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Application of finite element-eigenvalue method in the determination of transient temperature field in functionally graded materials

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Abstract

In this work, a finite element/eigen value method is formulated to solve the finite element models of time-dependent temperature field problems in non-homogenous materials such as functionally graded materials. The method formulates an eigen value problem from the original finite element model and proceeds to calculate the eigenvectors associated with the various eigen values from which the solution can be obtained thereby avoiding the use of time discretization that require lengthy calculations. The results obtained are exponential functions of time which when compared with the exact solution eventually tended to the steady state solution.

Keywords: Functionally Graded Materials, Finite Element Method, Transient Temperature Field,

1.0 Introduction

Recent experimental advances in material science have shown that a large percentage of materials in service are composites especially in high temperature and high heating rate service environments, hence the necessity to study the thermal behavior of composites in the form of functionally graded materials.

A functionally graded material is basically a combination of two material phases that has a gradual transition from one material at one surface to another material at the opposite surface. Functionally graded materials vary from homogenous ones in that their properties vary spatially which makes the thermal analysis of functionally graded materials considerably more complex than in the corresponding homogenous case. However since most of the functionally graded materials show a one dimensional heat non-homogeneity the temperature field can easily be obtained by the composite laminated plate model for a simple one dimensional heat conduction as has been done by various experts and design analysts.

Wang and Tian [1] gave a plausible solution to transient temperature field problems by means of the finite element/finite difference method. Finite element analysis is well treated in many standard texts see [2-6].

In this paper, the finite element method in addition to an eigenvalue method is employed to solve the system of time dependent equations that describe the transient temperature distribution in a functionally graded material, initially addressed by Wang and Tian, see [1] and the references therein

In the finite element-eigenvalue method, we develop the governing equation which is parabolic in nature; the variational form of the equation is obtained over each element followed by spatial approximation of the position-dependent variables of the problem, the end result is a set of ordinary differential equations in time (semi-discrete model). Finite element algebraic equations are then developed from the semi discrete finite element model by formulating the corresponding eigenvalue problem from which the eigenvalues and eigenvectors are obtained. The solution can now be obtained by applying fundamental eigenvalue/eigenvector solving scheme in mathematics.

The solutions obtained by the finite element/eigenvalue method are as accurate as those given by the finite element/finite difference method. The advantage of the finite element/eigenvalue method is that very few computations are needed to arrive at the steady state solution with high accuracy while the finite element/finite difference method requires a large number of computations due to time discretization in order to achieve the appropriate accuracy which limits the use of the method to the availability of a computational software. Thus this work produces highly accurate solutions with fewer computations and less time.

2.0 Formulation of the governing equations

Assuming the solid under investigation occupies a space within a coordinate system *x* which is surrounded by space *S*, Fourier's Law of heat conduction states that the heat flux vector of the solid is given as

$$q = -KA\frac{\partial T}{\partial x} \tag{2.1}$$

where $K = K_{ij}$ = thermal conductivity tensor. For an anisotropic solid

$$K_{\mathrm{ij}} = K_{\mathrm{ji}} \tag{2.2}$$

By the balance (conservation of energy) within the solid.

Rate of heat energy added to the elements = Rate of heat energy lost from the element

That is

Energy into element + energy generated within the element = change in internal energy + energy out of the element

$$-KA\frac{\partial T}{\partial x} + QAdx = \rho cA\frac{\partial T}{\partial x}dx - \left[KA\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(KA\frac{\partial T}{\partial x}\right)dx\right] \text{ or } QAdx = \rho cA\frac{\partial T}{\partial x}dx + \frac{\partial}{\partial x}\left(KA\frac{\partial T}{\partial x}\right)dx$$

i.e.

$$\frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) + Q = \rho cA\frac{\partial T}{\partial x} \tag{2.3}$$

This is the governing equation where material properties ρ, c and K are considered to be complex function of spatial coordinate, and the temperature is a function of spatial coordinate and time.

i.e.

$$T = T(x,t) \tag{2.4}$$

This equation can be solved for prescribed boundary and initial conditions as follows

Initial condition:

This specifies the temperature distribution at time zero

$$T(x,0) = T_0(x,0) \tag{2.5}$$

Boundary condition:

Essential boundary condition: specify temperature *T* at boundaries

Natural boundary conditions: specify heat flux *Q*

3.0 Finite element formulation.

In order to obtain the variational form of the heat equation assume that the medium undergoes a virtual temperature change δT i.e. δT now serves as the weight function in the variational form

$$\int_{\Omega}\left[-\frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) + \rho c\frac{\partial T}{\partial t} - Q\right]\delta T dv = 0 \tag{3.1}$$

The above integral yields

$$\int_{\Omega}\rho c\frac{\partial T}{\partial t}\sigma T dv+\int_{\Omega}\frac{\partial \sigma T}{\partial x}K\frac{\partial T}{\partial x}dv-\int_{\Omega}K\frac{\partial T}{\partial x}\delta T dv-\int_{\Omega}Q\delta T dv=0 \tag{3.2}$$

Let
$$\int_{s}K\frac{\partial T}{\partial x}\delta T dv+\int_{\Omega}Q\delta T dv=\hat{Q} \tag{3.3}$$

The model now takes the form
$$\int_{s}\rho c\frac{\partial T}{\partial t}\delta T dv+\int_{\Omega}\frac{\partial \delta T}{\partial t}K\frac{\partial t}{\partial x}dv-\hat{Q}=0 \tag{3.4}$$

Let the space around the solid be divided into a finite number of elements interconnected at the nodes of the elements. The temperature must then be expressed in terms of the values at the node thus

$$T(x,t)=[N_{\theta}]\{T\} \tag{3.5}$$

where $[N_{\theta}]$ is the shape function matrix which is a complex function of *x*. The temperature gradient at any point within the region Ω is given as

$$[\partial T]=\left[\partial T\big/\partial x_{j}\right]^{T}=[B_{\theta}]\{T\} \tag{3.6}$$

where
$$[B_{\theta}]=[1][N_{\theta}] \tag{3.7}$$

where $[1]$ is a differential operator matrix. Introducing this into the finite element model above yields:

$$[M]\{\dot{T}\}+[K]\{T\}=\{\hat{Q}\} \tag{3.8}$$

where
$$[M]=\int_{\Omega}\rho c[N_{\theta}]^{T}[N_{\theta}]d\Omega \tag{3.9}$$

$$[K]=\int_{\Omega}[B_{\theta}]^{T}[K][B_{\theta}]d\Omega \tag{3.10}$$

$[K]=$ Matrix of thermal conductivities of the medium. Thus the finite element model generates a system of parabolic differential equations in time.

The integrals involved in the determination of [M] and [K] can be evaluated using the numerical integration scheme such as Gauss-Legendre integration. Therefore the problem has been reduced from a partial differential equation in to variable x and T to a matrix of ordinary differential equations which we now solve using the eigenvalue-eigenvector solution method.

4.0 Eigenvalue problem formulation

This method involves the decomposition of the solution of the model $[M]\{\dot{T}\}+[K]\{T\}=\{\hat{Q}\}$ into

$$\{T\}=\{T\}_{h}+\{T\}_{p} \tag{4.1}$$

where $\{T\}_{h}$ is the homogeneous solution satisfying

$$[M]\{\dot{T}\}_{h}+[K]\{T\}_{h}=\{0\} \tag{4.2}$$

And $\{T\}_{p}$ is the particular solution satisfying

$$[M]\left\{\dot{T}\right\}_p + [K]\{T\}_p = \left\{\hat{Q}\right\} \tag{4.3}$$

4.1 Homogenous solution

Let matrices [M] and [K] be matrices of constants such that $[M]\left\{\dot{T}\right\}_h + [K]\{T\}_h = 0$ is a set of linear constants –coefficient ordinary differential equations. To solve this, we assume

$$[T]_h = V\exp(-\lambda t) \tag{4.4}$$

Substituting this in equation 4.4 yield $\quad (K-\lambda M)v\exp(-\lambda t) = 0 \quad (4.5)$

from which $\quad (K-\lambda M)v = 0 \quad (4.6)$

or $\quad \det(K-\lambda M)v = 0 \quad (4.7)$

From which the eigenvalues are obtained. Back substitution of the eigenvalues into equation 4.1 results in the eigenvectors $V_1^T, V_2^T, \Lambda\ , V_j^T$ from which the homogenous solution can be written as

$$[T]_h = c_1 v_1 \exp(-\lambda_1 t) + c_2 v_2 \exp(-\lambda_1 t) + \Lambda\ + c_j v_j \exp(-\lambda_1 t) \tag{4.8}$$

4.2 **Particular solution**

$$[M]\left\{\dot{T}\right\}_p + [K]\{T\}_p = \left\{\hat{Q}\right\} \tag{4.9}$$

In which case the vector $\{Q\}$ is made to cater for the initial conditions of the system. We assume that the particular solution is a constant, i.e.

$$\{T\}_p = \delta \tag{4.10}$$

It therefore follows that $\quad \left\{\dot{T}\right\}_p = 0 \quad (4.11)$

and $\quad [K]\{T\}_p = \left\{\hat{Q}\right\} \quad (4.12)$

or $\quad \{T\}_p = [K]^{-1}\left\{\hat{Q}\right\} \quad (4.13)$

The general solution can now be written as

$$[T]_h = c_1 v_1 \exp(-\lambda_1 t) + c_2 v_2 \exp(-\lambda_1 t) + \Lambda\ + c_j v_j \exp(-\lambda_1 t) + \{T\}_p \tag{4.14}$$

The constants $c_1, c_2, \Lambda\ .c_j^T$ can be determined by considering the initial condition of the system which leads to a set of linear algebraic equations in matrix form which can be solved using simple matrix operations.

5.0 Approximation of position–dependent material properties

Due to the non homogeneity of the material properties inside the functionally graded material, the material properties p,c, and [*K*] are complex functions of the spatial coordinates *x* this results in some difficulties in developing the integrals in equations (3.9) and (3.10). To resolve this, the material properties are specified at the nodes of the elements so that the properties can be approximated as

$$\rho(x) = [N_\theta]\{\rho\} \tag{5.1}$$

$$c(x) = [N_\theta]\{c\} \tag{5.2}$$

$$\mathrm{k}(\mathrm{x}) = [\mathrm{N}_\theta]\{\mathrm{k}\} \tag{5.3}$$

And so that substituting these approximation into equations (3.3) and (3.4)

$$[M] = \int_\Omega [N_\theta]\{\rho\}.[N_\theta]\{c\}.[N_\theta]^T [N_\theta] d\Omega \tag{5.4}$$

$$[K] = \int_\Omega [B_\theta][N_\theta]\{K\}[B_\theta] d\Omega \tag{5.5}$$

These integrals can then be solved with relative ease by the use of numerical integration schemes such as Gauss-Legendre integration scheme using 3 Legendre points.

6.0 Numerical examples.

The following examples are used to illustrate the method presented in this work:

6.1. *Example* 6.1

One-dimensional heat conduction in a functionally-graded material strip.

Determine the temperature history of a one dimensional functionally graded material strip of length L at points 0.25L, 0.5L and 0.75L. The functionally graded material strip is made of PSZ/Ti-6AL-4V composition system; their properties are:

$$K_{Ti} = 18.1 w/mk,\ \rho_{Ti} = 4420 kg/m^3,\ c_{Ti} = 808.3 J/kgK$$

$$K_p = 2.036 W/mk,\ \rho_p = 5600 kg/m^3,\ c_p = 615.6 J/kgK$$

The volume fraction of T*i*-6AL-4V in the functionally graded material is varied from 100% on the top surface $(x=0)$ to .0% on the bottom surface $(x=L)$ of the strip i.e. it contains pure PSZ on its bottom surface and pure T*i*-6AL-4V on the top surface. The material properties are expressed as an exponential function of position x as:

$$F_{FGM} = F_{Ti} \exp\left[\beta\left(\frac{x}{L}\right)\right] \text{ where } \beta = In\left(\frac{F_p}{F_{Ti}}\right) \tag{6.1}$$

where *f* is the density, specific heat capacity or coefficient of the thermal conductivity of the components of the functionally graded material.

Consider a case where the temperature at $x = 0$ is suddenly raised to $T_{0,}$ which is maintained thereafter. The temp at $x = L$ is kept at zero. The time interval is taken as [0, 0.5t_0] where

$$t_0 = \frac{\rho_{Ti} C_{Ti} L^2}{K_{Ti}} \tag{6.2}$$

(adopted from [1])

6.1.1 Exact solution

$$\frac{-\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) + \rho c \frac{\partial T}{\partial t} = 0 \tag{6.3}$$

for steady temperature field, $\dfrac{\partial T}{\partial t} = 0$

$$-\frac{d}{dx}\left(K\frac{dy}{dx}\right) = 0 \text{ or } -\frac{dK}{dx}\frac{dT}{dx} - K\frac{d^2K}{dx^2} = 0$$

or $$K\frac{d^2T}{dx^2}+\frac{dK}{dx}\frac{dT}{dx}=0 \tag{6.4}$$

But $K=K_{Ti}\exp\left(\frac{\beta x}{L}\right)$ and $\frac{dK}{dx}=\frac{\beta}{L}K_{Ti}\exp\left(\frac{\beta x}{L}\right)$

$$K_{Ti}\exp\left(\frac{\beta x}{L}\right)\frac{d^2T}{dx^2}+\frac{\beta K_{Ti}}{L}\frac{dT}{dx}=0$$

$$\frac{d^2T}{dx^2}+\frac{\beta}{L}\frac{dT}{dx}=0 \tag{6.5}$$

The characteristic equation is $M^2+\frac{\beta}{L}M=0$

$$M=0 \text{ or } M=-\frac{\beta}{L} \tag{6.6}$$

the general solution becomes

$$T(x)=\rho e^0+Qe^{-\frac{\beta}{L}} \quad \text{i.e. } T(x)=\rho+Qe^{-\frac{\beta}{L}} \tag{6.7}$$

Applying the initial boundary conditions

At $x=L,\ T=0$

$$0=\rho+Qe^{-\beta} \quad \rho=-Qe^{-\beta}$$

At $x=0,\ T=\ T_0,$

$$T_0=\rho+Q \quad or \quad T_0=-Qe^{-\beta}+Q$$

$$T_0=\ Q_0(1-e^{-\beta})$$

or $$Q_0=\frac{T_0}{(1-e^{-\beta})} \quad \text{and} \quad \rho=\frac{-T_0e^{-\beta}}{1-e^{-\beta}} \tag{6.8}$$

The particular solution therefore is

$$T(x)=\frac{-T_0e^{-\beta}}{1-e^{-\beta}}+\frac{T_0e^{-\frac{\beta x}{L}}}{1-e^{-\beta}}=T_0\left[\frac{-e^{-\beta}}{(1-e^{-\beta})}+\frac{e^{-\frac{\beta x}{L}}}{(1-e^{-\beta})}\right]$$

$$T(x)=T_0\left[\frac{1-1-e^{-\beta}+e^{\frac{-\beta x}{L}}}{1-e^{-\beta}}\right]=T_0\left[1+\frac{-1+e^{\frac{-\beta x}{L}}}{1-e^{-\beta}}\right]$$

$$T(x)=T_0\left[1-\frac{1-e^{-\frac{\beta x}{L}}}{(1-e^{-\beta})}\right] \tag{6.9}$$

Therefore $$T(x)=T_0\left[1-\frac{1-\exp\left(-\frac{\beta x}{L}\right)}{1-\exp(-\beta)}\right] \text{ where } \beta=In\left(\frac{K_p}{K_{Ti}}\right) \qquad (6.10)$$

At x = 0.25L; $T(x)=0.908T_0$

At x = 0.5L; $T(x)=0.749T_0$

At x = 0.75L; $T(x)=0.474T_0$

This result is displayed in Table 6.1:

Table 6.1: Exact solution

x/L	0.25	0.5	0.75
$T(x)/T_0$	0.908	0.747	0.475

6.1.2. 5-Node fnite eement-egenvalue slution: (Mesh of Two 1-D Quadratic elements)

The finite element model is

$$[M]\{\dot{T}\}+[K]\{T\}=0 \qquad (6.11)(6.11)$$

where $$[M]=\int_a^{b_1}[N_\theta]\{\rho\}.[N_\theta]\{c\}.[N_\theta]^T[N_\theta]dx \qquad (6.12)$$

and $$[K]=\int_a^{b_1}[B_\theta]^T.[N_\theta]\{K\}[B_\theta]dx \qquad (6.13)$$

$$[N_\theta]=\left[\left(1-\frac{\bar{x}}{h}\right)\left(1-\frac{2\bar{x}}{h}\right)\frac{4\bar{x}}{h}\left(1-\frac{\bar{x}}{h}\right)\frac{-\bar{x}}{h}\left(1-\frac{2\bar{x}}{h}\right)\right] \qquad (6.14)$$

$$[B_\theta]=\left[\left(\frac{-3}{h}+\frac{4\bar{x}}{h^2}\right)\left(\frac{4}{h}-\frac{8\bar{x}}{h^2}\right)\left(\frac{-1}{h}+\frac{4\bar{x}}{h^2}\right)\right] \qquad (6.15)$$

6.1.2.1 Element 1:

In order to obtain [M] and [K] using Gauss-Legendre, 3-point integration scheme, the variable $\bar{x}$ in $[N_\theta]$ and $[\mathrm{B}_\theta]$ is given as follows

$$\bar{x}=\frac{a+b}{2}+\frac{b-a}{2}x=\frac{L/2}{2}+\frac{L/2}{2}x=\frac{L}{4}(x+1) \qquad (6.16)$$

also $$d\bar{x}=\frac{b-a}{2}dx=\frac{L/2}{2}dx=\frac{L}{4}dx \qquad (6.17)$$

Introducing these new variable into $[N_\theta]$ and $[\mathrm{B}_\theta]$ yields :

$$[N_\theta]=\left[x\left(\frac{x-1}{2}\right)(1-x^2)\,x\left(\frac{x+1}{2}\right)\right] \qquad 6.18)$$

and
$$\left[B_\theta\right]=\left[\left(\frac{4x-2}{L}\,\frac{-8x}{L}\,\frac{4x+2}{L}\right)\right] \tag{6.19}$$

This change effectively converts the limit of integration from 0 and L/2 to –1 and 1 respectively
Hence

$$\left[M^1\right]=\int_{-1}^{1}\left[x\left(\frac{x-1}{2}\right)\left(1-x^2\right)x\left(\frac{x+1}{2}\right)\right]\begin{Bmatrix}4420\\4689.361\\4975.138\end{Bmatrix}\bullet$$
$$\left[x\left(\frac{x-1}{2}\right)\left(1-x^2\right)x\left(\frac{x+1}{2}\right)\right]\begin{Bmatrix}808.3\\755.099\\705.4\end{Bmatrix}\bullet \tag{6.20}$$
$$\begin{Bmatrix}x\left(\frac{x-1}{2}\right)\\\left(1-x^2\right)\\x\left(\frac{x+1}{2}\right)\end{Bmatrix}\left[x\left(\frac{x-1}{2}\right)\left(1-x^2\right)x\left(\frac{x+1}{2}\right)\right]\times\frac{L}{4}dx$$

This gives:

$$\left[M^1\right]=L\begin{bmatrix}237649.451 & 119088.37 & -59016.896\\119088.37 & 944252.124 & 116979.215\\-59016.896 & 116979.215 & 237649.451\end{bmatrix} \tag{6.21}$$

also

$$\left[K^1\right]=\int_{-1}^{1}\begin{Bmatrix}\frac{4x-2}{L}\\\frac{-8x}{L}\\\frac{4x+2}{L}\end{Bmatrix}\left[x\left(\frac{x-1}{2}\right)\left(1-x^2\right)x\left(\frac{x+1}{2}\right)\right]\begin{Bmatrix}18.1\\10.482\\6.071\end{Bmatrix}\left[\left(\frac{4x-2}{L}\,\frac{-8x}{L}\,\frac{4x+2}{L}\right)\right]\times\frac{L}{2}dx \tag{6.22}$$

$$\left[K^1\right]=\frac{1}{L}\begin{bmatrix}68.590 & -77.074 & 8.485\\-77.074 & 122.258 & -44.999\\8.485 & -44.999 & 36.512\end{bmatrix} \tag{6.23}$$

6.1.2.2 Element 2

For element 2, upon simplification, we have:

$$\left[M^2\right] = L\begin{bmatrix} 233441.371 & 116981.467 & -57972.573 \\ 116981.467 & 927542.776 & 114908.584 \\ -57972.513 & 114908.584 & 230338.732 \end{bmatrix} \tag{6.24}$$

and

$$\left[K^2\right] = \frac{1}{L}\begin{bmatrix} 23.009 & -25.856 & 2.846 \\ -25.856 & 40.949 & -15.094 \\ 2.846 & -15.094 & 12.248 \end{bmatrix} \tag{6.25}$$

Since the finite element model is $[M]\left\{\overset{\bullet}{T}\right\} + [K]\{T\} = 0$.

Then for a mesh of two quadratic elements

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 & 0 & 0 \\ K_{21}^1 & K_{22}^1 & K_{23}^1 & 0 & 0 \\ K_{31}^1 & K_{32}^1 & K_{33}^1 + K_{11}^2 & K_{12}^2 & K_{13}^2 \\ 0 & 0 & K_{21}^2 & K_{22}^2 & K_{23}^2 \\ 0 & 0 & K_{31}^2 & K_{32}^2 & K_{33}^2 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} + \begin{bmatrix} M_{11}^1 & M_{12}^1 & M_{13}^1 & 0 & 0 \\ M_{21}^1 & M_{22}^1 & M_{23}^1 & 0 & 0 \\ M_{31}^1 & M_{32}^1 & M_{33}^2 M_{11}^2 & M_{12}^2 & M_{13}^2 \\ 0 & 0 & M_{21}^2 & M_{22}^2 & M_{23}^2 \\ 0 & 0 & M_{31}^2 & M_{32}^2 & M_{33}^2 \end{bmatrix}\begin{Bmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \end{Bmatrix} = \begin{Bmatrix} \hat{Q}_1 \\ \hat{Q}_2 \\ \hat{Q}_3 \\ \hat{Q}_4 \\ \hat{Q}_5 \end{Bmatrix} \tag{6.26}$$

That is

$$\frac{1}{L}\begin{bmatrix} 68.59 & -77.076 & 8.485 & 0 & 0 \\ -77.076 & 122.258 & -44.999 & 0 & 0 \\ 8.485 & -44.999 & 59.523 & -25.856 & 2.846 \\ 0 & 0 & -25.856 & 40.949 & -15.094 \\ 0 & 0 & 2.846 & -15.094 & 12.248 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} +$$

(6.27)

$$L\begin{bmatrix} 237649.451 & 119088.370 & -59016.89 & 0 & 0 \\ 119088.370 & 944252.124 & 116979.215 & 0 & 0 \\ -59016.89 & 116979.215 & 467927.09 & 116891.467 & -57972.513 \\ 0 & 0 & 116891.467 & 927542.776 & 114908.584 \\ 0 & 0 & -57972.513 & 114908.584 & 230338.632 \end{bmatrix}\begin{Bmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{Bmatrix}$$

The condensed equations become:

$$\frac{1}{L^2}\begin{bmatrix} 122.258 & -44.999 & 0 \\ -44.999 & 59.523 & -25.856 \\ 0 & -25.856 & 40.949 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix} + \begin{bmatrix} 944252.124 & 116979.215 & 0 \\ 116979.215 & 467927.090 & 116981.467 \\ 0 & 116981.467 & 927542.776 \end{bmatrix}\begin{Bmatrix} \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \end{Bmatrix} =$$

$$\frac{1}{L^2}\begin{Bmatrix} 77.076\,T_0 \\ -8.485\,T_0 \\ 0 \end{Bmatrix} \tag{6.28}$$

since $T_1 = T_0$. The solution to the above the above is divided into the homogenous solution and particular solution

$$T = T_h + T_p \tag{6.29}$$

$T_h(t) = v\exp(-\lambda t)$. This gives $(K - \lambda M)v\exp(-\lambda t) = 0$.

From which $(K - \lambda M)v = 0$ or

$$\det\left|K - \lambda M\right| = 0 \tag{6.30}$$

in order to simplify the computation, let $\bar{\lambda} = L^2\lambda$

$$\begin{vmatrix} 122.258 - 944252.124\bar{\lambda} & -44.999 - 116979.215\bar{\lambda} & 0 \\ -44.999 - 116979.215\bar{\lambda} & 59.523 - 467927.09\bar{\lambda} & -25.856 - 116981\bar{\lambda} \\ 0 & -25.856 - 116981\bar{\lambda} & 40.949 - 927542.776\bar{\lambda} \end{vmatrix} = 0$$

$$133341.030 - 10055243978.353\bar{\lambda} + 136531713496475.661\bar{\lambda}^2 - 384212170660759369\bar{\lambda}^3 = 0 \tag{6.31}$$

Solving the cubic equation yields the eigenvalues as follows:

$$\bar{\lambda}_1 = 2.5974\times10^{-4}, \bar{\lambda}_2 = 1.6990\times10^{-5} \; and \; \bar{\lambda}_3 = 7.8619\times10^{-5}$$

since $\bar{\lambda} = L^2\lambda$, it follows therefore that;

$$\lambda_1 = \frac{2.5974\times10^{-4}}{L^2} \quad \lambda_2 = \frac{1.6990\times10^{-5}}{L^2} \quad \lambda_3 = \frac{7.8619\times10^{-5}}{L^2}$$

These are the eigenvalues associated with the solution from which the eigenvectors are obtained using equation (4.6):

For $\lambda_1 = \dfrac{2.5974\times10^{-4}}{L^2}$

$$\begin{bmatrix} -123.002 & -75.173 & 0 \\ -75.173 & -62.016 & -56.241 \\ 0 & -56.241 & -199.971 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{6.32}$$

$v_1^T = \begin{bmatrix} 1 & -1.636 & 0.467 \end{bmatrix}$.

For $\lambda_2 = \dfrac{1.6990\times10^{-5}}{L^2}$

$$\begin{bmatrix} 106.210 & -46.987 & 0 \\ -46.987 & 57.571 & -27.844 \\ 0 & -27.844 & 25.185 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{6.33}$$

$v_2^T = \begin{bmatrix} 1 & 2.260 & 2.498 \end{bmatrix}$.

For $\lambda_3 = \dfrac{7.8619\times10^{-5}}{L^2}$

$$\begin{bmatrix} 48.022 & -54.196 & 0 \\ -54.196 & 22.735 & -27.844 \\ 0 & -35.053 & 25.185 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{6.34}$$

$v_3^T = \begin{bmatrix} 1 & 0.886 & -0.971 \end{bmatrix}$.

The homogenous solution is written as

$$T_h = c_1 v_1 \exp(-\lambda_1 t) + c_2 v_2 \exp(-\lambda_2 t) + c_3 v_3 \exp(-\lambda_3 t) \tag{6.35}$$

In order to obtain the particular solution, the boundary conditions are now considered

$$\begin{bmatrix} 122.258 & -44.999 & 0 \\ -44.999 & 59.523 & -25.856 \\ 0 & -25.856 & 40.949 \end{bmatrix} T_p + L^2 \begin{bmatrix} 944252.124 & 116979.215 & 0 \\ 116979.215 & 467927.09 & 116981.467 \\ 0 & 116981.467 & 927542.776 \end{bmatrix} \dot{T}_p = \begin{Bmatrix} 77.076T_0 \\ -8.485T_0 \\ 0 \end{Bmatrix} \tag{6.36}$$

Let the particular solution be a constant i.e. $T_p = d$, so that $\dot{T}_p = 0$

It becomes obvious that

$$\begin{bmatrix} 122.258 & -44.999 & 0 \\ -44.999 & 59.523 & -25.856 \\ 0 & -25.856 & 40.949 \end{bmatrix} T_p = \begin{Bmatrix} 77.076T_0 \\ -8.485T_0 \\ 0 \end{Bmatrix}$$

$$T_p = \begin{bmatrix} 122.258 & -44.999 & 0 \\ -44.999 & 59.523 & -25.856 \\ 0 & -25.856 & 40.949 \end{bmatrix}^{-1} \begin{bmatrix} 77.076T_0 \\ -8.485T_0 \\ 0 \end{bmatrix} \tag{6.37}$$

$$T_p = \begin{Bmatrix} 0.905T_0 \\ 0.747T_0 \\ 0.471T_0 \end{Bmatrix}$$

In order to obtain the values of c_1, c_2, c_3 we must satisfy the initial condition $T(0) = 0$. The general solution can be written as

$$T = c_1 v_1 \exp(-\lambda_1 t) + c_2 v_2 \exp(-\lambda_2 t) + c_3 v_3 \exp(-\lambda_3 t) + T_p \tag{6.38}$$

since $T(0) = 0$. Then,

$$\begin{bmatrix} 1 & 1 & 1 \\ -1.636 & 2.26 & 0.886 \\ 0.467 & 2.498 & -0.971 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} 0.905T_0 \\ 0.747T_0 \\ 0.471T_0 \end{Bmatrix} \tag{6.39}$$

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1.636 & 2.26 & 0.886 \\ 0.467 & 2.498 & -0.971 \end{bmatrix}^{-1} \times \begin{Bmatrix} 0.905T_0 \\ 0.747T_0 \\ 0.471T_0 \end{Bmatrix} \tag{6.40}$$

$c_1 = -0.191T_0, c_2 = -0.310T_0 \; and \; c_3 = -0.404T_0$.

The solution can finally be expressed as:

$$\frac{T_2(t)}{T_0} = 0.905 - 0.191\exp(-\lambda_1 t) - 0.310\exp(-\lambda_2 t) - 0.404\exp(-\lambda_3 t)$$
$$\frac{T_3(t)}{T_0} = 0.747 + 0.312\exp(-\lambda_1 t) - 0.701\exp(-\lambda_2 t) - 0.358\exp(-\lambda_3 t) \qquad (6.41)$$
$$\frac{T_4(t)}{T_0} = 0.471 - 0.089\exp(-\lambda_1 t) - 0.774\exp(-\lambda_2 t) + 0.392\exp(-\lambda_3 t)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of the problem

$$\lambda_1 = \frac{2.5974\times10^{-4}}{L^2},\ \lambda_2 = \frac{1.6990\times10^{-5}}{L^2} \text{ and } \lambda_3 = \frac{7.8619\times10^{-5}}{L^2}$$

will be seen from the solution that as $t \to \infty$

$$T_2(t) = 0.905T_0,\ T_3(t) = 0.747T_0,\ T_4(t) = 0.471T_0$$

The numerical values of this result are as displayed below in Table 6.2. The variation of temperature with time at the positions of interest (x = 0.25L, x = 0.5L and x = 0.75L) along the functionally graded material strip is shown in Figures 6.1, 6.2 and 6.3.

Table 6.2: Transient temperatures in a FGM strip with five nodes.

X	${T(x)}/{T_0}$ steady values							
A	B	C	D	E	F	G	H	exact
0.25	0.386	0.592	0.728	0.788	0.823	0.847	0.905	0.908
0.5	0.013	0.172	0.372	0.487	0.563	0.616	0.747	0.749
0.75	-0.00994	0.00005	0.093	0.192	0.269	0.326	0.471	0.474

A = ${x}/{L}$, B = ${t}/{t_0} = 0.05$,

C = ${t}/{t_0} = 0.1$, D = ${t}/{t_0} = 0.2$,

E = ${t}/{t_0} = 0.3$, F = ${t}/{t_0} = 0.4$,

G = ${t}/{t_0} = 0.5$, H = ${t}/{t_0} = \infty$

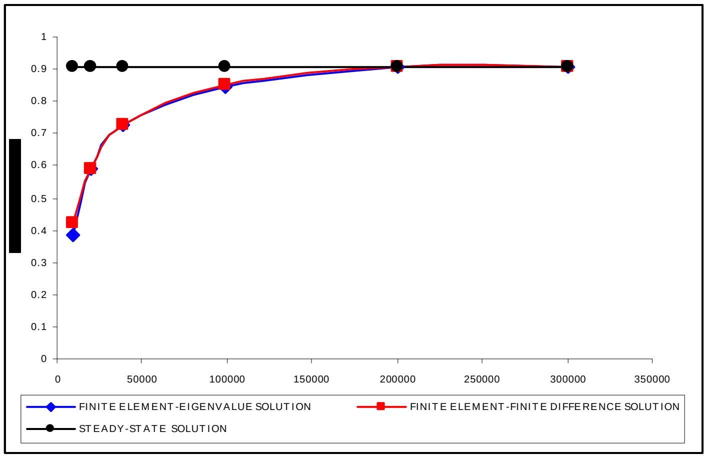


Figure 6.1 Transient temperatures at position x = 0.25L along the FGM strip with 5-nodes

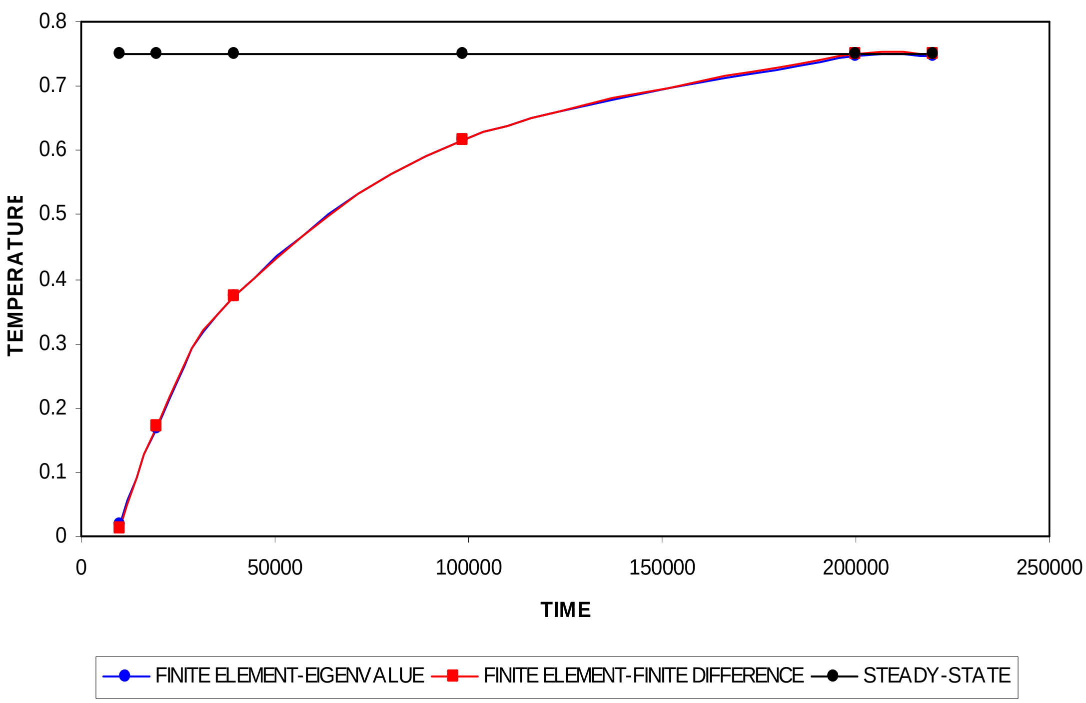
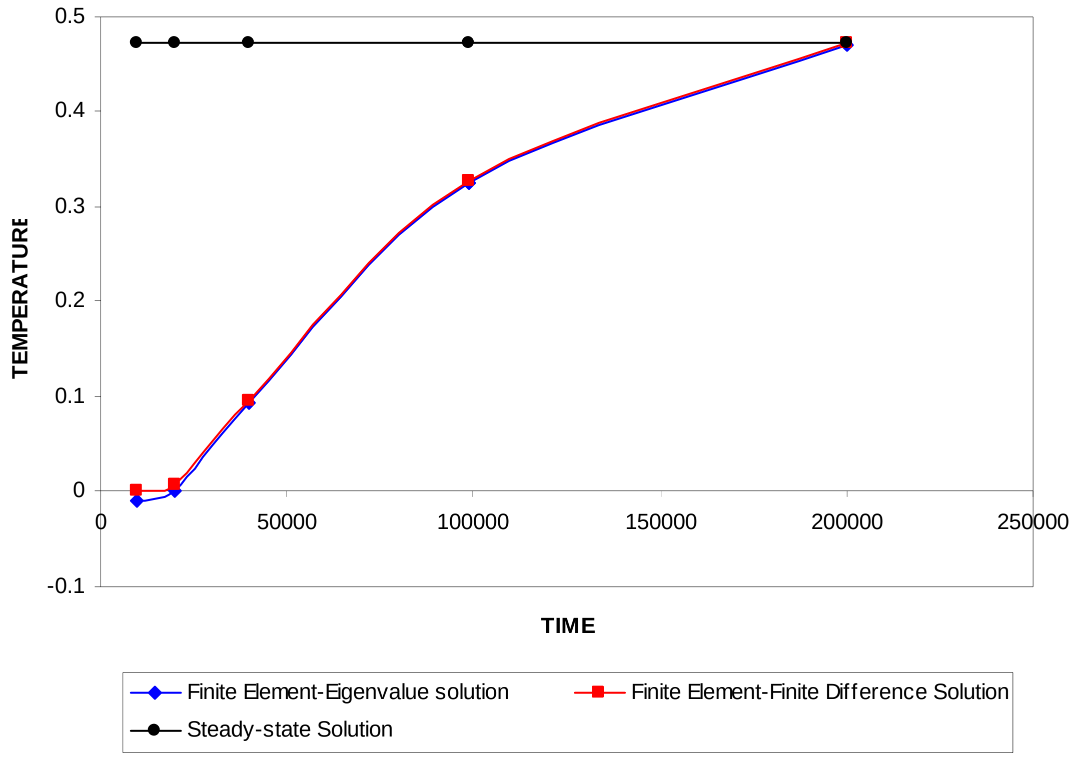


Figure 6.2: Transient temperatures at position x = 0.5L along the FGM strip using 5-nodes



0.5
0.4
0.3
0.2
0.1
0
-0.1
TEMPERATURE
0
50000
100000
150000
200000
250000
TIME
Finite Element-Eigenvalue solution
Finite Element-Finite Difference Solution
Steady-state Solution

Figure 6.3: Transient temperatures at position x=0.75L along the FGM strip with 5-nodes

6.1.3 9 Nodes finite element- eigenvalue solution (Mesh of Four 1-D Quadratic elements)

6.1.3.1 Element 1

Again using the model, after simplification, we have:

$$\left[M^{1} \right] = L \begin{bmatrix} 118956.721 & 59544.529 & -29639.928 \\ 59544.529 & 474235.976 & 599015.182 \\ -29639.928 & 59015.182 & 118162.701 \end{bmatrix} \tag{6.42}$$

$$\left[K^{1} \right] = \frac{1}{L} \begin{bmatrix} 151.216 & -170.546 & 19.330 \\ -170.546 & 300.463 & -129.917 \\ 19.330 & -129.917 & 110.586 \end{bmatrix} \tag{6.43}$$

6.1.3.2 Element 2:

Element 2 is simplified to:

$$\left[M^{2} \right] = L \begin{bmatrix} 117899.573 & 59015.361 & -29376.532 \\ 59015.361 & 470022.35 & 58490.765 \\ -2937.532 & 58490.765 & 117112.679 \end{bmatrix} \tag{6.44}$$

and

$$\left[K^{2}\right]=\frac{1}{L}\begin{bmatrix}87.572 & -98.767 & 11.195\\ -98.767 & 174.010 & -75.242\\ 11.195 & -75.242 & 64.047\end{bmatrix} \qquad (6.45)$$

6.1.3.3 Element 3:

Element 3 is simplified to:

$$\left[M^{3}\right]=L\begin{bmatrix}116851.736 & 58490.86 & -29115.446\\ 58490.86 & 465844.878 & 57970.922\\ -29115.446 & 57970.922 & 116071.829\end{bmatrix} \qquad (6.46)$$

$$\left[K^{3}\right]=\frac{1}{L}\begin{bmatrix}50.72 & -57.204 & 6.484\\ -57.204 & 100.781 & -43.577\\ 6.484 & -43.571 & 37.093\end{bmatrix} \qquad (6.47)$$

6.1.3.4 Element 4:

Element 4 is simplified to:

$$\left[M^{4}\right]=L\begin{bmatrix}115813.226 & 57971.027 & -28856.686\\ 57971.027 & 461704.725 & 57455.717\\ -28856.686 & 57455.717 & 115040.262\end{bmatrix} \qquad (6.48)$$

$$\left[K^{4}\right]=\frac{1}{L}\begin{bmatrix}29.371 & -33.126 & 3.755\\ -33.126 & 58.359 & -25.233\\ 3.755 & -25.233 & 21.478\end{bmatrix} \qquad (6.49)$$

The finite element model becomes:

$$\begin{bmatrix}K_{11}^{1} & K_{12}^{1} & K_{13}^{1} & 0 & 0 & 0 & 0 & 0 & 0\\ K_{21}^{1} & K_{22}^{1} & K_{11}^{1} & 0 & 0 & 0 & 0 & 0 & 0\\ K_{31}^{1} & K_{32}^{1} & K_{33}^{1}K_{11}^{2} & K_{12}^{2} & K_{13}^{2} & 0 & 0 & 0 & 0\\ 0 & 0 & K_{21}^{2} & K_{22}^{2} & K_{23}^{2} & 0 & 0 & 0 & 0\\ 0 & 0 & K_{31}^{2} & K_{32}^{2} & K_{33}^{2}+K_{11}^{3} & K_{12}^{3} & K_{13}^{3} & 0 & 0\\ 0 & 0 & 0 & 0 & K_{21}^{3} & K_{22}^{3} & K_{23}^{3} & 0 & 0\\ 0 & 0 & 0 & 0 & K_{31}^{3} & K_{32}^{3} & K_{33}^{3}K_{11}^{4} & K_{12}^{4} & K_{13}^{4}\\ 0 & 0 & 0 & 0 & 0 & 0 & K_{21}^{4} & K_{22}^{4} & K_{23}^{4}\\ 0 & 0 & 0 & 0 & 0 & 0 & K_{31}^{4} & K_{32}^{4} & K_{33}^{4}\end{bmatrix}\begin{Bmatrix}T_1\\ T_2\\ T_3\\ T_4\\ T_5\\ T_6\\ T_7\\ T_8\\ T_9\end{Bmatrix}+$$

$$\begin{bmatrix} M_{11}^1 & M_{12}^1 & M_{13}^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ M_{21}^1 & M_{22}^1 & M_{11}^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ M_{31}^1 & M_{32}^1 & M_{33}^1 + M_{11}^2 & M_{12}^2 & M_{13}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{21}^2 & M_{22}^2 & M_{23}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{31}^2 & M_{32}^2 & M_{33}^2 + M_{11}^3 & M_{12}^3 & M_{13}^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{21}^3 & M_{22}^3 & M_{23}^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{31}^3 & M_{32}^3 & M_{33}^3 + M_{11}^4 & M_{12}^4 & M_{13}^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & M_{21}^4 & M_{22}^4 & M_{23}^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & M_{31}^4 & M_{32}^4 & M_{33}^4 \end{bmatrix} \begin{Bmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \\ \dot{T}_9 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ Q_9 \end{Bmatrix} \tag{6.50}$$

The condensed equations are taking note that $T_1 = T_0$

$$\frac{1}{L}\begin{bmatrix} K_{22}^1 & K_{11}^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ K_{32}^1 & K_{33}^1 K_{11}^2 & K_{12}^2 & K_{13}^2 & 0 & 0 & 0 & 0 \\ 0 & K_{21}^2 & K_{22}^2 & K_{23}^2 & 0 & 0 & 0 & 0 \\ 0 & K_{31}^2 & K_{32}^2 & K_{33}^2 + K_{11}^3 & K_{12}^3 & K_{13}^3 & 0 & 0 \\ 0 & 0 & 0 & K_{21}^3 & K_{22}^3 & K_{23}^3 & 0 & 0 \\ 0 & 0 & 0 & K_{31}^3 & K_{32}^3 & K_{33}^3 + K_{11}^4 & K_{12}^4 & K_{13}^4 \\ 0 & 0 & 0 & 0 & 0 & K_{21}^4 & K_{22}^4 & K_{23}^4 \\ & & & & & & & \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ \end{Bmatrix} +$$

$$L\begin{bmatrix} M_{22}^1 & M_{11}^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ M_{32}^1 & M_{33}^1 + M_{11}^2 & M_{12}^2 & M_{13}^2 & 0 & 0 & 0 & \\ 0 & M_{21}^2 & M_{22}^2 & M_{23}^2 & 0 & 0 & 0 & 0 \\ 0 & M_{31}^2 & M_{32}^2 & M_{33}^2 + M_{11}^3 & M_{12}^3 & M_{13}^3 & 0 & 0 \\ 0 & 0 & 0 & M_{21}^3 & M_{22}^3 & M_{23}^3 & 0 & 0 \\ 0 & 0 & 0 & M_{31}^3 & M_{32}^3 & M_{33}^3 + M_{11}^4 & M_{12}^4 & M_{13}^4 \\ 0 & 0 & 0 & 0 & 0 & M_{21}^4 & M_{22}^4 & M_{23}^4 \\ & & & & & & & \end{bmatrix} \begin{Bmatrix} \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \\ \end{Bmatrix} = \begin{Bmatrix} -K_{12}^1 T_0 \\ -K_{31}^1 T_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \end{Bmatrix} \tag{6.51}$$

$$\frac{1}{L}\begin{bmatrix} 300.463 & -129.917 & 0 & 0 & 0 & 0 & 0 \\ -129.917 & 198.158 & -98.767 & 11.195 & 0 & 0 & 0 \\ 0 & -98.767 & 174.010 & -75.242 & 0 & 0 & 0 \\ 0 & 11.195 & -75.242 & 114.767 & -57.204 & 6.484 & 0 \\ 0 & 0 & 0 & -57.204 & 100.781 & -43.577 & 0 \\ 0 & 0 & 0 & 6.484 & -43.577 & 66.464 & -33.126 \\ 0 & 0 & 0 & 0 & 0 & -33.126 & 58.358 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{Bmatrix}+$$

$$L\begin{bmatrix} 474235.976 & 59015.182 & 0 & 0 & 0 & 0 & 0 \\ 59015.182 & 236062.274 & 59015.361 & -29376.532 & 0 & 0 & 0 \\ 0 & 59015.361 & 470022.35 & 58409.765 & 0 & 0 & 0 \\ 0 & -29376.532 & 58490.765 & 233964.415 & 58490.860 & -29115.446 & 0 \\ 0 & 0 & 0 & 58470.86 & 465844.878 & 57970.922 & 0 \\ 0 & 0 & 0 & -29115.446 & 57970.922 & 231885.005 & 57971.027 \\ 0 & 0 & 0 & 0 & 0 & 57971.027 & 461704.725 \end{bmatrix}\begin{Bmatrix} \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \end{Bmatrix}=$$

$$\frac{1}{L}\begin{Bmatrix} 170.546T_0 \\ -19.330T_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (6.52)$$

The solution is divided into the homogenous solution and the particular solution $T = T_h + T_p$

$$T_h = v\exp(-\lambda t) \text{ or } (K-\lambda M)v\exp(-\lambda t) = 0 \qquad (6.53)$$

from which $(K-\lambda M)v = 0$ or $\det|K-\lambda M| = 0$.

Solving this equation yields the eigenvalues with which the eigenvectors are set up. Since the finite-eigenvalue approach of solving transient temperature field problems has been effectively described above, it is of interest only to show that as the number of elements is increased, the solution obtained would ultimately tend to the exact solution.

$$[\mathrm{k}]\{\mathrm{T}\}_p = \{\hat{\mathrm{Q}}\} \qquad (6.54)$$

$$\begin{bmatrix} k_{22}^1 & k_{23}^1 & 0 & 0 & 0 & 0 & 0 \\ k_{32}^1 & k_{33}^1+k_{11}^2 & k_{12}^2 & k_{13}^2 & 0 & 0 & 0 \\ 0 & k_{21}^2 & k_{22}^2 & k_{23}^2 & 0 & 0 & 0 \\ 0 & k_{31}^2 & k_{32}^2 & k_{33}^2+k_{11}^3 & k_{12}^3 & k_{13}^3 & 0 \\ 0 & 0 & 0 & k_{21}^3 & k_{22}^3 & k_{23}^3 & 0 \\ 0 & 0 & 0 & k_{31}^3 & k_{32}^3 & k_{33}^3+k_{11}^4 & k_{12}^4 \\ 0 & 0 & 0 & 0 & 0 & k_{21}^4 & k_{22}^4 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{Bmatrix} = \begin{Bmatrix} -k_{21}^1 \\ -k_{31}^1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (6.55)$$

That is,

$$\begin{bmatrix} 300.463 & -129.917 & 0 & 0 & 0 & 0 & 0 \\ -129.917 & 198.158 & -98.767 & 11.195 & 0 & 0 & 0 \\ 0 & -98.767 & 174.010 & -75.242 & 0 & 0 & 0 \\ 0 & 11.195 & -75.242 & 114.767 & -57.204 & 6.484 & 0 \\ 0 & 0 & 0 & -57.204 & 100.781 & -43.577 & 0 \\ 0 & 0 & 0 & 6.484 & -43.577 & 66.464 & -33.126 \\ 0 & 0 & 0 & 0 & 0 & -33.126 & 58.359 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{Bmatrix} = \begin{Bmatrix} 170.546\ T_0 \\ -19.330\ T_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{6.56}$$

Solving the equations above gives rise to

$$\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{Bmatrix} = \begin{Bmatrix} 0.96017\ T_0 \\ 0.90789\ T_0 \\ 0.83912\ T_0 \\ 0.74884\ T_0 \\ 0.63011\ T_0 \\ 0.47424\ T_0 \\ 0.26919\ T_0 \end{Bmatrix} \tag{6.57}$$

Thus the particular solution yields.

$$T_p = \begin{Bmatrix} 0.96017\ T_0 \\ 0.90789\ T_0 \\ 0.83912\ T_0 \\ 0.74884\ T_0 \\ 0.63011\ T_0 \\ 0.47424\ T_0 \\ 0.26919\ T_0 \end{Bmatrix} \tag{6.58}$$

The nodes of interest are nodes 3, 5 and 7. The particular solution at these nodes of interest is:

$$T_p = \begin{Bmatrix} 0.908\ T_0 \\ 0.749\ T_0 \\ 0.474\ T_0 \end{Bmatrix} \tag{6.59}$$

The general solution can thus be written as:

$$T = T_p + c_1 v_1 \exp(-\lambda_1 t) + c_2 v_2 \exp(-\lambda_2 t) + \cdots + c_j v_j \exp(-\lambda_j t) \tag{6.60}$$

Therefore,

$$\frac{T_2}{T_0} = 0.908 + c_1 v_1 T_0 \exp(-\lambda_1 t) + c_2 v_2 T_0 \exp(-\lambda_2 t) + \cdots + c_j v_j T_0 \exp(-\lambda_j t) \tag{6.61}$$

$$\frac{T_3}{T_0} = 0.749 + c_1 v^1{}_1 T_0 \exp(-\lambda_1 t) + c_2 v^1{}_2 T_0 \exp(-\lambda_2 t) + \cdots + c_j v^1{}_j T_0 \exp(-\lambda_j t) \tag{6.62}$$

$$\frac{T_4}{T_0} = 0.474 + c_1 v^{11}{}_1 T_0 \exp(-\lambda_1 t) + c_2 v^{11}{}_2 T_0 \exp(-\lambda_2 t) + \cdots + c_j v^{11}{}_j T_0 \exp(-\lambda_j t) \tag{6.63}$$

It can easily be seen from the above solution that as $t \to \infty$;

$$\frac{T_2}{T_0} \to 0.908, \frac{T_3}{T_0} \to 0.749, \frac{T_4}{T_0} \to 0.474$$

It follows therefore that the steady state solution at nodes 3, 5 and 7 are:

$$T_2 = 0.908\ T_0,\ T_3 = 0.749\ T_0 \text{ and } T_4 = 0.474\ T_0$$

These correspond to the steady state solutions at these nodes.

6.2. Example 2: Two-dimensional heat conduction in a FGM square plate

Consider the conduction heat transfer in a non-homogeneous (functionally graded materials) square plate of dimensions $x = (0,2L)$ by $y = (0,2L)$, conductivity $k = k(x,y) = k_0\left[1+\left(\frac{x}{L}\right)^2+\left(\frac{y}{L}\right)^2\right]$, density ρ, specific heat capacity c. The plate is subjected to a sudden internal heat generation of Q_0. The edges of the plate are maintained at a temperature of T = 0. Determine the steady temperature distribution (T_c) at the centre of the plate using the finite element/eigenvalue method (adopted from [1])

The temperature distribution at the centre of the plate, solved with the finite element/eigenvalue method using a mesh of 2 × 2 rectangular elements, is given as:

$$T_c = 0.169 - 0.129\exp(-4.575t) - 0.029\exp(-19.794t) - 0.010\exp(-45.427t)$$

it follows therefore that as $T \to \infty$, $T_c \to 0.169$, this fact is shown in Figure 6.4.

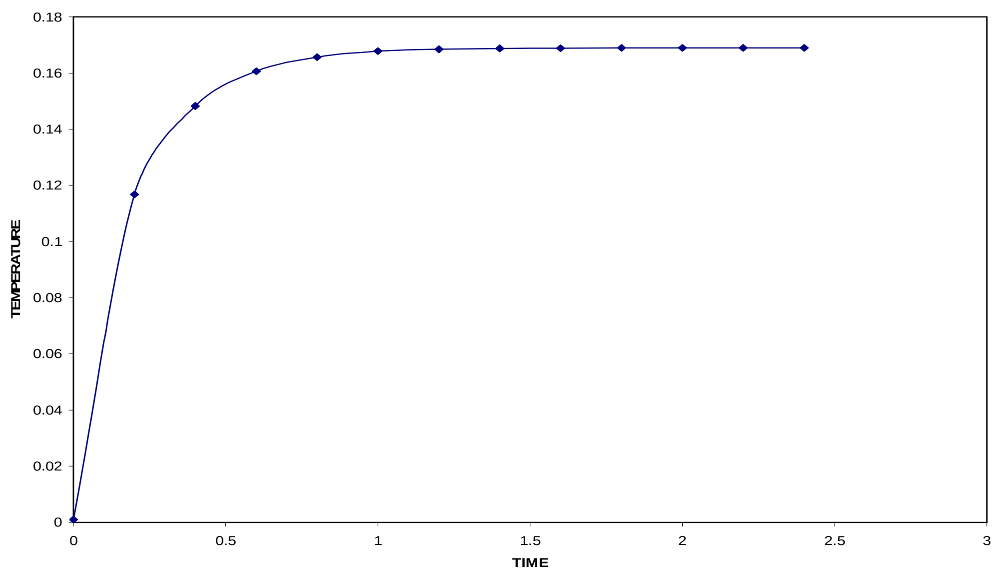


Figure 6.4: Transient temperature at the center of a non-homogeneous square plate under a sudden internal heat generation.

7.0 Discussion of results

A careful examination of the conclusions made in section 6.1.3, 6.2, Tables 1, 2 and Figures. 1, 2, 3, 4 shows that the method was able to solve the system of time dependent differential equations that describe the transient temperature distribution in a functionally graded material with very high accuracy.
The efficiency of the method is clearly seen as the number of finite elements is increased. It was observed that the solution tended fast to the exact solution when the number of elements was increased to four 1-D quadratic elements (9 nodes). The results generated are exponential functions of time.

8.0 Conclusion

Due to the non-homogeneity of the properties of functionally graded materials, it is often difficult to obtain the exact solutions of their thermal conductivity equations, hence the need to develop a numerical method to obtain solutions that are as accurate as possible. In this work, a finite element-eigenvalue method was discussed, analyzed and used to solve thermal conductivity problem associated with a functionally graded material (non-homogeneous material). The solving scheme developed is simple. It finds its basis in fundamental eigenvalue/eigenvector problem-solving scheme in mathematics and it does not involve time discretization which, in complex cases like this, would have been very difficult to solve unless with the use of a software.

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