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On the wave equations of shallow water with rough bottom topography

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Abstract

We had earlier investigated the case of Shallow Water flow over a bottom with rough topography. However, in the solution and thus the graphs shown, we had only considered the real parts of the phase speed, C. In this present study, we included these neglected imaginary or complex parts as parts of the value of the phase speed so as to see its impact on the flow. With this therefore, we saw that the flow form in terms of the velocity and wave profile did not resemble at all. This therefore shows that the complex nature of the phase speed of the flow must be retained so as to get the true nature of the flow characteristics and pattern whenever any flow analysis has to be carried out about shallow water flow.

1.0 Introduction

For quite some time, Shallow water problem in fluid dynamics have attracted the attentions of researchers. In most of these works, assumptions were made of simpler bottom topography of such shallow waters. This is simply to avoid the complex nature of the physical features peculiar to such shallow water. However, it is these physical features that controls the mechanism of wave propagation and probably its breaking along the line of flow, development of singularities along the line of flow and where possible, turbulence.

It is these important qualities of the bottom topography that led to our attempt to model the real nature of the bottom of shallow water. In our study therefore, the analytical study of the development were based mainly on the traditional shallow water approximations with some little modifications.

Okeke (1983 [1]) and Okeke (1985 [2]) had worked on shallow water waves where he assumed that the shallow water has slopping bottom. He equally assumed the waves to be linear which in reality is not true. In our work, we assumed that the wave is generally non-linear and considered two cases where it is dispersive and non-dispersive.

In his later work, Okeke (1985 [2]), he even assumed that such shallow water has constant Water depth. A comparison of the results we obtained faired very favourably with Okeke (1983 [1]) if we assume the terms of the model the way he did.

To be able to obtain a closed solution for the water wave, we used the expression for the water depth as we obtained in the non linear dispersive wave train of the shallow water model here. Peregrine (1972 [3]), Whitham (1975 [4]), Leibovich (1974 [5]) and Okeke (1983 [1]) were very helpful in the understanding of the assumptions made so as to obtain reasonable solution. Also, the solution we obtained in Mbah and Ezeorah (2007 [6]) which is redeveloped here showed a very good result which on our attempt on including the complex part of C in the graphing, gave us a totally different set of graphs. It is this behaviour that necessitated this current study.

2.0 A Model of the bottom topography of shallow water.

In Mbah and Ezeorah (2007 [6]), we had considered the general bottom topography of Shallow water flow as:

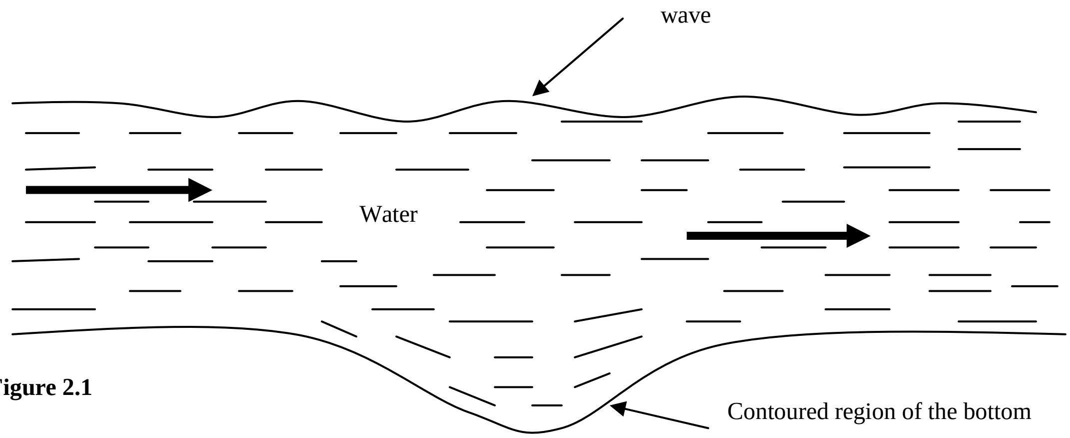


Figure 2.1

We had considered the contoured region as below:

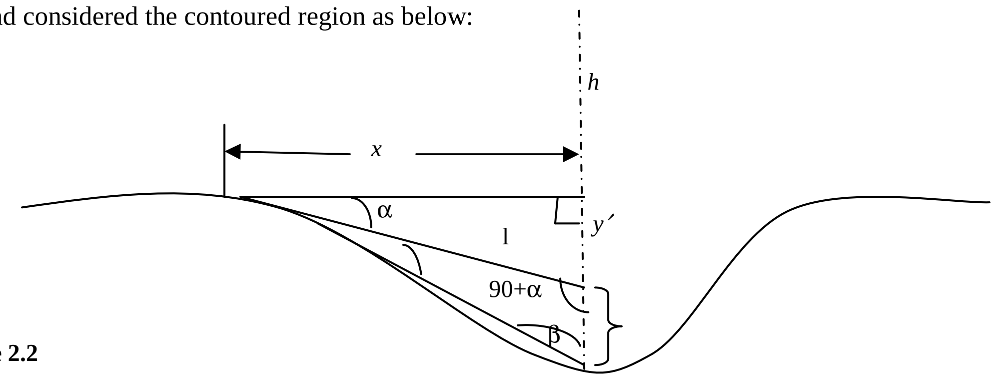


Figure 2.2

From this diagram, we obtained the general expression for the height of the water at all points of the bottom of flow as:

$$H = y' + \delta y' + h \Rightarrow H = x\tan\alpha + x\{\cot\beta - \tan\alpha\} + h = h + x\cot\beta \qquad (2.1)$$

where h = height at the level bottom (no contour), x = the width of the contoured region and β is the angle of inclination of the center of the contoured region with respect to the starting point of the width ($x = 0$). In this case, we have assumed that β is not a function of x so that we consider it constant. Later, we shall study a case where the angle β is a function of x.

3.0 Governing equations for shallow water waves with rough bottom topography

Our development of the model here is based essentially on the shallow water equation governing the weakly non-linear waves on the surface of the shallow water with rough bottom topography. We shall incorporate the vertical component of water particles and the related energy transfer as the wave progresses. The effect of linear dispersion is included as against what mostly obtains in literatures.

Thus, we take the x–axis as the horizontal and normal to shore line, the z-axis is the vertically upward direction where $z = \eta(x, t)$ represents the wave profile occurrence on the water surface, $z = h$ is the constant water depth of the shallow water as measured from the undisturbed water level. Thus, the equation governing the evolution of using $H = h + x \cot\beta$ as:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[U(h + x\cot\beta)\right] = O \text{ , } \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} + \lambda\frac{\partial^3 u}{\partial x^3} = \frac{(h + x\cot\beta)^2}{3}\left(\frac{\partial^3 u}{\partial x^2 \partial t}\right) \quad (3.1)$$

Here $u(x,t)$ is the component of the particle velocity along the x – axis for t > 0 as the time. λ is the constant associated with the wave dispersion in the shallow water, g is the constant due to gravity with

the term
$$\frac{\left(h + x\cot\beta^2\right)}{3}\frac{\partial^3 u}{\partial x^{2\partial t}} \quad (3.2)$$
representing the vertical velocity effect on the horizontal velocity as well as the pressure.

3.1 Solution Procedure

Let us define another function w(x,t) such that:

$$U = \frac{1}{htx\cot\beta}\frac{\partial w}{\partial t} \quad (3.3)$$

and
$$\eta = -\frac{\partial w}{\partial x} \quad (3.4)$$

and they satisfy equations (3.1) and (3.2) when substituted into them. Thus, substituting (3.3) and (3.4) into (3.1) and (3.2) have both satisfied provided that

$$\frac{1}{h + x\cot\beta}\frac{\partial^2 w}{\partial t^2} + \frac{1}{h + x\cot\beta}\left\{\frac{1}{h + x\cot\beta}\left[\frac{\partial^2 w}{\partial x \partial t} - u\cot\beta\right]\right\} - g\frac{\partial^2 w}{\partial x^2} + \lambda$$
$$\left\{\frac{1}{h + x\cot\beta}\frac{\partial^4 w}{\partial x^3 \partial t} - \frac{3\cot\beta}{(h + x\cot\beta)^2}\frac{\partial^3 w}{\partial x^2 \partial t} + \frac{2\cot^2\beta}{(h + x\cot\beta)^3}\frac{\partial^2 w}{\partial x \partial t} - \frac{6u\cot^2\beta}{(h + x\cot\beta)^3}\right\}$$
$$= \frac{(h + x\cot\beta)^2}{3}\left[\frac{1}{h + x\cot\beta}\frac{\partial^4 w}{\partial x^2 \partial t^2} - \frac{2\cot\beta}{(h + x\cot\beta)^2}\frac{\partial^3 w}{\partial x \partial t^2}\right] \quad (3.5)$$

Equation (3.5) simplifies to

$$\frac{\partial^2 w}{\partial t^2} + \lambda\frac{\partial^2 w}{\partial x^3 \partial t} - \frac{3\lambda\cot\beta}{h + x\cot\beta}\frac{\partial^2 w}{\partial x^2 \partial t} + \frac{2\lambda\cot^2\beta}{(h + x\cot\beta)^2}\frac{\partial^2 w}{\partial x \partial t} + \frac{1}{h + x\cot\beta}\frac{\partial^2 w}{\partial x^2 \partial t}$$
$$- g(h + x\cot\beta)\frac{\partial^2 w}{\partial x^2} - \frac{u\cot\beta}{h + x\cot\beta} - \frac{6u\lambda\cot^3\beta}{(h + x\cot\beta)^2} = \frac{(h + x\cot\beta)}{3}\frac{\partial^4 w}{\partial x^2 \partial t^2} \quad (3.6)$$
$$- 2(h + x\cot\beta)\cot\beta\frac{\partial^3 w}{\partial x \partial t^2}$$

Since periodic wave trains are possible in nonlinear dispersive system, let us define a characteristic curve $\Gamma : \xi = x - ct$ which is the phase of the oscillation propagating with constant phase speed c and wave number k such that we can write $w = w(\xi)$ (3.7)

Now let us make the substitution $r = h + x\cot\beta.$ (3.8)

Then using equation (3.7) and (3.8) in equation (3.6), we obtain

$$-\left\{\lambda c+\frac{r^2c^2}{3}\right\}\frac{d^4w(\xi)}{d\xi^4}+\left\{\frac{3\lambda c}{r}\cot\beta+\frac{2r^2c^2}{r}\cot\beta\right\}\frac{d^3w(\xi)}{d\xi^3}+\left[c^2-gr-\frac{2\lambda c\, cot^{2\beta}}{r^2}-\frac{c}{r}\right]$$

$$+\left\{\frac{c}{r^2}\cot\beta+\frac{\partial\tau}{r2}\cot^3\beta\right\}\frac{\partial w[\xi]}{\partial\xi}=0 \qquad (3.9)$$

In equation (3.9), the non-linear term is included so as to know its effect no matter how small on the phase speed. Thus, we shall obtain the phase speed c for the two cases:

(1) where the dispersive effect λ is neglected that is, $\lambda = o$

(2) where the dispersive effect λ is not neglected, that is, $\lambda \neq o.$

Case 1

If λ= 0 and we explicitly define $w(\xi)=Ae^{ik\xi}$ then equation (3.9) reduces to

$$\frac{Ak^4r^2c^2}{3} - \frac{2iAk^3r^2c^2}{r}-Ak^2c^2+Ak^2gr+\frac{2\lambda cAk^2}{r2}\cot^2\beta+\frac{Ack^2}{r}+\frac{iAck\cot\beta}{r^2}=0$$

that is, $c^2\left[\frac{Ak^4r^2}{3}-2iAk^3r-Ak^2\right]$ + c $\left[\frac{Ak^2}{r}+\frac{iAk}{r2}\cot\beta\right]$ + $Ak^2gr=0$

$$\Rightarrow \quad c^2\left[\frac{k^3r^2}{3}-2ik^2r-k\right] + c\left[\frac{k}{r}+\frac{i\cot\beta}{r^2}\right]+grk=0,$$

Solving for *c*, we obtain

$$c=\frac{-\left[\frac{k}{r}+\frac{i\cot\beta}{r^2}\right]\pm\left\{\left(\frac{k}{r}+\frac{i\cot\beta}{r^2}\right)^2-4grk\left[\frac{k^3r^2}{3}-2ik^2r-k\right]\right\}^{1/2}}{2\left[\frac{k^3r^2}{3}-2ik^2r-k\right]}$$

In shallow water wave, only low wave numbers are involved so that we can neglect $k^4$and k^3 to obtain

$$c=\frac{-\left[\frac{k}{r}+\frac{i\cot\beta}{r^2}\right]\pm\left\{\left(\frac{k}{r}+\frac{i\cot\beta}{r^2}\right)^2+4grk^2\right\}^{1/2}}{-2\left[2ikr+1\right]k} \qquad (3.10)$$

Case II

We consider here a non-linear dispersive wave in which now $\lambda\neq 0.$ Thus substituting for w(ξ) in equation (9), we obtain

$$-\left\{\lambda c+\frac{r^2c^2}{3}\right\}(Ak^4)+\left\{\frac{3\lambda c}{r}\cot\beta+\frac{2r^2c^2\cot\beta}{r}\right\}\left\{-iAk^3\right\}+$$

$$\left\{c^2-gr-\frac{2\lambda c}{r^2}\cot^2\beta-\frac{c}{r}\right\}(-Ak^2)+\left\{\frac{c}{r^2}\cot\beta+\frac{6\lambda}{r^2}\cot^3\beta\right\}(iAk)=0.$$

$$\Rightarrow \qquad c^2\left[-\frac{Ak^4r^2}{3}-\frac{2iAr^2k^3}{r}\cot\beta-Ak^2\right]+$$

$$c\left\{-\lambda Ak^4-\frac{3\lambda iAk^3}{r}+\frac{2\lambda Ak^2}{r^2}\cot^2\beta+\frac{Ak^2}{r}+\frac{iAk}{r^2}\cot\beta\right\}+grAk^2+\frac{6\lambda iAk}{r^2}\cot^3\beta=0$$

Provided the roughness of the bottom of the shallow water is not a step slope, we can neglect $Cot^3\beta$ since in this case β is large. Hence, the above equation reduces to

$$c^2\left\{\frac{k^3r^2}{3}+2irk^2\cot\beta+k\right\}+c\left[\lambda k^3+\frac{3\lambda ik^2}{r}-\frac{2\lambda k}{r^2}\cot^2\beta-\frac{k}{r}-\frac{i\cot\beta}{r2}\right]-grk=0.$$

We can solve this equation to obtain a value for c which on neglecting k^3 and k^4 gives:

$$c=\frac{-\left[\frac{3\lambda ik^2}{r}-\frac{2\lambda k}{r^2}\cot^{2\beta}-\frac{k}{r}\frac{i\cot\beta}{r^2}\right]+\left\{\left(\frac{3\lambda ik^2}{r}-\frac{2\lambda k}{r^2}\cot^{2\beta}-\frac{k}{r}-\frac{i\cot\beta}{r^2}\right)^2+4grk\left[2irk^2\cot\beta+k\right]\right\}^{1/2}}{2\left[2irk^2\cot\beta+k\right]}$$

Since k is small and β is large, we readily see that the value of c obtained in this case II is greater than that obtained in case 1. If we had linearised the wave, we would obtain c = $\left\{g\left[h+x\cot\beta\right]\right\}^{1/2}$.

Therefore, for this non-linear dispersive wave, c as obtained is greater than the c for a linear wave speed given by c = $\left\{g\left(h+x\cot\beta\right)\right\}^{1/2}$ and this is as a result of additional terms that depended on the wave amplitude. Hence, we shall adopt that c in our case here is greater than $\left\{g\left(h+x\cot\beta\right)\right\}^{1/2}$ so that we will able to solve equation (3.9) to then obtain values for the velocity and the wave profile. Hence, integrating equation (3.9) once and neglecting the constant of integration, we obtain:

$$\left\{\lambda c+\frac{r^2c^2}{3}\right\}\frac{d^3w(\xi)}{d\xi^3}-\left\{\frac{3\lambda c}{r}\cot\beta+2rc^2\cot\beta\right\}\frac{d^2w(\xi)}{d\xi^2}-\left\{c^2-gr-\frac{2\lambda c}{r2}\cot^2\beta-\frac{c}{r}\right\}\frac{dw(\xi)}{d\xi}-$$

$$\left\{\frac{c}{r^2}\cot\beta+\frac{6\lambda}{r2}\cot^3\beta\right\}=0. \qquad (3.11)$$

Suppose we let $P=\dfrac{dw(\xi)}{d\xi}$ and then substitute in the equation above. Then we have a second order non-homogenous differential equation with constant coefficients given as :

$$\left\{\lambda c+\frac{r^2c^2}{3}\right\}\frac{d^2p}{d\xi^2}-\left\{\frac{3\lambda c}{r}\cot\beta+\frac{6\cot\beta}{r}\right\}\frac{dp}{d\xi}-\left\{c^2-gr-\frac{2\lambda c}{r2}\cot^{2\beta}-\frac{c}{r}\right\}p-\left\{\frac{c}{r^2}\cot\beta\,\&\,\frac{6\lambda}{r2}\cot^3\beta\right\}=0 \qquad (3.12)$$

To simplify this equation further, let us make some substitutions like:

$$\sigma = \lambda c + \frac{r^2 c^2}{3}, \quad \rho = \frac{3\lambda c}{\partial}\cot\beta + \frac{6\cot\beta}{r,}, \quad \tau = c^2 - gr - \frac{2\lambda c}{r^2}\cot^2\beta - \frac{c}{r}$$

$$and \quad \delta = \frac{c}{r^2}\cot\beta + \frac{6\lambda}{r^2}\cot^3\beta.$$

With these, we get equation (3.12) as: $$\frac{\partial^2 P}{\partial \xi^2} - \rho\frac{\partial p}{\partial \xi} - \tau p = \delta \qquad (3.13)$$

and this is non-homogenous. When solved, we get the complementary solutions as:

$$m = \frac{\frac{\rho}{\sigma} \pm \left\{\left(\frac{\rho}{\sigma}\right)^2 + 4\frac{\tau}{\sigma}\right\}^{1/2}}{2} = \frac{\rho}{2\sigma} \pm \frac{1}{2}\left\{\left(\frac{\rho}{\sigma}\right)^2 + 4\frac{\tau}{\sigma}\right\}^{1/2}$$

$$\Rightarrow \quad P_{cf} = 1^{\frac{\rho}{\sigma}\xi i}\left\{A\cos \tfrac{1}{2}\left(\left(\frac{\rho}{\sigma}\right)^2 + 4\frac{\tau}{\sigma}\right)^{1/2}\right\}\xi + B\sin\left\{\tfrac{1}{2}\left(\left(\frac{\rho}{\sigma}\right)^2 + 4\frac{\tau}{\sigma}\right)^{1/2}\right\}\xi \qquad (3.14)$$

We can equally solve for the particular part solution of the equation (3.13) to get that

$$P_{pI} = -\delta/\tau. \qquad (3.15)$$

Thus, the general solution of equation (3.13) is:

$$P = P_{cf} + P_{pI}$$

$$\Rightarrow p = 1^{\frac{\rho}{2\sigma}\xi i}\left[A\cos\left\{\frac{1}{2}\left(\left(\frac{\rho}{\sigma}\right)^2 + 4\frac{\tau}{\sigma}\right)^{\frac{1}{2}}\right\}\xi + B\sin\left\{\frac{1}{2}\left(\left(\frac{\rho}{\sigma}\right)^2 + 4\frac{\tau}{\sigma}\right)^{\frac{1}{2}}\right\}\xi\right] - \frac{\delta}{\tau} \qquad (3.16)$$

But $P = \dfrac{dw(\xi)}{d\xi} \quad and \quad U = -\dfrac{c}{r}\dfrac{dw(\xi)}{d\xi},$

then we have that

$$U = -\frac{c}{r}1^{\frac{\rho}{2\sigma}\xi i}\left[ACos\left\{\frac{1}{2}\left(\left(\frac{\rho}{\sigma}\right)^2 + 4\frac{\tau}{\sigma}\right)^{\frac{1}{2}}\right\}\xi + BSin\left\{\frac{1}{2}\left(\left(\frac{\rho}{\sigma}\right)^2 + 4\frac{\tau}{\sigma}\right)^{\frac{1}{2}}\right\}\xi - \frac{\delta}{\tau}\right] \qquad (3.17)$$

$$\eta = \frac{\partial w(\xi)}{\partial \xi} = \rho$$

so that also,

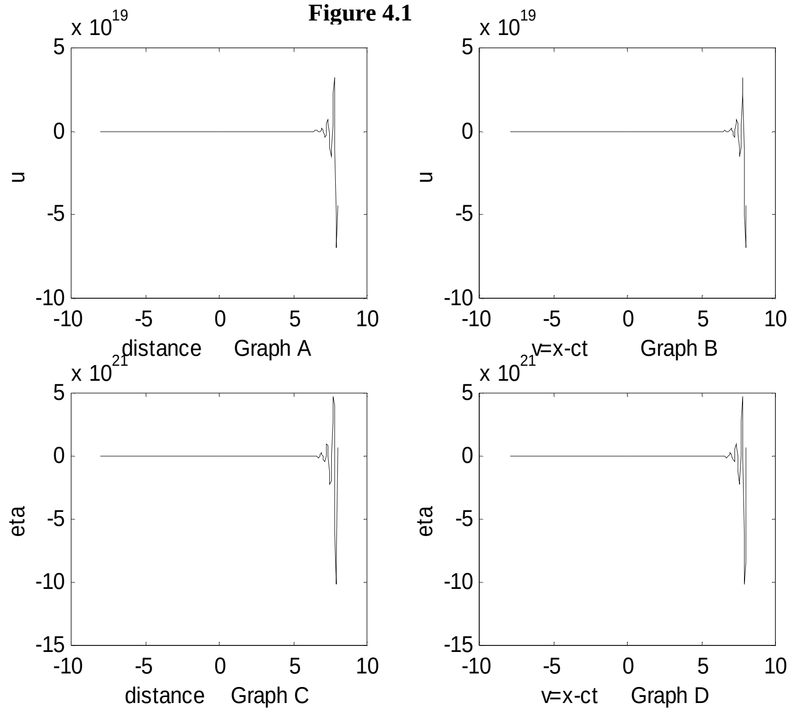
$$\eta = \lambda^{\frac{\rho}{2\sigma}\xi i}\left[A\cos\left\{\frac{1}{2}\left(\left(\frac{\rho}{\sigma}\right)^2 + 4\frac{\tau}{\sigma}\right)^{\frac{1}{2}}\right\}\xi + B\sin\left\{\frac{1}{2}\left(\left(\frac{\rho}{\sigma}\right)^2 + 4\frac{\tau}{\sigma}\right)^{\frac{1}{2}}\right\}\xi - \frac{\delta}{\tau}\right] \qquad (3.18)$$

Equation (3.17) and (3.18) the expression for the velocity and the wave profile of the shallow water wave over rough bottom topography as shown in the figures below.

4.0 Analysis and discussion of the result.

In this analysis, we shall look at what happens to the phase speed c for both the dispersive and the non-dispersive cases. The graphs in figures 1 & 2 so presented therefore reflect the effect of the dispersive term λ on the flow velocity as well as on the wave profile. Particularly, figure 1 is for the case when the flow is dispersive and shown , for chosen values for the constants, as:

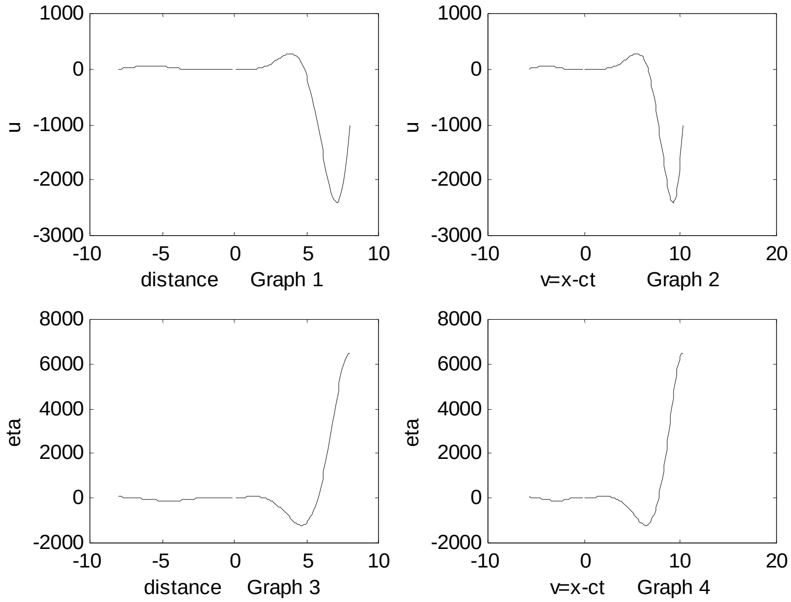
Figure 4.1



For the same values of the constants, we have the case where the flow is non-dispersive as:

Figure 4.2

Figure 2



Comparing these two figures, we can see the great effect of dispersiveness on the flow system. Even when we compare these two figures with what obtained in our former work, Mbah & Ezeorah (2007), where we considered only the real parts of the phase speed, there is no much resemblance in the flow graphs. This goes to show that dispersiveness of the flow system has much effect on the flow field and results and equally accepting only the real parts of the phase speed in analysis may not give the true situation in the flow results or characteristics.

Equally tested in this analysis is the effect of depth of the water on the dispersiveness of the flow. Thus, retaining the values for the parameters but only varying the angle of the contoured region from 0.5 t0 0.85 will give us the following graphs shown in Figures 3 & 4.

Figure 4.3

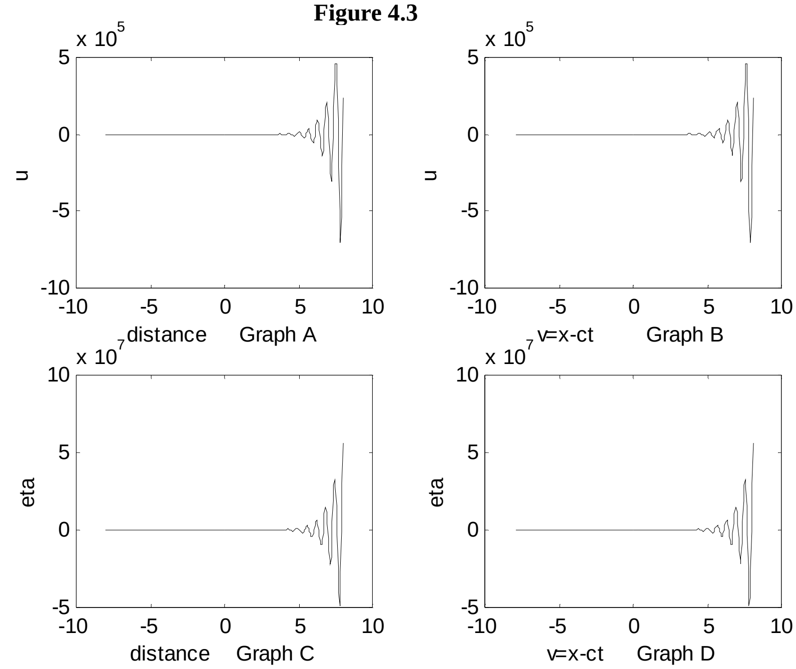
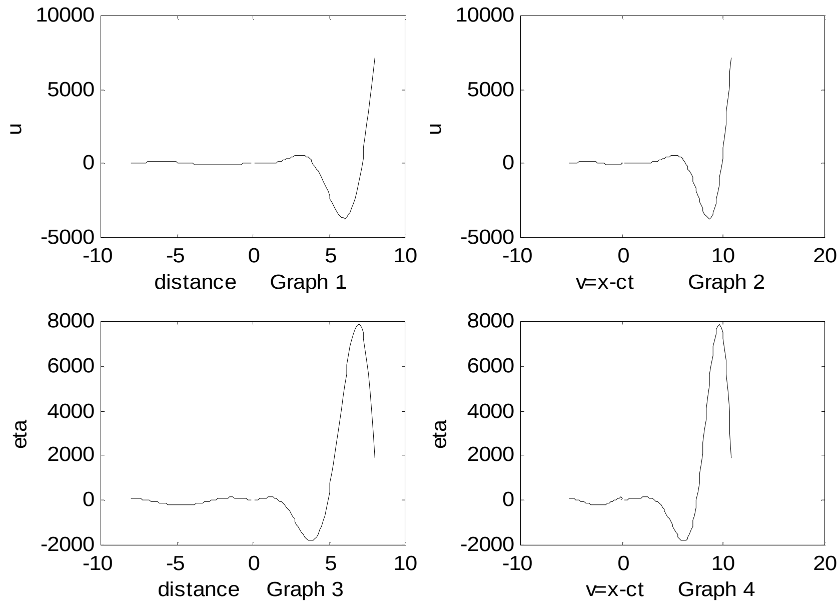


Figure 4.4



From comparison of Figures 4.1, 4.2 and Figures 4.3, 4.4, we can see clearly the effect of the height of the water on the dispersiveness of the flow. Thus, when the water is very shallow, h very small, we expect less effect of the dispersiveness on the flow.

In general, when we consider the same cases as we considered in Mbah & Ezeorah (2007 [6]) and Mbah (2007 [7]) we get the same observations although the graphs are not the same showing the effect of the complex (non-real) nature of the phase speed. Thus we conclude that a true study of the effect of the dispersive term on the flow mechanism of shallow water must consider the phase speed as complex number. Also, as shown and will be shown in our subsequent works, one must state clearly the flow parameters before going on to state the result of ones investigations. Shallow water flow is very common and clear understanding of the mechanism of its flow is necessary. It is always found in our Agricultural practices and in erosion controls in terms of channel flows and therefore very relevant in our day to day activities.

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