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Stability of critical points in a resistant medium under the influence of electric and magnetic fields

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Abstract

In this work, we consider the stability of critical points in a resistant medium for a set of coupled, autonomous, non linear system of differential euations under the influence of electric and magnetic fields. The results obtained from associated linear system and auxillary equations of the critical point show that the critical points are either stable or saddle (unstable) or unstable pointsdepending on some conditions.

***Keywords*:** Stability, critical points, resistant medium, electric and magnetic field**.**

1.0 Introduction

In [1], Andreev considered the problem of asymptotic stability of non autonomous functional differential equation under the supposition that the derivative of the Lyapunov functional is a non negative scalar function. The results obtained modified and extended a number of well known results. In another paper [5], Gil formulated explicit stability for non linear retarded systems with separate autonomous linear parts in terms of the roots of the characteristic polynomials based on recent estimates of the matrix resolvent. He discussed the global stability, considered the estimates for the norm of the Green's function and then derive a bound for a region of attraction of the zero solution. In another work by Derricks and Grossman [4], they considered the stability of critical points for some nonlinear differential equations by obtaining the associated linear system for the differential equations and the auxiliary equation obtained from these associated linear system showed the stability of the critical points.

2.0 Mathematical formulation

The set of coupled, autonomous, non linear system of differential equations in [2] is:

$$\left.\begin{aligned} &\frac{dv}{dt} = \sin\phi + \beta v - av^2, v(0) = 1 \\ &v\frac{d\phi}{dt} = -\cos\phi, \phi(0) = \phi_0 \end{aligned}\right\} \qquad (2.1)$$

Where $v(t)$ is the velocity $\phi(t)$ *is the local path angle,* α is the electric field; β is the imposed magnetic field and t(time) is an independent variable.

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3.0 Method of solution

Let

$$\left.\begin{aligned} \frac{dv}{dt} &= f(v,\phi),\ v(0) = v_0 \\ \frac{d\phi}{dt} &= g(v,\phi),\ \phi(0) = \phi_0 \end{aligned}\right\} \tag{3.1}$$

be a system of equations as in (2.1).

If $f(v,\phi) = g(v,\phi) = 0$, we obtain the critical points for different α and β values. We assume that $f(v,\phi)$ and $g(v,\phi)$ posses continuous third parital derivative so that each can be expanded in a Taylor series up to second degree about a critical point, and then selecting the first degree terms of the expansion, we obtain an associated linear system. The auxiliary equation obtained from the associated linear system gives different λ values which will then be used to determine the stability of the critical points.

***Theorem* 3.1**

Let (v,ϕ) be a critical points of (2.1) for a particular α and β values and let λ_1 and λ_2 be two auxiliary roots corresponding to (v,ϕ). Then the critical point (v,ϕ) is said to be:

(i) Stable if λ_1 and λ_2 are real; distinct and of negative signs or if λ_1 and λ_2 are real equation and of negative signs.

(ii) Saddle (unstable) if λ_1 and λ_2 are real, distinct and of positive signs or if λ_1 and λ_2 are real equal and of positive signs.

4.0 Exact solution

With $\alpha = 1,\ \beta = \sqrt{5}$ in (2.1), the critical points are $\left(0.62, \pi/2\right)$ and $\left(1.62, \pi/2\right)$. Then the associated linear system for $\left(0.62, \pi/2\right)$ is

$$v' = f\left(0.62, \pi/2\right) + \frac{\partial f}{\partial v}\left(0.62, \pi/2\right)(v - 0.62) + \frac{\partial f}{\partial \phi}\left(0.62, \pi/2\right)\left(\phi - \pi/2\right) + \mathrm{K}$$

and

$$\phi' = g\left(0.62, \pi/2\right) + \frac{\partial g}{\partial v}\left(0.62, \pi/2\right)(v - 0.62) + \frac{\partial f}{\partial \phi}\left(0.62, \pi/2\right)\left(\phi - \pi/2\right) + \mathrm{K}\ .$$

Then selecting the first degree terms of the above expansions, we obtain

$$\begin{pmatrix} v' \\ \phi' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1.61 \end{pmatrix}\begin{pmatrix} v \\ \phi \end{pmatrix} + \begin{pmatrix} -0.62 \\ -0.81\pi \end{pmatrix} \tag{4.1}$$

Let $x = v' = v - 0.62$, $y = \phi' = 1.615\phi - 0.81\pi$ then

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\cdot 61 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{4.2}$$

The auxiliary equation $(1-\lambda_1)(1.61-\lambda_2)=0$ for (4.1) gives $\lambda_1=1$ and $\lambda_2=1\cdot 61$. Similarly the associated linear system for $\left(1.62, \pi/2\right)$ is

$$\begin{pmatrix} v' \\ \phi' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0\cdot 62 \end{pmatrix} \begin{pmatrix} v \\ \phi \end{pmatrix} + \begin{pmatrix} 1\cdot 62 \\ 0\cdot 31\pi \end{pmatrix} \tag{4.3}$$

Let $x=v'=-v+1\cdot 62,\ y=\phi'=0\cdot 62\phi+0\cdot 31\pi$, then

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1\cdot 62 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.4}$$

The auxiliary equation $(-1-\lambda_3)(0\cdot 62-\lambda_4)=0$ for (4.3) gives $\lambda_3=-1$ and $\lambda_4=0\cdot 62$. Also with $\alpha=1, \beta=-3$ in (2.1), the critical points are $\left(-2\cdot 62, \frac{\pi}{2}\right)$ and $\left(-0\cdot 38, \frac{\pi}{2}\right)$. The associated linear system for $\left(-2\cdot 62, \frac{\pi}{2}\right)$ is

$$\begin{pmatrix} v' \\ \phi' \end{pmatrix} = \begin{pmatrix} 2\cdot 24 & 0 \\ 0 & -0\cdot 38 \end{pmatrix} \begin{pmatrix} v \\ \phi \end{pmatrix} + \begin{pmatrix} 5\cdot 87 \\ 0\cdot 19\pi \end{pmatrix} \tag{4.5}$$

Let $x=v'=2\cdot 24v+5\cdot 87,\ y=\phi'=0\cdot 38\phi+0\cdot 19\pi$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2\cdot 4 & 0 \\ 0 & -0\cdot 38 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.6}$$

The auxiliary equation $(2\cdot 24-\lambda_5)(-0\cdot 38-\lambda_6)=0$ for (4.3) gives $\lambda_5=2\cdot 24$ and $\lambda_6=-0\cdot 38$. Similarly, the associated linear system $\left(0.38, \pi/2\right)$ is

$$\begin{pmatrix} v' \\ \phi' \end{pmatrix} = \begin{pmatrix} -2\cdot 24 & 0 \\ 0 & -2\cdot 63 \end{pmatrix} \begin{pmatrix} v \\ \phi \end{pmatrix} + \begin{pmatrix} -0\cdot 85 \\ 1\cdot 32\pi \end{pmatrix} \tag{4.7}$$

Let $x=v'=-2\cdot 24v-0\cdot 85,\ y=\phi'=-2\cdot 63\phi+1\cdot 32\pi$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2\cdot 24 & 0 \\ 0 & -2\cdot 63 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.8}$$

The auxiliary equation $(-2.24-\lambda_7)(-2.63-\lambda_8)=0$ for (4.7) gives $\lambda_7=-2.24$ and $\lambda_8=-2.63$

Moreover, with $\alpha=-3, \beta=1$ in (2.1), the critical points are $\left(-0.77, \pi/2\right)$ and $\left(0.44, \pi/2\right)$. The associated linear system for $\left(-0.77, \pi/2\right)$ is

$$\begin{pmatrix} v' \\ \phi' \end{pmatrix} = \begin{pmatrix} -3.62 & 0 \\ 0 & -0.30 \end{pmatrix} \begin{pmatrix} v \\ \phi \end{pmatrix} + \begin{pmatrix} -2.79 \\ 0.65\pi \end{pmatrix} \tag{4.9}$$

Let $x=v'=-3\cdot 62v-2\cdot 79,\ \ y=\phi'=-0\cdot 30\phi+0\cdot 65\pi$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -3.62 & 0 \\ 0 & -0.30 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.10}$$

The auxiliary equation $(-3.62-\lambda_9)\ (-0.30-\lambda_{10})=0$ for (4.9) gives $\lambda_9=-3.62$ and $\lambda_{10}=-0\cdot 30$.

In a similar manner, the associated linear system for $\left(0\cdot 44, \frac{\pi}{2}\right)$ is

$$\begin{pmatrix} v' \\ \phi' \end{pmatrix} = \begin{pmatrix} +3.64 & 0 \\ 0 & 2\cdot 79 \end{pmatrix} \begin{pmatrix} v \\ \phi \end{pmatrix} + \begin{pmatrix} -1\cdot 60 \\ -1\cdot 14\pi \end{pmatrix} \tag{4.11}$$

Let $x = v' = 3.64v - 1.60$, $y = \phi' = -2.79\phi - 1.14\pi$

$$\begin{pmatrix} v' \\ y' \end{pmatrix} = \begin{pmatrix} +3.64 & 0 \\ 0 & +2\cdot 79 \end{pmatrix} \begin{pmatrix} v \\ \phi \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.12}$$

The auxiliary equation $(+3\cdot 64 - \lambda_{11})(2\cdot 79 - \lambda_{12})$ for (4.11) gives $\lambda_{11} = -3.64$ and $\lambda_{12} = 2.79$.

5.0 Analysis of results

With $\alpha = 1,\ \beta = \sqrt{5}$ the critical points are $\left(0.62, \pi/2\right)$ and $\left(1.62, \pi/2\right)$ the auxiliary roots is $\lambda_1 = 1$ and $\lambda_2 = 1.61$ corresponding to the critical point $\left(0.62, \pi/2\right)$ are real, distinct and of positive sign. Hence the critical points $\left(0.62, \pi/2\right)$ is an unstable point. Similarly, the auxiliary roots $\lambda_3 = -1$ and $\lambda_4 = 0.62$ corresponding to the critical point $\left(1.62, \pi/2\right)$ are real, distinct and of opposite signs. Hence the critical point $\left(1.62, \pi/2\right)$ is a saddle (unstable) point. The same results were obtained for $\alpha = 1, \beta = 2$ or $\alpha = 2, \beta = 1$ or $\alpha = 1, \beta = -2$ or $\alpha = -2, \beta = 1$. However, when $\alpha = 1, \beta = -3$ or $\alpha = -3, \beta = 1$ there was a stable critical point. In this case, when $\alpha = 1, \beta = -3$, the critical points are $\left(-2.62, \pi/2\right)$ and $\left(-0.38, \pi/2\right)$. The auxiliary roots $\lambda_5 = 2.24$ and $\lambda_6 = -0\cdot 38$ corresponding to the critical point $\left(-2.62, \pi/2\right)$ are real, distinct and of opposite signs; hence the critical point $\left(-2.62, \pi/2\right)$ is a saddle (unstable) point whereas the auxiliary roots $\lambda_7 = -2.24$ and $\lambda_8 = -2.63$ corresponding to the critical point $\left(-0.38, \pi/2\right)$ are real, distinct and of negative sign; hence the critical point $\left(-0.38, \pi/2\right)$ is a stable point. A similar result was obtained when $\alpha = -3$, and $\beta = 1$ as in later part of Section (4) above. These results further show that both electric and magnetic fields have effect on the stability of critical points in a resistant medium which can then lead to any of the results above.

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