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Radial flow of slightly compressible fluids: A finite element-finite difference approach

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Abstract

In this work, we develop a finite element-finite difference method to solve the differential equation governing the radial flow of slightly compressible fluids. The finite element method is used to carry out spatial approximations so as to study the variation of fluid properties at the various nodes to which effect we divided the entire radial domain of the fluid into a mesh of four radial 1-D quadratic elements which exposes nine nodes to intense study. Time approximation is done with the aid of the Crank-Nicolson finite difference scheme.

***Keywords*:** Finite element method, Crank-Nicolson, Finite difference scheme, Time approximation, Radial 1-D quadratic element.

1.0 Introduction

Fluids, generally, are compressible so that they will change with pressure, but, under steady flow conditions and provided that the changes in density are small, it is usually possible to simplify the analysis of a problem by assuming that the fluid is incompressible and of constant density. Since liquids are relatively difficult to compress, it is normal to treat them as if they are incompressible for all cases of steady flow. However for unsteady flow conditions, high pressure conditions can develop and the compressibility of liquids must then be taken into consideration.

Gases are easily compressed and, except when changes of pressure and, therefore density are very small, the effect of compressibility and changes of internal energy must be taken into account.

The necessity to consider compressibility, changes of pressure and internal energy effectively complicates the analysis of fluid flow hence the necessity to develop a robust numerical scheme which would enable the observation of the fluid properties at various points since real fluid flow is always turbulent and unsteady. Previously research had been done on: Finite Element Analysis of the distribution of Velocity in incompressible fluids, using the Lagrange interpolation function, see [1-6].

In this paper, we propose the finite element-finite difference method to solve the differential equation which governs the radial flow of slightly incompressible fluids. The method is developed to provide more accurate and broader spectrum of analysis for slightly compressible fluids.

2.0 Mathematical modelling

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)=\frac{\Phi uc}{0.000264k}\cdot\frac{\partial p}{\partial t} \tag{2.1}$$

where
u: volumetric flow rate per unit cross sectional area in the radial direction
Φ: porosity
r: radius/length
c: compressibility

k: permeability

Let $\dfrac{\Phi uc}{0.000264k} = a = cons\tan t$. Therefore

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = a\frac{\partial p}{\partial t} \tag{2.2}$$

Boundary conditions:

$$p(0,t) = p_o \text{ and } \frac{\partial p}{\partial x}(0,t) = 0$$

Initial condition: $$p(x,0) = p_o$$

The following assumptions were made in developing this model thus

- The single phase liquid has a small and constant compressibility
- Permeability is constant and same in all directions i.e. isotropic
- Porosity is constant
- Pressure gradients are small
- The geometry and boundary conditions are dependent only on the radial direction and independent of the other two directions

2.1 Weak formulation

The weak form of the above equation is obtained by multiplying the equation by a weight function $w = w(r)$ and integrating it over the domain of the element.

$$0 = \int_v w\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + a\frac{\partial p}{\partial t}\right] r dr d\theta dz \tag{2.3}$$

$$0 = \int_0^1\int_0^{2\pi}\int_{r_A}^{r_B} w\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + a\frac{\partial p}{\partial t}\right] r dr d\theta dz$$

$$0 = 2\pi\int_{r_A}^{r_B} w\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + a\frac{\partial p}{\partial t}\right] r dr \tag{2.4}$$

$$0 = 2\pi\int_{r_A}^{r_B}\left[-\frac{1}{r}w\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + wa\frac{\partial p}{\partial t}\right] r dr \tag{2.5}$$

Using the integration by parts principle

$$\int_{r_A}^{r_B} w\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)dr = -\int_{r_A}^{r_B} r\frac{\partial w}{\partial r}\frac{\partial p}{\partial r}dr + \left[wr\frac{\partial p}{\partial r}\right]\Bigg|_{r_A}^{r_B} \tag{2.6}$$

Substituting equation 6 into equation 5 yields

$$0 = 2\pi\int_{r_A}^{r_B}\left[r\frac{\partial w}{\partial r}\frac{\partial p}{\partial r} + wra\frac{\partial p}{\partial t}\right] r dr - \left[2\pi wr\frac{\partial p}{\partial r}\right]\Bigg|_{r_A}^{r_B}$$

$$0 = 2\pi\int_{r_A}^{r_B}\left[r\frac{\partial w}{\partial r}\frac{\partial p}{\partial r} + wra\frac{\partial p}{\partial t}\right] dr - w(r_A)Q_1^e - w(r_B)Q_2^e \tag{2.7}$$

where $$Q_1^e = -2\pi\left(r\frac{\partial p}{\partial r}\right)\Bigg|_{r_A} \quad and \quad Q_2^e = 2\pi\left(r\frac{\partial p}{\partial r}\right)\Bigg|_{r_B}$$

Equation (2.7) is the weak form of equation 1

2.2 Finite element modelling

Let the solution to equation (2.7) be of the separable variable form

$$p(r,t) = \sum_{j=1}^{n} p_j^e(t)\psi_j^e(r) \tag{2.8}$$

where $\psi_j^e =$ Lagrange radial interpolation function at the jth node and $p_j^e =$ pressure at the jth node of the element. Since we are applying the Rayleigh-Ritz finite element method in this paper, we assume that the weight function is equal to the interpolation function.

$$w(r) = \psi_i(r) \tag{2.9}$$

Substitute equations (2.8) and (2.9) into equation (2.7)

$$0 = 2\pi\int_{r_A}^{r_B}\left[r\frac{\partial\psi_i}{\partial r}\frac{\partial}{\partial r}\sum_{j=1}^{n}p_j^e\psi_j^e + \psi_i ra\frac{\partial}{\partial t}\sum_{j=1}^{n}p_j^e\psi_j^e\right]dr - \psi_i(r_A)Q_1^e - \psi_i(r_B)Q_2^e$$

$$0 = \sum_{j=1}^{n}\left\{\left[2\pi\int_{r_A}^{r_B}r\frac{\partial\psi_i}{\partial r}\frac{\partial\psi_j}{\partial r}dr\right]p_j^e + \left[2\pi\int_{r_A}^{r_B}ar\psi_i\psi_j dr\right]\dot{p}_j^e\right\} - \psi_i(r_A)Q_1^e - \psi_i(r_B)Q_2^e$$

Let $$2\pi\int_{r_A}^{r_B}r\frac{\partial\psi_i}{\partial r}\frac{\partial\psi_j}{\partial r}dr = k_{ij},\ 2\pi\int_{r_A}^{r_B}ar\psi_i\psi_j dr = m_{ij} \tag{2.10}$$

and $\psi_i(r_A)Q_1^e + \psi_i(r_B)Q_2^e = Q_i$

$$0 = \sum_{j=1}^{n}\left\{k_{ij}p_j^e + m_{ij}\dot{p}_j^e\right\} - Q_i \tag{2.11}$$

In matrix form, equation (2.11) becomes $0 = [K]\{p\} + [M]\{\dot{p}\} - \{Q\}$ or

$$[M]\{\dot{p}\} + [K]\{p\} = \{Q\} \tag{2.12}$$

where

$[M] = mass\ matrix$

$[K] = stiffness\ matrix$

$\{Q\} = flux\ vector$

The radial 1-D Lagrange quadratic interpolation functions are

$$\psi_1(r) = \frac{1}{h_e^2}(r_B - r)(r_B - r_A - 2r)$$

$$\psi_2(r) = \frac{4}{h_e^2}(r - r_A)(r_B - r) \tag{2.13}$$

$$\psi_3(r) = -\frac{1}{h_e^2}(r - r_A)(r_B - r_A - 2r)$$

where $r_B = h + r_A$. Substitute equation (2.13) into equation (2.9) to get the stiffness and mass matrices

$$[K]=\frac{2\pi}{6h}\begin{bmatrix} 3h+14r_A & -(4h+16r_A) & h+2r_A \\ -(4h+16r_A) & 16h+32r_A & -(12h+16r_A) \\ h+2r_A & -(12h+16r_A) & 11h+14r_A \end{bmatrix} \tag{2.14}$$

$$[M]=\frac{2\pi ah}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \tag{2.15}$$

We assume that in this paper that the initial point corresponds to the centre of the pipe, which implies that $r_A = 0$.

It follows therefore that the stiffness and mass matrices are given by

$$[K]=\frac{2\pi}{6h}\cdot\frac{r_{e+1}+r_e}{2}\begin{bmatrix} 3 & -4 & 1 \\ -4 & 16 & -12 \\ 1 & -12 & 11 \end{bmatrix} \tag{2.16}$$

In this paper, we divide the domain into four 1-D quadratic finite elements and the finite element model over an element is given as

$$\begin{bmatrix} m_{11}^e & m_{12}^e & m_{13}^e \\ m_{21}^e & m_{22}^e & m_{23}^e \\ m_{31}^e & m_{32}^e & m_{33}^e \end{bmatrix}\begin{Bmatrix} \dot{p}_1^e \\ \dot{p}_2^e \\ \dot{p}_3^e \end{Bmatrix}+\begin{bmatrix} k_{11}^e & k_{12}^e & k_{13}^e \\ k_{21}^e & k_{22}^e & k_{23}^e \\ k_{31}^e & k_{32}^e & k_{33}^e \end{bmatrix}\begin{Bmatrix} p_1^e \\ p_2^e \\ p_3^e \end{Bmatrix}=\begin{Bmatrix} Q_1^e \\ 0 \\ Q_3^e \end{Bmatrix} \tag{2.17}$$

For a mesh of four 1-D quadratic elements the assembled equations are:

$$
\begin{bmatrix}
m_{11}^1 & m_{12}^1 & m_{13}^1 & 0 & 0 & 0 & 0 & 0 & 0 \\
m_{21}^1 & m_{22}^1 & m_{23}^1 & 0 & 0 & 0 & 0 & 0 & 0 \\
m_{31}^1 & m_{32}^1 & m_{33}^1 + m_{11}^2 & m_{12}^2 & m_{13}^2 & 0 & 0 & 0 & 0 \\
0 & 0 & m_{21}^2 & m_{22}^2 & m_{23}^2 & 0 & 0 & 0 & 0 \\
0 & 0 & m_{31}^2 & m_{32}^2 & m_{33}^2 + m_{11}^3 & m_{12}^3 & m_{13}^3 & 0 & 0 \\
0 & 0 & 0 & 0 & m_{21}^3 & m_{22}^3 & m_{23}^3 & 0 & 0 \\
0 & 0 & 0 & 0 & m_{31}^3 & m_{32}^3 & m_{33}^3 + m_{11}^4 & m_{12}^4 & m_{13}^4 \\
0 & 0 & 0 & 0 & 0 & 0 & m_{21}^4 & m_{22}^4 & m_{23}^4 \\
0 & 0 & 0 & 0 & 0 & 0 & m_{31}^4 & m_{32}^4 & m_{33}^4
\end{bmatrix}
\begin{Bmatrix}
\dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \\ \dot{p}_5 \\ \dot{p}_6 \\ \dot{p}_7 \\ \dot{p}_8 \\ \dot{p}_9
\end{Bmatrix} +
$$

$$
\begin{bmatrix}
k_{11}^1 & k_{12}^1 & k_{13}^1 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_{21}^1 & k_{22}^1 & k_{23}^1 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_{31}^1 & k_{32}^1 & k_{33}^1 + k_{11}^2 & k_{12}^2 & k_{13}^2 & 0 & 0 & 0 & 0 \\
0 & 0 & k_{21}^2 & k_{22}^2 & k_{23}^2 & 0 & 0 & 0 & 0 \\
0 & 0 & k_{31}^2 & k_{32}^2 & k_{33}^2 + k_{11}^3 & k_{12}^3 & k_{13}^3 & 0 & 0 \\
0 & 0 & 0 & 0 & k_{21}^3 & k_{22}^3 & k_{23}^3 & 0 & 0 \\
0 & 0 & 0 & 0 & k_{31}^3 & k_{32}^3 & k_{33}^3 + k_{11}^4 & k_{12}^4 & k_{13}^4 \\
0 & 0 & 0 & 0 & 0 & 0 & k_{21}^4 & k_{22}^4 & k_{23}^4 \\
0 & 0 & 0 & 0 & 0 & 0 & k_{31}^4 & k_{32}^4 & k_{33}^4
\end{bmatrix}
\begin{Bmatrix}
p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9
\end{Bmatrix} =
\begin{Bmatrix}
Q_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ Q_9
\end{Bmatrix}
\qquad (2.18)
$$

3.0 Numerical example

Consider the flow of a slightly compressible fluid in a cylinder pipe of radius 1m. The properties of the fluid are as shown below

$$\Phi = 0.5,\ c = 0.0002,\ u = 10\, m^3/s,\ k = 2.5$$

so that

$$a = \frac{\Phi u c}{0.000264k} = \frac{0.5 \times 10 \times 0.0002}{0.000264 \times 2.5} = 1.5$$

Also $$radial\ \ length\ of\ \ an\ element = h = \frac{1}{4}$$

$$\left[K^1\right] = \frac{\pi}{6} \cdot \begin{bmatrix} 3 & -4 & 1 \\ -4 & 16 & -12 \\ 1 & -12 & 11 \end{bmatrix},$$

$$\left[K^2\right] = \frac{\pi}{6} \cdot \begin{bmatrix} 9 & -12 & 3 \\ -12 & 48 & -36 \\ 3 & -36 & 33 \end{bmatrix},$$

$$\left[K^3\right]=\frac{\pi}{6}\cdot\begin{bmatrix}15 & -20 & 5\\ -20 & 80 & -60\\ 5 & -60 & 55\end{bmatrix},$$

$$\left[K^4\right]=\frac{\pi}{6}\cdot\begin{bmatrix}21 & -28 & 7\\ -28 & 112 & -84\\ 7 & -84 & 77\end{bmatrix}.$$

Also $$\left[M^1\right]=\left[M^2\right]=\left[M^3\right]=\left[M^4\right]=\frac{2\pi}{80}\begin{bmatrix}4 & 2 & -1\\ 2 & 16 & 2\\ -1 & 2 & 4\end{bmatrix}$$

The assembled equation thus becomes

$$\frac{2\pi}{80}\begin{bmatrix}4 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 2 & 16 & 2 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 2 & 8 & 2 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 2 & 16 & 2 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 2 & 8 & 2 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 2 & 16 & 2 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 2 & 8 & 2 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 16 & 2\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 4\end{bmatrix}\begin{Bmatrix}\dot{p}_1\\ \dot{p}_2\\ \dot{p}_3\\ \dot{p}_4\\ \dot{p}_5\\ \dot{p}_6\\ \dot{p}_7\\ \dot{p}_8\\ \dot{p}_9\end{Bmatrix}+$$

$$\frac{\pi}{6}\begin{bmatrix}3 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ -4 & 16 & -12 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & -12 & 20 & -12 & 3 & 0 & 0 & 0 & 0\\ 0 & 0 & -12 & 48 & -36 & 0 & 0 & 0 & 0\\ 0 & 0 & 3 & -36 & 48 & -20 & 5 & 0 & 0\\ 0 & 0 & 0 & 0 & -20 & 80 & -60 & 0 & 0\\ 0 & 0 & 0 & 0 & 5 & -60 & 76 & -28 & 7\\ 0 & 0 & 0 & 0 & 0 & 0 & -28 & 112 & -84\\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & -84 & 77\end{bmatrix}\begin{Bmatrix}p_1\\ p_2\\ p_3\\ p_4\\ p_5\\ p_6\\ p_7\\ p_8\\ p_9\end{Bmatrix}=\begin{Bmatrix}Q_1^1\\ Q_2^1\\ Q_3^1+Q_1^2\\ Q_2^2\\ Q_3^2+Q_1^3\\ Q_2^3\\ Q_3^3+Q_1^4\\ Q_2^4\\ Q_3^4\end{Bmatrix}$$

Due to the balance of internal fluxes, it follows that

$$Q_3^1+Q_1^2=Q_3^2+Q_1^3=Q_3^3+Q_1^4=0 \;\; and$$

$$Q_2^2=Q_2^3=Q_2^4=0$$

$$\frac{2\pi}{80}\begin{bmatrix} 4 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 16 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 8 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 16 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 8 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 16 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 8 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 16 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 4 \end{bmatrix}\begin{Bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \\ \dot{p}_5 \\ \dot{p}_6 \\ \dot{p}_7 \\ \dot{p}_8 \\ \dot{p}_9 \end{Bmatrix} +$$

$$\frac{\pi}{6}\begin{bmatrix} 3 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 16 & -12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -12 & 20 & -12 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & 48 & -36 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -36 & 48 & -20 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -20 & 80 & -60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & -60 & 76 & -28 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & -28 & 112 & -84 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & -84 & 77 \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ Q_9 \end{Bmatrix} \tag{3.1}$$

We now consider the boundary conditions

$$p(0,t) = p_o$$

which implies that

$$p_1 = p_o \text{ and } \frac{\partial p}{\partial x}(0,t) = 0$$

which implies that $$Q_1^1 = Q_1 = 0$$

$$\frac{2\pi}{80}\begin{bmatrix} 4 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 16 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 8 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 16 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 8 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 16 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 8 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 16 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 4 \end{bmatrix}\begin{Bmatrix} 0 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \\ \dot{p}_5 \\ \dot{p}_6 \\ \dot{p}_7 \\ \dot{p}_8 \\ \dot{p}_9 \end{Bmatrix} +$$

$$\frac{\pi}{6}\begin{bmatrix} 3 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 16 & -12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -12 & 20 & -12 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & 48 & -36 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -36 & 48 & -20 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -20 & 80 & -60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & -60 & 76 & -28 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & -28 & 112 & -84 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & -84 & 77 \end{bmatrix}\begin{Bmatrix} p_0 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ Q_9 \end{Bmatrix}$$

The condensed equations are

$$\frac{2\pi}{80}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 8 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 16 & 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 8 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 16 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 8 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 16 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{p}_2 \\ \ddot{p}_3 \\ \ddot{p}_4 \\ \ddot{p}_5 \\ \ddot{p}_6 \\ \ddot{p}_7 \\ \ddot{p}_8 \\ \ddot{p}_9 \end{Bmatrix} +$$

$$\frac{\pi}{6}\begin{bmatrix} -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16 & -12 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & 20 & -12 & 3 & 0 & 0 & 0 & 0 \\ 0 & -12 & 48 & -36 & 0 & 0 & 0 & 0 \\ 0 & 3 & -36 & 48 & -20 & 5 & 0 & 0 \\ 0 & 0 & 0 & -20 & 80 & -60 & 0 & 0 \\ 0 & 0 & 0 & 5 & -60 & 76 & -28 & 7 \\ 0 & 0 & 0 & 0 & 0 & -28 & 112 & -84 \end{bmatrix}\begin{Bmatrix} p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \end{Bmatrix} = \frac{\pi}{6}\begin{Bmatrix} -3p_o \\ 4p_o \\ -p_o \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.2}$$

Recall that the finite element model was of the form

$$[M]\{\ddot{p}\} + [K]\{p\} = \{Q\}$$

Thus

$$[M]=\frac{2\pi}{80}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 8 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 16 & 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 8 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 16 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 8 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 16 & 2 \end{bmatrix} \tag{3.3}$$

$$[K]=\frac{\pi}{6}\begin{bmatrix} -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16 & -12 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & 20 & -12 & 3 & 0 & 0 & 0 & 0 \\ 0 & -12 & 48 & -36 & 0 & 0 & 0 & 0 \\ 0 & 3 & -36 & 48 & -20 & 5 & 0 & 0 \\ 0 & 0 & 0 & -20 & 80 & -60 & 0 & 0 \\ 0 & 0 & 0 & 5 & -60 & 76 & -28 & 7 \\ 0 & 0 & 0 & 0 & 0 & -28 & 112 & -84 \end{bmatrix} \tag{3.4}$$

$$\{Q\}=\frac{\pi}{6}\begin{Bmatrix} -3p_o \\ 4p_o \\ -p_o \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.5}$$

4.0 Finite difference modelling

In this work, we use the α family of approximation, in which a weighted average of the time derivative of the dependent variable, *p*, is approximated at two consecutive time steps by linear interpolation of the values of the variable at the two steps:

$$(1-\alpha)\{\dot{p}\}_s+\alpha\{\dot{p}\}_{s+1}=\frac{\{p\}_{s+1}-\{p\}_s}{\Delta t_{s+1}} \qquad for\ 0\le\alpha\le 1 \tag{4.1}$$

where $\{\ \}_s$ refers to the value of the enclosed quantity at time $t=t_s=\sum_{i=1}^{s}\Delta t_i$. Since the finite element model is valid for any $t>0$, it is valid for $t=t_s$ and $t=t_{s+1}$

$$[M]\{\dot{p}\}_s+[K]\{p\}_s=\{Q\}_s \tag{4.2}$$

$$[M]\{\dot{p}\}_{s+1}+[K]\{p\}_{s+1}=\{Q\}_{s+1} \tag{4.3}$$

We multiply both sides of equation (4.1) by $\Delta t_{s+1}[M]$ to get

$$\Delta t_{s+1}\alpha[M]\{\dot{p}\}_{s+1}+\Delta t_{s+1}(1-\alpha)[M]\{\dot{p}\}_s=[M]\left(\{p\}_{s+1}-\{p\}_s\right)$$

We substitute for $[M]\{\dot{p}\}_{s+1}$ and $[M]\{\dot{p}\}_{s}$ from equations (4.2) and (4.3) respectively

$$\Delta t_{s+1}\alpha\left(\{Q\}_{s+1}-[K]\{p\}_{s+1}\right)+\Delta t_{s+1}(1-\alpha)\left(\{Q\}_{s}-[K]\{p\}_{s}\right)=[M]\left(\{p\}_{s+1}-\{p\}_{s}\right)$$

Rearranging the terms into known and unknown, we get

$$\left([M]+\Delta t_{s+1}\alpha[K]\right)\{p\}_{s+1}=\left([M]-\Delta t_{s+1}(1-\alpha)[K]\right)\{p\}_{s}+\Delta t_{s+1}\left(\alpha\{Q\}_{s+1}+(1-\alpha)\{Q\}_{s}\right)$$

But

$$\{Q\}_{s+1}=\{Q\}_{s}=\{Q\}$$

Therefore writing

$$\Delta t_{s+1}=\Delta t$$

$$\left([M]+\Delta t\alpha[K]\right)\{p\}_{s+1}=\left([M]-\Delta t(1-\alpha)[K]\right)\{p\}_{s}+\Delta t\{Q\} \tag{4.4}$$

we apply the Crank-Nicholson finite difference scheme i.e. we take $\alpha=0.5$ equation (4.4) becomes

$$\left([M]+\frac{\Delta t[K]}{2}\right)\{p\}_{s+1}=\left([M]-\frac{\Delta t[K]}{2}\right)\{p\}_{s}+\Delta t\{Q\}$$

$$\{p\}_{s+1}=\left[\left[[M]+\frac{\Delta t[K]}{2}\right]^{-1}\left[[M]-\frac{\Delta t[K]}{2}\right]\right]\{p\}_{s}+\left[[M]+\frac{\Delta t[K]}{2}\right]^{-1}\left[\Delta t\{Q\}\right]$$

Let

$$\left[\left[[M]+\frac{\Delta t[K]}{2}\right]^{-1}\left[[M]-\frac{\Delta t[K]}{2}\right]\right]=[S] \tag{4.5}$$

$$\left[[M]+\frac{\Delta t[K]}{2}\right]^{-1}\left[\Delta t\{Q\}\right]=\{C\} \tag{4.6}$$

where $\Delta t = time\ step$

The Crank-Nicholson finite difference scheme can thus be written as

$$\{p\}_{s+1}=[S]\{p\}_{s}+\{C\} \tag{4.7}$$

Considering the initial condition, it follows therefore that

$$p(0)=p_0\begin{Bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{Bmatrix}$$

The solutions are then obtained by substituting equations (3.3), (3.4) and (3.5) into equations (4.5) and (4.6) which are in turn substituted into equation (4.7). To achieve this, a code is written in the MatLab programming environment.

5.0 Exact solution

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)=\frac{\Phi uc}{0.000264k}\cdot\frac{\partial p}{\partial t}$$

Let

$$\frac{\Phi uc}{0.000264k}=a=cons\tan t$$

Therefore

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)=a\frac{\partial p}{\partial t}$$

It is obvious that as $t\to\infty$, $\frac{\partial p}{\partial t}\to 0$, the steady state solution is obtained by solving the differential equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)=0$$

Applying the principles of integration by parts yields

$$\frac{1}{r}\left[\frac{\partial p}{\partial r}+r\frac{\partial^2 p}{\partial r^2}\right]=0$$

$$\frac{1}{r}\frac{\partial p}{\partial r}+\frac{\partial^2 p}{\partial r^2}=0$$

or

$$r\frac{\partial^2 p}{\partial r^2}+\frac{\partial p}{\partial r}=0 \qquad (5.1)$$

$$p(0,t)=p_o$$

and

$$\frac{\partial p}{\partial x}(0,t)=0$$

Solving the above differential equation with its boundary conditions yields

$$p(r)=p_o(1-r) \qquad (5.2)$$

The repeated use of equation (4.7) can cause the temporal approximation error to grow with time, depending on the value of α. The critical time step is given by

$$\Delta t_{cr}=\frac{2}{\lambda_{\max}}$$

where $\lambda_{\max}$ is the maximum eigenvalue associated with equation (2.12)

$$-\lambda[M]\{p\}+[K]\{p\}=0$$

For the model under scrutiny, this reduces to

$$-\lambda\frac{\pi}{40}p_2+\frac{\pi}{6}p_1=0 \text{ or } \lambda=\frac{20}{3}$$

This yields

$$\Delta t_{cr}=\frac{3}{10}$$

6.0 Results

Table 6.1: Variation of pressure of fluid with time at the various nodes

$t(s) \rightarrow$	0.1	0.5	1.0	5.0	10.0	15.0	$t \rightarrow \infty$	Steady state solution
$r = 0.25m$	0.95	0.92	0.89	0.84	0.81	0.78	0.75	0.75
$r = 0.50m$	0.80	0.72	0.68	0.64	0.61	0.57	0.50	0.50
$r = 0.75m$	0.62	0.59	0.52	0.49	0.41	0.36	0.25	0.25
$r = 1.0m$	0.36	0.32	0.28	0.21	0.17	0.11	0	0

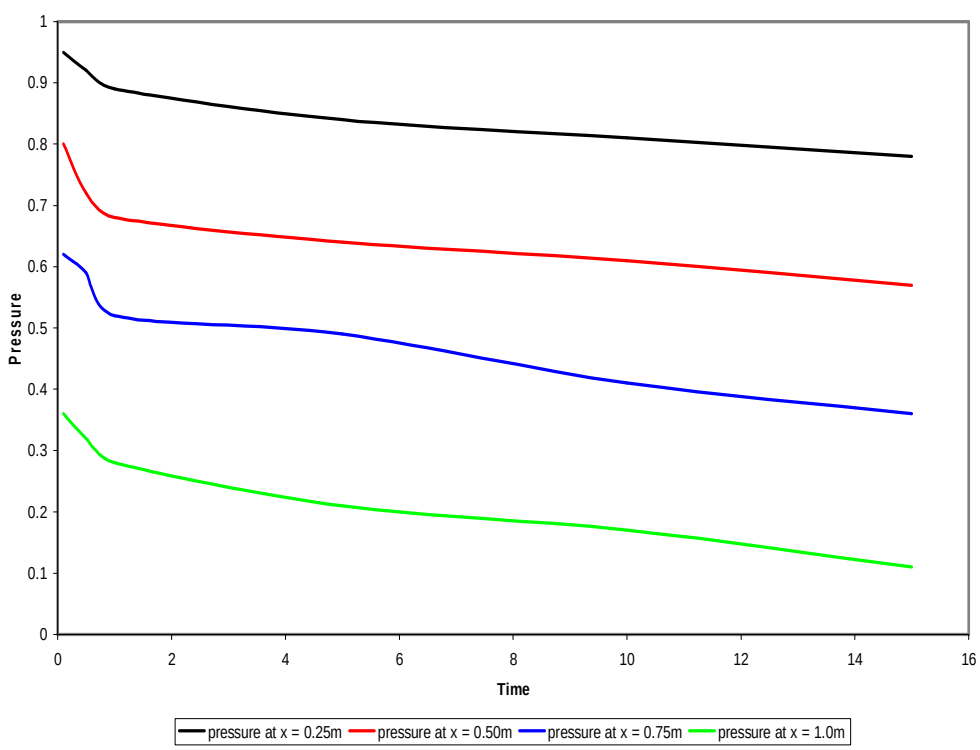

Figure 6.1: Graph showing the variation of fluid pressure with time at various points

6.1 Discussion of results

The graphs shown in figure 6.1 indicate that the pressure at the various nodes of interest tended towards the steady state solution which is an indication that the solutions are accurate and the method very robust.

The critical time step was computed as $\Delta t_{cr} = \frac{3}{10}$. Thus in order for the solution to be stable, the time step should be smaller than $\Delta t_{cr} = \frac{3}{10}$; otherwise the solution will be unstable. Hence to obtain a sufficiently accurate solution, the time step must be taken as a fraction of Δt_{cr}.

7.0 Conclusion

We have presented in this work an accurate model for solving the differential equations governing radial flow of slightly compressible fluids using the finite element-finite difference method which can thus be used in characterizing the behaviour of these flows.

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