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Numerical methods for solving non-linear integral equations

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Abstract

This paper presents the numerical methods of Non-linear Integral Equations by Cubic Spline Collocation Tau Methods. Two Numerical collocation methods are applied to some nonlinear integral Equations after the non-linear Integral Equations have been linearized using Taylor's Series linearization scheme. Then the linearized Integral Equation is then evaluated for the purpose of comparison the computational cost, accuracy and the errors obtained for each method. The two numerical methods are Standard Cubic Spline Collocation Method (SCSCM) and Perturbed Cubic Spline Collocation Tau method (PCSCTM). These methods have been used by ref. [2] ***for solving singularly Perturbed second Order Differential Equations. Numerical examples are given which show that the errors obtained by PCSCTM are smaller than that of SCSCM.***

1.0 Introduction

We consider the general nonlinear volterra integral equation of the form

$$y(s) + \lambda \int_{x_{j-1}}^{x_j} k(x\,,\,t)\; y(t)\; dt = f(s) \tag{1.1}$$

Where λ is a scalar parameter, y(*s*) is an unknown function, *f*(*s*) is a given function and *k(x , t , y(t))* is the kernel which is always given. Many numerical techniques have been used successfully for equation (1.1) and in this section; we discussed in details a straight forward yet generally applicable technique the "Cubic Spline Collocation" method (see Ref. [1]). The Newton's scheme from the Taylor's series expansion may be represented by the first three terms of around (x_n , t_n , y_0) in the following form:

$$\begin{aligned} K(S,t,y) &= K(S_n\,,\,t_n\,,\,y_n) + (s - s_n)\frac{\partial K(S_n\,,\,t_n\,,\,y_n)}{\partial S} \\ &\quad + (t - t_n)\frac{\partial K(S_n\,,\,t_n\,,\,y_n)}{\partial t} + (y - y_n)\frac{\partial K(S_n\,,\,t_n\,,\,y_n)}{\partial y} \end{aligned} \tag{1.2}$$

By substituting equation (1.2) in equation (1.1), we obtain

$$\begin{aligned} y(s) + \lambda \int_{x_{j-1}}^{x_j} \Bigg[K(S_n\,,\,t_n\,,\,y_n) + (x - x_n)\;\frac{\partial K(S_n\,,\,t_n\,,\,y_n)}{\partial S} + \\ (t - t_n)\frac{\partial K(S_n\,,\,t_n\,,\,y_n)}{\partial t} + (y - y_n)\;\frac{\partial K(S_n\,,\,t_n\,,\,y_n)}{\partial y}\Bigg] dt = f(s)\;\; x_{j-1} \le S \le x_j \end{aligned} \tag{1.3}$$

The integral part of (1.3), t is an independent variable, y is a dependent variable, and s is a parameter, therefore by integrating equation (1.3) with respect to t, we obtain

$$y(s)+\lambda\int_{x_{j-1}}^{x_j}\frac{\partial K(S_n,t_n,y_n)}{\partial y}y(t)\,dt+\lambda\left[(s-s_n)\frac{\partial K(S_n,t_n,y_n)}{\partial x}\right.$$
$$\left.-y_n\frac{\partial K(S_n,t_n,y_n)}{\partial y}\right]\int_{x_j}^{s}(t-t_n)\,dt=f(s) \qquad (1.4)$$

Thus, equation (1.4) is our linearized form of equation (1.1). Now, it is convenient to begin by considering techniques based on the use of approximate Cubic Spline solution (see Ref [1]) on equation (1.4) to obtain.

$$y(s)+\lambda\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j}\frac{\partial K(S_n,t_n,y_n)}{\partial y}\left[M_{j-1}\frac{(t_j-t)^3}{6h}+M_j\frac{(t-t_{j-1})^3}{6h}+\left(y_{j-1}-\frac{h^2}{6}M_{j-1}\right)\right.$$
$$\left.\left(\frac{(t_j-t)}{h}\right)+y_j-\frac{h^2}{6}M_j\right)\frac{(t_j-t)}{h}\Bigg]dt$$
$$+\lambda\left[K(S_n,t_n,y_n)+(s-s_n)\frac{\partial K(S_n,t_n,y_n)}{\partial s}-y_n\frac{\partial K(S_n,t_n,y_n)}{\partial y}\right]=f(s) \qquad (1.5)$$

In equation (1.5), the integrals have to be evaluated.

2.0 Standard cubic spline collocation method (SCSCM)

Details of this method can be found in [3]. However, after the evaluation of the integrals in equation (1.5), the left over are then collocated at point $s = s_k$, hence equation (1.5) becomes

$$y(s_k)+\lambda\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j}\frac{\partial K(S_n,t_n,y_n)}{\partial y}\left[M_{j-1}\frac{(t_j-t)^3}{6h}M_j\frac{(t-t_{j-1})^3}{6h}+\left(y_{j-1}-\frac{h^2}{6}M_{j-1}\right)\right.$$
$$\left.\left(\frac{(t_j-t)}{h}+y_j-\frac{h^2}{6}M_j\right)(t_j-t)\right]dt=f(s_k) \qquad (2.1)$$

where

$$s_k=\frac{a+(b-a)\,i}{n+1}\ ;\ i=1,2,\dots,n+1 \qquad (2.2)$$

Thus, equation (2.1) gives a system of $(n + 1)$ linear algebraic equations, which together with the recursive Cubic Spline relation (see Ref. [1]) and the two end, conditions,

$$M_0 = M_n = 0 \qquad (2.3)$$

Altogether, comprise a complete system to solve for the $(2n + 2)$ unknowns

$$y_0, y_1, \dots, y_n, m_0, m_1 \dots, m_n$$

3.0 Perturbed cubic spline collocation method (PCSCTM)

In this method, after the evaluation of the integrals in equation (1.5), again, the left over are then slightly perturbed to give

$$y(s)+\lambda\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j}\frac{\partial K(S_n,t_n,y_n)}{\partial y}\left[M_{j-1}\frac{(t_j-t)^3}{6h}+M_j\frac{(t-t_{j-1})^3}{6h}\right.$$
$$\left.+\left(y_{j-1}-\frac{h^2}{6}M_{j-1}\right)\left(\frac{(t_j-t)}{h}+y_j--\frac{h^2}{6}M_j\right)(t_j-t)\right]dt=f(s)+H_n(s) \tag{3.1}$$

where $H_n(s)=\sum_{i=1}^{n}Y_i\,T_{n-i+1}(s)\ and\ T_n(s)$

is the Chebyshev polynomial of degree n valid in $[a,b]$ is defined by

$$T_n(s)=\cos\left[n\cos^{-1}\left\{\frac{2s-b-a}{b-a}\right\}\right];\ a\le s\le b \tag{3.2}$$

and it satisfies the recurrence relation

$$T_{n+1}(s)=2\left\{\frac{2s-b-a}{b-a}\right\}T_n(s)-T_{n-1}(s)$$

The Chebyshev polynomial oscillates with equal amplitude in the range under consideration and this makes the Chebyshev polynomial more suitable in function approximation problems.

Thus, equation (3.1) are then collocated at point $s = s_k$, hence, equation (3.1) becomes

$$y(s_k)+\lambda\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j}\frac{\partial K(S_n,t_n,y_n)}{\partial y}\left[M_{j-1}\frac{(t_j-t)^3}{6h}+\frac{M_j(t-t_{j-1})^3}{6h}\right.$$
$$\left.+\left(y_{j-1}-\frac{h^2}{6}M_{j-1}\right)\left(\frac{(t_j-t)}{h}+y_j-\frac{h^2}{6}M_j\right)(t_j-t)\right]dt=f(s_k)+H_n(s_k) \tag{3.3}$$

where,

$$s_k=a+\frac{(b-a)k}{N+2},\ k=1,2,\dots,N+2$$

Thus, equation (3.3) gives rise to a system of $(N + 2)$ linear equations, which together with the recurrence Cubic Spline (see Ref. [1]) and the two end conditions in equation (2.3). altogether, comprise a complete system to solve for the $3(N + 1)$ unknowns $y_0, y_1,\dots,y_n$ $m_0, m_1, m_2\dots,\tau_0, \tau_1,\dots,\tau_N$.

The error is defined by

$$error=\max_{a\le x_j\le b}\left|y(x_j)-y_{i,N}\right|$$
$$x_j=a+\frac{(b-a)j}{100};\ j=0,1,\dots,99,100.$$

where $y(x)$ is given in a closed form.

4.0 Numerical results and discussion

We consider two test examples

Example **4.1**

Consider the nonlinear weekly singular Volterra integral equation of the form:

$$y(x) = \frac{-x^4}{10} + \frac{5}{6}x^2 + 3/8 + \int_0^x \frac{1}{2x} y^2(t)\,dt; \; x \in [0,1] \tag{4.1}$$

with exact solution $y(x) = x^2 + \frac{1}{2}$ and singular point $x_0 = 0$, following the linearization techniques discussed earlier, the above nonlinear in equation (4.1) will be reduced to the linear Volterra integral equation of the form:

$$y(x) = \frac{-x^4}{10} + \frac{5x^2}{6} + \frac{3}{8} - \frac{y_n^2}{2x_{n+1}}\left[1 + \frac{x - x_{n+1}}{x_{n+1}}\right]x + \frac{y_n}{x_{n+1}}\int_0^x \frac{1}{2x} y^2(t)\,dt, \; x_0 \le x \le x_{n+1} \tag{4.2}$$

Table 4.1: Error for Example 4.1 at the seven iteration for case N = 5.

h	x	Standard Cubic Spline Collocation Method	Perturbed Cubic Spline Collocation Method
$1/5$	$1/5$	1.043216×10^{-4}	4.3289156×10^{-6}
	$2/5$	2.0048299×10^{-4}	3.8923452×10^{-6}
	$3/5$	5.6721891×10^{-4}	5.6328731×10^{-6}
	$4/5$	8.7159238×10^{-4}	6.0123494×10^{-6}
	$5/5$	8.9672320×10^{-4}	6.4349732×10^{-6}

Table 4.2: Errors for Example 4.1 at the seven iterations for case N = 8

h	x	Standard Cubic Spline Collocation Method	Perturbed Cubic Spline Collocation Method
$1/8$	$1/8$	7.0247320×10^{-4}	2.6732954×10^{-8}
	$2/8$	4.9346722×10^{-3}	3.692936×10^{-8}
	$3/8$	3.8134567×10^{-3}	3.942349×10^{-8}
	$4/8$	3.2173298×10^{-3}	4.632185×10^{-8}
	$5/8$	3.004329×10^{-3}	5.0421173×10^{-8}
	$6/8$	2.994238×10^{-3}	6.3347213×10^{-8}
	$7/8$	1.8739256×10^{-3}	6.0034321×10^{-8}
	$8/8$	1.623459×10^{-3}	$7.00034997 \times 10^{-8}$

Example **4.2**

The nonlinear Volterra integral equation

$$y(x) = \exp(x) - \frac{1}{2}(\exp(2x) - 1) + \int_0^x y^2(t)dt; \quad x \in [0 , 1] \tag{4.3}$$

with exact solution y(x) = exp(x).
The linearized form of equation (4.3) is of the form:

$$y(x) = \exp(x) - \frac{1}{2}(\exp(2x) - 1) - y_n^2 x + 2y_n \int_0^x y(t)dt; \; x_0 \le x \le x_{n+1} \tag{4.4}$$

Table 4.3: Error for Example 4.2 at the seven iterations for case *N* = 5.

h	*x*	Standard Cubic Spline Collocation Method	Perturbed Cubic Spline Collocation Method
$1/5$	$1/5$	1.072437×10^{-5}	1.033425×10^{-7}
	$2/5$	$8.992639 \text{ X } 10^{-4}$	1.432119×10^{-7}
	$3/5$	7.8872351×10^{-4}	6.734892×10^{-6}
	$4/5$	4.8993281×10^{-4}	9.665928×10^{-6}
	$5/5$	9.004927×10^{-4}	9.721351×10^{-6}

Table 4-4: Error for Example 4.2 at the seven iterations for case *N* = 8.

h	*x*	Standard Cubic Spline Collocation Method	Perturbed Cubic Spline Collocation Method
$1/8$	$1/8$	4.342189×10^{-4}	1.0563289×10^{-9}
	$2/8$	6.349231×10^{-4}	2.6234951×10^{-9}
	$3/8$	6.994327×10^{-4}	2.4213895×10^{-9}
	$4/8$	8.0021635×10^{-4}	3.6789348×10^{-9}
	$5/8$	1.9967387×10^{-3}	5.0072381×10^{-9}
	$6/8$	2.0045931×10^{-3}	5.732895×10^{-9}
	$7/8$	4.6623996×10^{-3}	6.1132819×10^{-9}
	1	5.843218×10^{-3}	7.0004328×10^{-9}

5.0 Conclusion

In this paper, two collocation methods namely, standard cubic spline collocation and perturbed cubic spline collocation methods (which is well known as orthogonal) are examined for solving nonlinear Integral Equations.

Tables 4.1 – 4.4 show the numerical solutions in terms of the errors obtained for the two nonlinear integral equations at the seven iterations. It is observed that the perturbed cubic spline collocation method converges faster than the standard cubic spline collocation method. The SCSCM and PCSCM involves large matrix system of algebraic equations of different degrees. It is interesting to compare the accuracy and the cost of computation involved in the two methods. In table 4.1, for example, for the case $N = 5$, SCSCM involved 12 systems of algebraic equations with maximum error 8.967232 E – 04 while the PCSCM involved 18 systems of algebraic equation with maximum error 6.4349732E-06. in the case of PCSCM, extra work and computations are involved and these are compensated for in terms of the errors obtained. We also observed that as N increases, PCSCM converges faster in all cases considered.

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