left(F^{-1}\right)^T\psi(x) \qquad (2.12)$$

from (2.10). Observe that (2.3) can be write as $a^{[s]}=Hv^{[s]}$ where now

$$H=\begin{pmatrix} \alpha^{[2]}_{0,0} & \alpha^{[2]}_{0,1} & \cdots & \alpha^{[2]}_{0,k-1} & h\beta^{[1]}_{0,0} & \cdots & h\beta^{[2]}_{0,m_1-1} & h^2\beta^{[2]}_{0,0} & \cdots & h^2\beta^{[2]}_{0,m_2-1} & \cdots & h^s\beta^{[s]}_{0,0} & \cdots & h^s\beta^{[s]}_{0,m_s-1} \\ \alpha^{[2]}_{1,0} & \alpha^{[2]}_{1,1} & \cdots & \alpha^{[2]}_{0,k-1} & h\beta^{[1]}_{1,0} & \cdots & h\beta^{[1]}_{1,m_1-1} & h^2\beta^{[2]}_{1,0} & \cdots & h^2\beta^{[2]}_{1,m_2-1} & \cdots & h^s\beta^{[s]}_{1,0} & \cdots & h^s\beta^{[s]}_{1,m_s-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha^{[2]}_{k+m_1+...+m_s-1,0} & \alpha^{[2]}_{k+m_1+...+m_s-1,1} & \cdots & \alpha^{[2]}_{k+m_1+...+m_s-1,k-1} & h\beta^{[1]}_{k+m_1+...+m_s-1,0} & \cdots & h\beta^{[1]}_{k+m_1+...+m_s-1,m_1-1} & h^s\beta^{[s]}_{k+m_1+...+m_s-1,0} & \cdots & h^s\beta^{[s]}_{k+m_1+...+m_s-1,m_2-1} & \cdots & h^s\beta^{[s]}_{k+m_1+...+m_s-1,0} & \cdots & h^s\beta^{[s]}_{k+m_1+...+m_s-1,m_s-1} \end{pmatrix} \quad \textbf{(}2.12$$

This H is of dimension $(k+m_1+...+m_s)(k+m_1+...+m_s)$. Similarly, $Y(x)=\left(v^{[s]}\right)^T\left(F^{-1}\right)^T\psi(x)$ in (2.8) implying finally that $H = F^{-1}$, provided non-singularity of F. This is all the computational tool needed to find the unknown constants

$$\left\{\alpha^{[s]}_{j,i}\right\}_{j=0,i=0}^{k-1,k+m_1+...+m_s-1},\left\{\beta^{[s]}_{j,i}\right\}_{j=0,i=0}^{m_1-1,k+m_1+...+m_s-1},...,\left\{\beta^{[s]}_{j,i}\right\}_{j=0,i=0}^{m_s-1,k+m_1+...+m_s-1};\quad s\geq 1 \qquad (2.13)$$

and thus to characterize a multi-derivative GLM of the type in (2.3) completely. The next section seeks to show how this strategy in (2.12) of constructing GMLM can be use as a veritable tool in the design of efficient, high order and highly stable GLM.

3.0 Recovering some existing LMM through interpolation and collocation

The above highlighted approach of deriving the class of methods

$$Y(x)=\left(v^{[s]}\right)^T\left(F^{-1}\right)^T\psi(x) \qquad (3.1)$$

in (2.3) and (2.12) recovers the following well established and well know families of methods by fixing s, F, and H appropriately. This is illustrated in what is now to follow.

3.1 Adams-Bashforth LMM

This is the implicit methods (3.1) given by setting

$$s=1, m_1=l, k=1, x^{[1]}_{n+j}=x_{n+j};\ j=0(1)l-1 \qquad (3.2)$$

$$F=\begin{pmatrix} 1 & x_{n+l-1} & x^2_{n+l-1} & x^3_{n+l-1} & \cdots & x^{l+1}_{n+l-1} \\ 0 & 1 & 2x_n & 3x_n & \cdots & (l+1)x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{n+l-1} & 3x_{n+l-1} & \cdots & (l-1)x_{n+l-1} \end{pmatrix}, H=\begin{pmatrix} \alpha^{[2]}_{0,l-1} & h\beta^{[1]}_{0,0} & \cdots & h\beta^{[1]}_{0,l-1} \\ \alpha^{[2]}_{1,l-1} & h\beta^{[1]}_{1,0} & \cdots & h\beta^{[1]}_{1,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{[2]}_{l,l-1} & h\beta^{[1]}_{l,0} & \cdots & h\beta^{[1]}_{l,l-1} \end{pmatrix}$$

They are of limited stability region and this region deteriorates in size as k increases.

3.2 Adams-Moulton LMM

This is the implicit methods (3.1) given by setting

$$s=1, m_1=l, k=1, x^{[1]}_{n+j}=x_{n+j};\ j=0(1)l-1 \qquad (3.3)$$

$$F=\begin{pmatrix} 1 & x_{n+l-1} & x^2_{n+l-1} & x^3_{n+l-1} & \cdots & x^{l+1}_{n+l-1} \\ 0 & 1 & 2x_n & 3x_n & \cdots & (l+1)x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{n+l-1} & 3x_{n+l-1} & \cdots & (l-1)x_{n+l-1} \end{pmatrix}, H=\begin{pmatrix} \alpha^{[2]}_{0,l-1} & h\beta^{[1]}_{0,0} & \cdots & h\beta^{[1]}_{0,l-1} \\ \alpha^{[2]}_{1,l-1} & h\beta^{[1]}_{1,0} & \cdots & h\beta^{[1]}_{1,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{[2]}_{l,l-1} & h\beta^{[1]}_{l,0} & \cdots & h\beta^{[1]}_{l,l-1} \end{pmatrix}$$

The stability region of these methods improves that of the above, but again the size of this shrinks as k increases indefinitely.

3.3 The LMM of Fatunla [15]

This is the implicit method (3.1) given by setting

$$s=1, m_1=l+1, k=1, x_{n+j}^{[1]}=x_{n+j};\ j=0(1)l \tag{3.4}$$

$$F=\begin{pmatrix} 1 & x_{n+l-3} & x_{n+l-3}^2 & x_{n+l-3}^3 & \cdots & x_{n+l-3}^{l+1} \\ 0 & 1 & 2x_n & 3x_n & \cdots & (l+1)x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{n+l-1} & 3x_{n+l-1} & \cdots & (l-1)x_{n+l-1} \end{pmatrix}, H=\begin{pmatrix} \alpha_{0,l-1}^{[2]} & h\beta_{0,0}^{[1]} & \cdots & h\beta_{0,1}^{[1]} \\ \alpha_{1,l-1}^{[2]} & h\beta_{1,0}^{[1]} & \cdots & h\beta_{1,1}^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{l,l-1}^{[2]} & h\beta_{l,0}^{[1]} & \cdots & h\beta_{l,+l,1}^{[1]} \end{pmatrix}$$

The explicit case of the method (3.1) is given by setting

$$s=1, m_1=l, k=1, x_{n+j}^{[1]}=x_{n+j};\ j=0(1)l-1 \tag{3.5}$$

$$F=\begin{pmatrix} 1 & x_{n+l-3} & x_{n+l-3}^2 & x_{n+l-3}^3 & \cdots & x_{n+l-3}^{l+1} \\ 0 & 1 & 2x_n & 3x_n & \cdots & (l+1)x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{n+l-1} & 3x_{n+l-1} & \cdots & (l-1)x_{n+l-1} \end{pmatrix}, H=\begin{pmatrix} \alpha_{0,l-1}^{[2]} & h\beta_{0,0}^{[1]} & \cdots & h\beta_{0,1-1}^{[1]} \\ \alpha_{1,l-1}^{[2]} & h\beta_{1,0}^{[1]} & \cdots & h\beta_{1,l-1}^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{l,l-1}^{[2]} & h\beta_{l,0}^{[1]} & \cdots & h\beta_{l,+l,1}^{[1]} \end{pmatrix}.$$

These methods are weakly stable and again the region of stability shrinks as k increases.

3.4 Backward Differentiation LMM

This is the implicit methods (3.1) given by setting

$$s=1, m_1=l, k=1, x_{n+j}^{[1]}=x_{n+l} \tag{3.6}$$

$$F=\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^{l+1} \\ 0 & 1 & 2x_n & 3x_n & \cdots & (l+1)x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{n+l-1} & 3x_{n+l-1} & \cdots & (l-1)x_{n+l-1} \end{pmatrix}, H=\begin{pmatrix} \alpha_{0,0}^{[1]} & h\beta_{0,l-1}^{[1]} & \cdots & h\beta_{0,l-1}^{[1]} \\ \alpha_{1,0}^{[1]} & h\beta_{1,l-1}^{[1]} & \cdots & h\beta_{1,l}^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{l,l-1}^{[1]} & h\beta_{l,l-1}^{[1]} & \cdots & h\beta_{l,l}^{[1]} \end{pmatrix}$$

This class of methods is of better stability characteristics compared to those above. In fact, they are stiffly-stable for k = 1,2,3,4,5,6, and instability sets in when $k \geq 7$. They have been implemented in the MATLAB software package with Adam-Bashforth starters to resolve the inherent implicitness in the methods. A direct extension of this is the one- leg LMM.

3.5 The One-Leg LMM

$$Y(x)=\sum_{j=o}^{k-1} \alpha_j(x)Y_{n+j}+hf\left(\sum_{j=o}^{m-1}\beta_j(x)x_{n+j}^{[1]}, \sum_{j=o}^{m-1}\beta_j(x)Y_{n+j}^{[1]}\right) \tag{3.7}$$

It is stiffly-stable for $m=1, k=l, x_n^{[1]}=x_{n+l}, l=1(1)5$ and unstable for $l \geq 6$. The $\boldsymbol{H}$ and F are as in the above. It is instructive to see Onumanyi et al [33] and Ikhile and Okuonghae [29] and Otunta at el [28] and Okuonghae [32] for details.

3.6 Multi-Derivative Methods: Second Derivative LMM

The stiffly stable implicit LMM (3.1) of Enright [14] is recovered completely by setting.

$$s=2, m_1=l+1, m_2=l, k=1,\ x_{n+j}^{[1]}=x_{n+l},\ j=0(1)l;\ x_n^{[2]}=x_{n+j} \tag{3.8}$$

$$F=\begin{pmatrix} 1 & x_{n+l-1} & x_{n+l-1}^2 & x_{n+l-1}^3 & \cdots & x_{n+l-1}^{l+2} \\ 0 & 1 & 2x_n & 3x_n^2 & \cdots & (l+2)x_n^{l+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{n+l-1} & 3x^2_{{}_{n+l}} & \cdots & (l-2)(l+1)x_{n+l}^{/+1} \\ 0 & 0 & 2 & 6x_{n+l} & \cdots & (1+2)(1+2)x_{n+/}^{/} \end{pmatrix},$$

$$H=\begin{pmatrix} \alpha_{0,l-1}^{[2]} & h\beta_{0,0}^{[1]} & \cdots & h\beta_{0,l-1}^{[1]} & h\beta_{0,l}^{[2]} \\ \alpha_{1,l-1}^{[2]} & h\beta_{1,0}^{[1]} & \cdots & h\beta_{1,l-1}^{[1]} & h\beta_{1,l}^{[2]} \\ \vdots & \vdots & & \vdots & \\ \alpha_{l+2,l-1}^{[2]} & h\beta_{l+2,0}^{[1]} & \cdots & h\beta_{l+2,l-/}^{[1]} & \beta_{l+2,/}^{[2]} \end{pmatrix}$$

This resultant LMM is stiffly stable for $k=1(1)7$ and unstable for $k=8$ and conjectured to be unstable for $k>8$.

3.7 Backward Differentiation Type Second Derivative LMM

This is the stiffly stable second derivative implicit LMM given by setting,

$$s=2, m_1=l+1, m_2=l, k=1, x_n^{[1]}=x_{n+l}=x_n^{[2]}=x_{n+l} \tag{3.9}$$

$$F=\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^{l+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n+l-1} & x_{n+l-1} & x_{n+l-1}^3 & & x_{n+/-1}^{/+1} \\ 0 & 1 & 2x_{n+l} & 3x_{n+l}^2 & \cdots & (l+1)x_{n+l}^{l+1} \\ 0 & 0 & 2 & 6x_{n+l} & \cdots & (l+1)lx_{n+/}^{l-1} \end{pmatrix}, H=\begin{pmatrix} \alpha_{0,0}^{[2]} & \cdots\ h\beta_{0,j-1}^{[2]} & h\beta_{0,j}^{[1]} & h\beta_{0,/}^{[2]} \\ \alpha_{1,0}^{[1]} & \cdots\ \alpha_{1,l-1}^{[1]} & h\beta_{1,l}^{[1]} & h\beta_{1,l}^{[2]} \\ \vdots & \vdots & \vdots & \\ \alpha_{l+1,0}^{[2]} & \cdots\ \alpha_{l+1,l-1}^{[2]} & h\beta_{l+1,/}^{[1]} & \beta_{l+1,/}^{[2]} \end{pmatrix}.$$

This resultant LMM are stiffly stable for $k=1(1)10$ and unstable for $k=11$ and to be unstable for $k>11$. Further new LMM, hybrid LMM and other interesting useful methods are derived by other choices of s, ***H, F***, but the interest is on the A-stable and stiffly –stable ones, these are the desirable methods for stiff initial value problems of (1.1). One is pleased to remark that there have been success in the search of such methods in Ikhile and Okuonghae [29], Otunta at el Okuonghae [23].

4.0 Directions for investigation for new continuous methods

This section highlights and points to directions of investigation that may lead to more useful methods with stronger stability characteristics.

4.1 The multi-derivative and hybrid LMM

Hybrid methods are known for attainable higher order and smaller error constants than the conventional LMM and more fundamentally, they provide a means of bypassing the Dahlquist order barrier theorem. Hybrid methods are given from the foregoing as

$$Y(x)=\sum_{j=0}^{k-1}\alpha_j^{[s]}(x)Y_{n+j}+\sum_{j=0}^{k-1}\alpha_j^{*[s]}(x)Y_{n+v_j} \tag{4.1}$$

$$+\sum_{r=1}^{s}h^r\left(\sum_{j=0}^{m_r-1}\beta_j^{[r]}(x)f^{(r-1)}\left(x_{n+j}^{[r]},Y\left(x_{n+j}^{[r]}\right)\right)+\sum_{j=0}^{m_r-1}\beta_j^{*[r]}(x)f^{(r-1)}\left(x_{n+j}^{[r]},Y\left(x_{n+w_j}^{[r]}\right)\right)\right);$$

where

$$Y_{n+v_q}=\sum_{j=0}^{m_r-1}\alpha_{j1}^{[r]}(x)Y_{n+j}+\sum_{r=1}^{s}h^r\left(\sum_{j=0}^{m_r-1}\beta_{j1}^{[r]}(x)f^{(r-1)}\left(x_{n+j}^{[r]},Y\left(x_{n+j}^{[r]}\right)\right)\right);\ 0<v_q<1, q=0(1)k-1$$

$$Y_{n+w_q} = \sum_{j=0}^{m_r-1} \alpha_{j2}^{[r]}(x) Y_{n+j} + \sum_{r=1}^{s} h^r \left(\sum_{j=0}^{m_r-1} \beta_{j2}^{[r]}(x) f^{(r-1)}\left(x_{n+j}^{[r]}, Y\left(x_{n+j}^{[r]}\right)\right)\right); 0 < v_q < 1, q = 0(1)m_r - 1$$

4.2 The multi-derivative predictor-corrector LMM

Predictor-Corrector methods (PC) are given by

$$Y^{[p]}(x) = \sum_{j=0}^{k_r-1} \alpha_{j1}^{[r]}(x) Y_{n+j} + \sum_{r=1}^{s} h^r \left(\sum_{j=0}^{m_r-2} \beta_{j2}^{[r]}(x) f^{(r-1)}\left(x_{n+j}^{[r]}, Y\left(x_{n+j}^{[r]}\right)\right)\right);$$

$$Y^{[c]}(x) = \sum_{j=0}^{k_r-1} \alpha_{j2}^{[r]}(x) Y_{n+j} + \sum_{r=1}^{s} h^r \left(\sum_{j=0}^{m_r-2} \beta_{j2}^{[r]}(x) f^{(r-1)}\left(x_{n+j}^{[r]}, Y^{[p]}\left(x_{n+j}^{[r]}\right)\right)\right); \tag{4.2}$$

This composite method is well suitable for non-stiff IVPs (1.1) in ODEs. When the corrector is iterated to convergence, the stability of this composite method is strictly that of the corrector.

4.3 Multi-derivative parallel multi-block GMLM

This class of methods has the general form

$$Y^*(x) = \sum_{j=0}^{m_r-1} A_j^{[r]}(x) Y_{n+j}^* + \sum_{r=1}^{s} h^r \left(\sum_{j=0}^{m_r-2} \beta_{j2}^{[r]}(x) f^{(r-1)}\left(Y * \left(x_{n+j}^{[r]}\right)\right)\right) \tag{4.3}$$

where

$$A_j^{[s]}(x) = \left[\alpha_{j.g.h}^{[s]}(x)\right]_{g,h=1(1)l} \qquad B_j^{[s]}(x) = \left[\beta_{j.g.h}^{[s]}(x)\right]_{g,h=1(1)l}$$

$$Y*(x) = \left(Y_{n+1}, \ldots, Y_{n+1}, Y(x)\right)^T; \quad Y_{n-j+l}^* = \left(Y_{n-j}, \ldots, Y_{n-l+l-1}\right)^T; \tag{4.4}$$

$$F^{(r-1)}\left(Y^*\left(x_{n-j+l}\right)\right) = \left(f^{(r-1)}\left(x_{n-j+1}^{[r]}, Y\left(x_{n-j+1}^{[r]}\right)\right), \ldots, f^{(r-1)}\left(x_{n-j+l}^{[r]}, Y\left(x_{n-j+l}^{[r]}\right)\right)\right)^T$$

The simplest is one block methods. Parallel block methods offer the potential of obtaining A-stable, stiffly-stable and L-stable methods. They are capable of implementation on a parallel computer with multi-processor capabilities and thus computational speed up is expected from this family of methods. See Burrage [36].

4.4 Multi-derivative cyclic LMM

This class of methods has the general form

$$Y^*(x) = A_j^{[s]}(x) Y_{2n+j}^* + \sum_{r=1}^{s} h^r \left(\sum_{j=0}^{m_r-1} B_j^{[r]}(x) f^{(r-1)}\left(Y_{2n+j}^{[r]}\right)\right); s \geq 1 \tag{4.5}$$

where

$$A_j^{[s]}(x) = \left[\alpha_{j.g.h}^{[s]}(x)\right]_{g,h=1(1)l} \qquad B_j^{[s]}(x) = \left[\beta_{j.g.h}^{[s]}(x)\right]_{g,h=1(1)l}$$

$$Y*(x) = \left(Y_{2n+1}, \ldots, Y_{2n+1}, Y(x)\right)^T; \quad Y_{n-j+l}^* = \left(Y_{2n-j}, \ldots, Y_{2n-l+l-1}\right)^T; \tag{4.6}$$

$$F^{(r-1)}\left(Y^*\left(x_{n-j+l}\right)\right) = \left(f^{(r-1)}\left(x_{2n-j+1}^{[r]}, Y\left(x_{2n-j+1}^{[r]}\right)\right), \ldots, f^{(r-1)}\left(x_{2n-j+l}^{[r]}, Y\left(x_{2n-j+l}^{[r]}\right)\right)\right)^T$$

Cyclic LMM like Parallel block methods offer the potential of obtaining A-stable, stiffly-stable and L-stable methods. They are capable also of implementation on a parallel computer with multi-processors and thus computational speed up is expected from this family of methods. More so, they offer a means of bypassing the Dahlquist order limitation. It is on its own a useful tool for stabilizing zero-unstable LMM. Parallel Block methods can also be used to achieve this.

4.5 The multi-derivative linear multi-step RKM

This is the multi-derivative multi-step RKM

$$Y^*(x)=\sum_{j=0}^{k-1}\alpha_j^{[s]}(x)Y_{n+j}+\sum_{r=1}^{s}h^r\left(\sum_{j=0}^{m_r-1}\beta_j^{[r]}(x)K_{r,j}^{[n]}\right);s\geq 1 \tag{4.7}$$

with the definition $K_{r,j}^{[n]}=K_r^{[n]}\left(x_{n+j}^{[r]}\right)$ then the stages are given as

$$\begin{aligned}
&K_{1,0}^{[n]}=f\left(x_n^{[1]},Y\left(x_n^{[1]}\right)\right)\\
&K_{21,0}^{[n]}=f\left(x_n^{[2]},Y\left(x_n^{[2]}\right)\right)\\
&K_{v,j}^{[n]}=f^{(v-1)}\left(x_{n+j}^{[v]},\sum_{i=0}^{k-1}\alpha_i^{[s]}(x)Y_{n+i}+\sum_{r=1}^{s}h^r\left(\sum_{j=0}^{m_r-1}\beta_i^{[r]}(x)f^{(v-1)}\left(x_{n+j}^{[r]},K_{r,i}^{[n]}\right)\right)\right);\\
&\qquad j=1(1)m_r-1;v=1(1)s
\end{aligned} \tag{4.8}$$

the $x_{n+i}^{[r]}$ are the collocation points. They have the potential of L-stability.

5.0 Extension to second order ODEs:

Extension of the foregoing methods to the numerical solution of second order IVPs

$$y''=f(x,y),\ \ y(a)=y_0,y'(a)=y_{00} \tag{5.1}$$

in ODEs is possible. In fact, a class of continuous LMM for second order ODEs is given as

$$Y(x)=\sum_{j=0}^{k-1}\alpha_j(x)Y_{n+j}+h^2\sum_{j=0}^{m-1}\beta_j(x)f\left(x_{n+j}^{[1]},Y\left(x_{n+j}^{[1]}\right)\right)$$

5.1 Symmetric LMM

P-stable LMM are symmetric, that is $\alpha_j=\alpha_{k-j};\beta_j=\beta_{k-j}$, which are recommended methods for oscillatory second order ODEs (5.1) with an initial value. Symmetric continuous LMM requires that $k-l$, and $m-l$ are even integers. Symmetric implicit LMM are given by setting

$$m=l+1,\quad k=l,x_{n+j}^{[1]},\ j=0(1)l \tag{5.2}$$

$$F=\begin{pmatrix}1 & x_n & x_n^2 & x_n^3 & \ldots & x_n^{2l}\\ \vdots & \vdots & \vdots & \vdots & & \vdots\\ 1 & x_{n+l-1} & x_{n+l-1}^2 & x_{n+l-1}^3 & & x_{n+l-1}^{2l}\\ 0 & 0 & 2 & 6x_{n+l}^2 & \ldots & 2l(2l-1)x_n^{2l-2}\\ 0 & 0 & \vdots & \vdots & \ddots & \vdots\\ & & 2 & 6x_{n+l} & \ldots & 2l(2l-1)x_{n+l}^{2l-2}\end{pmatrix},H=\begin{pmatrix}\alpha_{0,0} & \ldots\ \alpha_{0,l-1} & h^2\beta_{0,0} & h^2\beta_{0,1}\\ \alpha_{1,0} & \ldots\ \alpha_{1,l-1} & h^2\beta_{1,0} & h^2\beta_{1,l}\\ \vdots & \vdots & & \vdots\\ \alpha_{l+1,0} & \alpha_{l+1,l-1} & h^2\beta_{l+1,0} & h^2\beta_{l+1,l}\end{pmatrix}\text{P}$$

-stable methods are limited to $k=2,$ a particular example is the P-stable method

$$y_{n+2}-2y_{n+1}+y_n=h^2\left(\frac{1}{4}f_{n+2}+\frac{1}{2}f_{n+1}+\frac{1}{4}f_n\right) \tag{5.3}$$

with $k=2,$ of order $p=2$ for oscillatory problem of (5.1), see Fatunla [15]. Higher orders P-stable LMM are obtained by hybrid of methods. Further extension is to the RKNM.

5.2 The Stormer-Cowell LMM

The well known Stormer-Cowell class of LMM is

$$m=l+1,\quad k=2,x_{n+j}^{[1]},\ j=0(1)l \tag{5.4}$$

$$F=\begin{pmatrix} 1 & x_n & x_{n+l-1}^2 & x_{n+l-1}^3 & \cdots & x_{n+l-1}^{2l} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{n+l-2} & x_{n+l-2}^2 & x_{n+l-2}^3 & & x_{n+l-2}^{2l} \\ 0 & 0 & 2 & 6x_{n+l}^2 & \cdots & 2l(2l-1)x_n^{2l-2} \\ 0 & 0 & \vdots & \vdots & \ddots & \vdots \\ & & 2 & 6x_{n+l} & \cdots & 2l(2l-1)x_{n+l}^{2l-2} \end{pmatrix}, H=\begin{pmatrix} \alpha_{0,l,2} & \cdots \alpha_{0,l-1} & h^2\beta_{0,0} & h^2\beta_{0,1} \\ \alpha_{1,2} & \cdots \alpha_{1,l-1} & h^2\beta_{1,0} & h^2\beta_{1,l} \\ \vdots & \vdots & & \vdots \\ \alpha_{l+1,2} & \alpha_{l+1,l-1} & h^2\beta_{l+1,0} & h^2\beta_{l+1,l} \end{pmatrix}^T$$

hey however, suffer from orbital instability; see Hairer et al [31]

5.3 Parallel multi-block methods

Block method for (5.1) is,

$$Y^*(x)=\sum_{j=0}^{k-1}A_j^{[2]}(x)y_{n+j}^*+h^2\sum_{j=0}^{m-1}\beta_j^{[2]}(x)F^{(r-1)}\left(Y^*\left(x_{n+j}^{[2]}\right)\right)$$

The blocks $Y_{n+j,}^*F^{(r-1)}\left(Y^*\left(x_{n+j}^{[2]}\right)\right)$ are defined as accordingly; see (4.3) and Burrage [36].

5.4 Extension to linear multi-step RKNM

This is simply,

$$\begin{aligned} Y(x)&=\sum_{j=0}^{k-1}\alpha_j^{[2]}(x)Y_{n+j}+h^2\sum_{j=0}^{m-1}\beta_j^{[2]}(x)K_{r,j}^{[n]} \\ Y'(x)&=\sum_{j=0}^{k-1}\alpha_j^{[2]}(x)Y'_{n+j}+h^2\sum_{j=0}^{m-1}\beta_j^{[2]}(x)K_{r,j}^{[n]} \end{aligned} \qquad (5.5)$$

in the sense of the GLM with the definition $K_{r,j}^{[n]}=K_r^{[n]}\left(x_{n+j}^{[r]}\right)$ then the stages are given as

$$K_{1,0}^{[n]}=f\left(x_n^{[1]},Y\left(x_n^{[1]}\right)\right)$$

$$K_{v,j}^{[n]}=f\left(x_{n+j}^{[2]},\sum_{j=0}^{k-1}\alpha_j^{[2]}(x)Y_{n+j}+h^2\sum_{j=0}^{m-1}\beta_j^{[2]}(x)K_{r,j}^{[n]}\right);\quad j=1(1)m-1 \qquad (5.5a)$$

The $x_{n+i}^{[r]}$ are the collocation points. They have potential of P-stability needed to integrate highly oscillatory IVPs in ODEs in (5.1). This class has an easy extension to the more general problem

$$y''=f(x,y,y'),\ \ y(a)=y_0,\ y'(a)=y_{00} \qquad (5.6)$$

A continuous RKNM for this is

$$\begin{aligned} Y(x)&=\sum_{j=0}^{k-1}\alpha_j^{[2]}(x)Y_{n+j}+h\sum_{j=0}^{m-1}\beta_{j1}^{[2]}(x)K_{r,j}^{[n]}+h^2\sum_{j=0}^{m-1}\beta_{j2}^{[2]}(x)K_{r,j}^{[n]} \\ Y'(x)&=\sum_{j=0}^{k-1}\alpha_j^{[2]}(x)Y'_{n+j}+h\sum_{j=0}^{m-1}\beta_{j3}^{[2]}(x)K_{r,j}^{[n]} \end{aligned} \qquad (5.7)$$

with

$$K_{1,0}^{[n]}=f\left(x_n^{[1]},Y\left(x_n^{[1]}\right),Y'\left(x_n^{[1]}\right)\right);\ \ j=1(1)m-1 \qquad (5.8)$$

$$K_{v,j}^{[n]}=f\left(x_{n+j}^{[2]},\sum_{j=0}^{k-1}\alpha_j^{[2]}(x)Y_{n+j}+h\sum_{j=0}^{m-1}\beta_j^{*[2]}(x)K_{r,j}^{[n]}+h^2\sum_{j=0}^{m-1}\beta_j^{*[2]}(x)K_{r,j}^{[n]},\sum_{j=0}^{k-1}\alpha_j^{[2]}(x)Y'_{n+j}+h\sum_{j=0}^{m-1}\beta_j^{*[2]}(x)K_{r,j}^{[n]}\right)$$

6.0 The numerical applications, discussion and conclusion.

Consider the class of backward differentiation type (BDT) second derivative continues linear multi-step methods (CLMM), (3.9);

$$y_{n+k}\sum_{j=0}^{k-1}\alpha_j(t)y_{n+j} = h\lambda_{1,k}(t)f_{n+k} + h^2\lambda_{2,k}(t)f_{n+k}^{/}, \alpha_k \neq 0 \tag{6.1}$$

of section (3), a subject of investigation in Okuonghae [32], where now $y^{/}(x_{n+j}) = f_{n+j}$, $h = x_{n+1} - x_n$ $t = (x - x_{n+1})/h$, $\alpha_k(t) = 1$, $\{\alpha_k(t)\}_{j=0}^{k-1}$, $\lambda_{1,k}(t)$ and $\lambda_{2,k}$ are the coefficients so that the numerical methods in (6.1) are exact when $y(x)$ is an arbitrary polynomial of degree $k+1$. The order of this CLMM is $k+l$ and the stability polynomial is

$$\prod(R|z,t) = R^k + \sum_{j=0}^{k-1}\alpha_j(t)R^j - \left(z\lambda_{1k}(t)R^k + z^2\lambda_{2,k}(t)R^k\right),$$

The method in (6.1) is stable for $k \leq 10$ and instability sets in when $k \geq 11$. A practical problem of interest is a stiff nonlinear chemical kinetics problem: Higham et al [20, p.158]

$$y_1^{/} = -0.04y_1 + 10^4 y_2 y_3$$

$$y_2^{/} = 400y_1 + 10^4 y_2 y_3 - 10^7 y_2^2,\ y(0) = \begin{pmatrix}1\\0\\0\end{pmatrix} \tag{6.2}$$

$$y_3^{/} = 3\times 10^7 y_2^2,$$

with *x* being the rang $x \in 0(0.001)3$. For $k=3$ (6.1) is the second derivative CLMM

$$y_{n+3} - \left(-\frac{22t}{85} + \frac{39t^2}{85} - \frac{41t^3}{170} + \frac{7t^4}{170}\right)y_n - \left(1 - \frac{64t}{85} + \frac{72t^2}{85} - \frac{64t^3}{85} + \frac{13t^4}{85}\right)y_{n+1}$$

$$-\left(\frac{86t}{85} + \frac{33t^2}{85} - \frac{87t^3}{170} + \frac{19t^4}{85}\right)y_{n+3} = h\left(-\frac{23t}{85} + \frac{6t^2}{85} - \frac{23t^3}{85} + \frac{6t^4}{85}\right)f_{n+3}$$

$$= h^2{}_n\left(\frac{14t}{85} - \frac{11t^2}{170} - \frac{14t^3}{85} + \frac{11t^4}{170}\right)f_{n+3}^{/} \tag{6.3}$$

this case when $k=3$ gives this method of order four and zero = stable for all values of t in $\Omega = \{t : t \in (-\infty - 1.416) \cup (0.4, \infty)\}$. Setting *t = 2* in (6.3) gives the equivalent discrete form of the SDBCLMM (6.1) to be.

$$y_{n+3} - \frac{4}{85}y_n + \frac{27}{85}y_{n+1} - \frac{108}{85}y_{n+2} = h\frac{66}{85}f_{n+3} - h^2\frac{18}{85}f_{n+3}^{/}, p = 4: \qquad \text{Bdtsdclmm} \tag{6.4}$$

The root locus plot of the stability polynomial of this is in figugre 6.1, showing that it is stiffly stable. The numerical results from the method in (6.4) is compared with that from Enright [14, *k*=3] of the same order and of the state-of-the art MATLAB Ode 15s code. The result is shown for the solution component $y_2(x)$ tend to zero as *x* increase its magnitude, the graphs of the numerical solution of this component from the second derivative methods in (6.4) coincide with the method of Enright [14] for $k=3$ and shows that the methods out perform the state-of-the-art MATLAB ode 15s code on this stiff problem. Furthermore, the numerical applications of these methods to problems of practical applications have

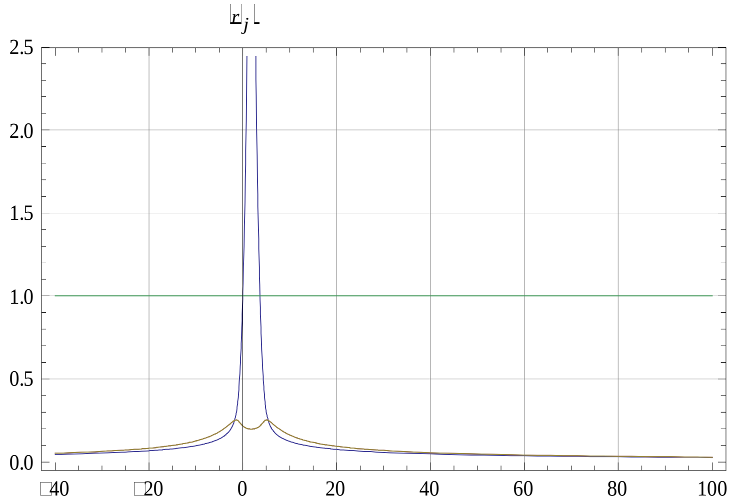

Figure 6.1: Root locus plot for k = 3.

already be presented in Ikhile and Okuonghae [29], Otunta et al [28] and more results are to be found in Okuonghae [32].

Conclusively, the search for higher accuracy, reliable and efficient numerical methods for IVPs in ODEs necessitates the need to incorporate more analytic information of derivatives of the underlining ODEs into the design of methods. This presentation highlights a strategy of interpolation and collocation

for deriving multi-derivative GMLM for the numerical solution of ordinary deferential equations. The proposed approach uses continuous polynomial interpolation and collocation of the solution of the initial value problem. The purpose is to show that this can be made into a veritable tool in the design of efficient, high order and highly stable GLM compared to the means of Taylor series expansion, integration, and differentiation method amongst others of deriving computational methods. More methods are easily obtained from a perturbation of the continuous interpolants. The areas of extensions considered are by no means exhaustive and there is therefore no limitation to other directions of further extensions.

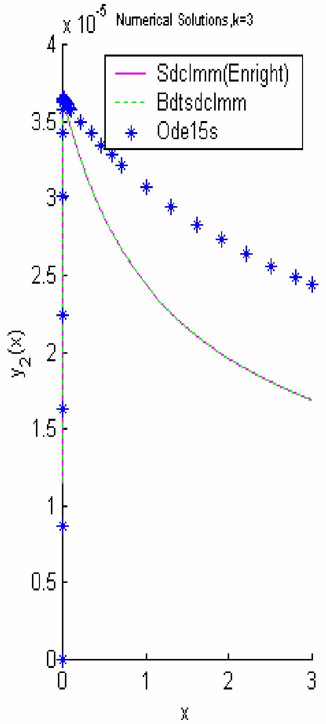

Figure 6.2: The plot of numerical solution from (6.4) of the component $y_2(x)$ of the problem for k = 3.

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