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Duffing oscillator as model for predicting earthquake occurrence I

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Abstract:

We consider the continental crust under damage. Using the observed results of microseism in many seismic stations we study a model – Duffing oscillator - which shows the same similarity with the time series of prevailing seismic waves that signals the occurrence of earthquakes in many earthquake prone areas of the world. In our study we use two models – the case where we include the effect of noise and the case where we consider the coda waves as the dominant force. We exhibit interesting results and their significance to earthquake prediction.

***Keywords*: microseism, damage mechanics, coda waves, Duffing oscillator, noise, resonance.**

***Classification*: 74A60, 74J15, 74L05.**

1.0 Introduction

In many experimental studies of rock fracture for example Eberhardt (1998 [6]) the presence of backscattered noise is established with or without acoustic event. Nawa et al. (1998 [16]) discovered the existence of incessant excitation of the Earth's free oscillations in absence of earthquakes.

From the installation of the first seismic stations, it has been widely observed that, in the absence of earthquakes, the seismic records display the presence of a continuous ground motion, of variable amplitudes. It is obvious today and it is well established that this permanent activity is due to the combined activities of atmospheric and oceanic as well as human activities. This continuous activity is known as microseism or seismic noise. Macia et al (2003 [13]) showed also that the base level noise spectrum can be interpreted as the resonant response of the solid Earth to atmospheric and oceanic activities.

As espoused by Thompson and Margetan (2002 [25]) when a flaw is present a distribution of signals in the presence of noise should be considered. Within the single scattering model we can consider the total signal to be the linear, phased superposition of the noise signals (formed by the randomly phased sum of the contribution of many individual flaws) and the flaw signal. Under certain assumptions of instantaneous noise distribution normally, envelope detection of narrow band signals, the envelope r of the superposition of signal plus noise is known to obey a Rician probability density functional (Rice 1944, Yalda et al 1998, Haykin 1994 [7, 23, 28],) given by

$$f(r)=\left(\frac{r}{\sigma^2}\right)e^{-\frac{1}{2}\frac{\left(r^2+A^2\right)}{\sigma^2}}I_0\left(\frac{rA}{\sigma^2}\right) \qquad (1.1)$$

where A is the noise-free envelope of the signal which is assumed to be the same for all grain ensembles, r the standard deviation of the noise distribution p® and I_0 is the modified Bessel function of the first kind and zero order. The signals can then be taken as

$$I=R\cos(qt+\theta)\ldots.\quad(a)\quad \left(I=Q\cos qt+I_N\right)\ldots..(b) \qquad (1.2)$$

We note that in all the existing models – Continuum damage model (Kachanov 1986, Krajcinovic 1996, Voyiadjil and Kattan 1999, Nanjo et al. 200 [8, 9, 15, and 275)]. Elasticity based damage model (Lamaitre and Chaboche 1996 [11]), Pore fluid model (Costin 1987, Lockner 1993, 1998 [4, 11, 12]) - there is no consideration of noise in signals and no indication of being able to predict the possible time or rather a warning time of occurrence of earthquake. Seismic stations are established to study the time series of the activities of the continental crust with a view to predicting possible time of occurrence of earthquake. There is need therefore for a study that can reveal such features. We are therefore considering the time series for micro seismic observations and find a suitable model that fitly exhibits all the features of seismic time series. Correig has done a lot of work on this (see for example Correig and Urquizu (2002 [2]), Corrieg et al (2005 [3])). We here build on those foundational models and do an in-depth analysis to help us relate these physical occurrences to the prediction of earthquake in areas of the world where this is prone.

The interpretations of our model give a great insight to these occurrences.

2.0 A Proposed model

Correig (2002 [2]) proposed two models of microseism time series given by

$$\dot{q} = p$$
$$\dot{p} + \frac{\partial V_0}{\partial q} + \delta p = \sum_{i=1}^{2} \gamma_i \cos(\omega_i t) + \varepsilon F(t) \qquad (2.1)$$
$$V_0(q) = -\alpha \frac{q^2}{2} + \beta \frac{q^4}{4}$$

where $V_0(q)$ is the potential and $\alpha = \alpha_0 + \eta f(t)$. $F(t)$ is a random noise, δ is the coefficient of damping, β, the coefficient of nonlinearity. Correig [2] also conjectured that volcanic tremor, be modeled in terms of an additive force component in the microseism model i.e.

$$\dot{q} = p$$
$$\dot{p} + \frac{\partial V_0}{\partial q} + \delta p = \sum_{i=1}^{2} \gamma_i \cos(\omega_i t) + \varepsilon F(t) + \gamma F_{tr}(t) \qquad (2.2)$$

where $F_{tr}(t)$ stands for a chaotic source as expounded by Julian 1994 and γ is its strength. γ_i the amplitudes of the external harmonic forces of frequency $\acute{\omega}_i$ and ε the noise amplitude $\alpha = \alpha_0 \left(1 + \eta \cos \omega_0 t\right)$ where η is the amplitude and $\acute{\omega}_0$ the frequency of the parametric resonance. This model took note of the noise in signals of flaws.

In his review on microseism studies, Bath (1973) states that the studies of microseisms, the steady unrest of the ground, is a borderline field between meteorology, oceanography, and seismology. Microseisms are, no doubt of greater concern to seismologists, but when their generation is to be explained, recourse must be taken to meteorology and oceanographic conditions. As a consequence, microseism constitutes a random process, like atmospheric turbulence and ocean surface waves.

We can look at microseism phenomenon by considering the 3-phase system atmosphere, hydrosphere (ocean or lake) and solid earth as a coupled nonlinear dynamical system that generates microseism oscillations as a result of its complex dynamics.

As a first approximation to the mathematical description of microseisms, the model of a nonlinear damped oscillator with multifrequency external excitation has turned out to be useful.

In a study by Correig and Urquist (1979) the following results were highlighted

(i) Microseism time series are nonstationary
(ii) Microseism time series are stochastic

(iii) From the point of view of data analysis, there is strong evidence in favour of a nonlinear character of microseism time series.

The same results – nonstationarity, stochasticity and nonlinearity – were also obtained for time series generated by a Duffing oscillator (Guckenheimar and Holmes 1997) as well as for a n-well potential forced oscillator, having added, in both cases, additive noise to account for stochasticity. It is worth pointing out that the results were the same for both observations and generated time series for all applied tests. Hence a Duffing oscillator with noise is an adopted model for the study of microseism time series.

Inland observations provide the following constraint:

(iv) For a given seismic station, the central frequency of the main spectral peak may suffer slight variations, following the time variations of the source of cyclonic storms.

(v) For a cyclonic storm fixed in space, the central frequency of the main spectral peak may be shifted when comparing different seismic stations.

(vi) By comparing records corresponding to stormy and quiet days, the location of the spectral peaks is preserved, and for frequencies higher than 2 Hz. The corresponding power spectra tend to coalesce to the same level.

(vii) Microseisms propagate incoherently. As observed above any nonlinear forced oscillator with additive noise could be used to simulate the observed microseism time series. Thus we study the Duffing oscillator

$$\ddot{q} + \delta\dot{q} - \alpha q + \beta q^3 = \gamma\cos(\omega t) \tag{2.3}$$

where δ is the coefficient of damping, α the proper or resonant frequency of the systems in the absence of external forces, β the coefficient of nonlinearity, γ the amplitude of the external harmonic force.

Observation of the classical Duffing oscillator shows that it can generate time series that may be periodic, quasi periodic or chaotic, but not stochastic. Hence the need to add white noise to the external force. The white noise can account for observation (vi) in the sense that local high frequency noise contents may act a driving force.

As noted in Correig and Urguizu (2002 [2]) it was found that to generate a time series qualitatively similar to the observed one, there is need to add a second harmonic force with a driving frequency of about 0.015 Hz (corresponding to the 70s period wave packet, the infra-gravity wave) added to an harmonic force with driving frequency of 0.2 Hz (the secondary microseism peak) as observed in the recorded microseism. This last frequency is related to the oceanic standing wave, the infra-gravity waves with a predominant frequency of 0.015 Hz could be related to wind waves and our equation becomes

$$\ddot{q} + \delta\dot{q} - \alpha q + \beta q^3 = \sum_{i=1}^{2}\gamma_{ii}\cos(\omega_i t) + \varepsilon F(t) \tag{2.4}$$

As noted in Correig et al (2005 [3]) coda waves are the main source of energy so that in the equipartioned region the forces of atmospheric storms or fiord resonances became negligible. This finding was supported by the findings of Okeke and Asor (2001 [20]). What ensues then is the model of a simple exponential relaxation process $N(t) = N_0 e^{-\lambda t}$. But since coda waves are continuously generated a summation of the exponential processes is considered with the inter-event time following a Poisson distribution. In this case our equation takes the form of the system

$$\begin{aligned} p &= \dot{q} \\ \dot{p} + \frac{\partial V(q)}{\partial q} + \delta p &= N_0 \sum_{i=1}^{n} e^{\lambda(t-t_i)} \end{aligned} \tag{2.5}$$

where $V(q)$ is the classical bi-stable potential $V = -\alpha_0 \frac{q^2}{2} + \beta \frac{q^4}{4}$ and *i* stands for each coda wave contribution.

3.0 Stability analysis

It is our intention to study the stability of the equations in the two models. In this first part we consider the first model. We shall study the second model in a subsequent paper. We study the general equation of a nonlinear Duffing oscillator

$$\ddot{x} + p(x,\dot{x})\dot{x} + q(x) = f(t) \tag{3.1}$$

In order to analyze the model correctly if we take q(x) to be of the form

$$q(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + O(x^4) \tag{3.2}$$

We demand that x q(x) ≥ 0 so that $a_0 = a_2 = 0$ and we have

$$q(x) = a_1 x + a_3 x^3 + O(x^4) \tag{3.3}$$

We can therefore take

$$q(x) = b(x + \eta x^3) \quad b > 0,\ |\eta| << 1 \tag{3.4}$$

A suitable *f(t)* is *Fcosωt* so that the Duffing equation we have is

$$\ddot{x} + 2k\dot{x} + b(x + \eta x^3) = F\cos\omega t \tag{3.5}$$

This equation is in every sense the same as (2.3) since we can have a signal plus noise represented by $R\cos\omega t$ (see Rice 1948 [23] for this derivation). We believe that a study of this equation particularly the stability analysis will give very interesting clue to the behaviour of the seismic activities that eventually culminate in the occurrence of earthquake. Let us note here that adding noise to the forcing function does not change the form as we noted above as depicted in Rice (1948 [23]).

First we note that equation (3.5) has a solution close to

$$x(t) = A\cos(\omega t + \theta) \tag{3.6}$$

where *A* satisfy the cubic equation

$$A^2\left(\left(b\left(1 + \frac{3\eta A^2}{4}\right) - \omega^2\right)^2 + 4k^2\omega^2\right) = F^2 \tag{3.7}$$

The question now is: what are the consequences of this for our problem?

In order to simplify calculations let us set the following values

$$b = 1, \quad \varepsilon = \frac{3\eta}{4}, \quad \kappa = 2k, \quad u = A^2, \quad f(u,v) = F^2 \tag{3.8}$$

This simplification does not change the features of our model and hence the results. Equation (14) becomes

$$f(u,v) = \varepsilon^3 u^3 + 2\varepsilon(1-v)u^2 + (\kappa^2 v + (1-v)^2)u \tag{3.9}$$

Considering the lightly damped case taking κ of the same order as ε > 0 for various fixed values of $v > 0$ we find that for $v << 1$, $f(u,v') < f(u,v) \quad for \quad v' > v$ and for $v >> 1$ we have that the equation $\frac{\partial f}{\partial u} = 0$ has two positive roots $v_1(u) < v_2(u)$. Since *f(u,v)* is cubic in *v* with positive leading coefficient then we must have that $f(u, v_1(u))$ is a maximum and $f(u, v_2(u))$ is the minimum which increase with u in such a way that

$$f(u', v_1(u')) > f(u, v_1(u)),\ f(u', v_2(u')) > f(u, v_2(u)),\ v_1(u') > v_1(u),\ v_2(u') > v_2(u)\ for\, u' > u \tag{3.10}$$

Let us note that what we are interested in here that will benefit us in our analysis of seismic waves is to know the behavior of $u(=A^2)$ vis-à-vis $v(\omega^2)$for fixed F^2 (= $f(u,v)$). If F^2 is small we see that the line $y = F^2$ crosses the curve $y = f(u,v)$ at a unique point $P_1(v) = (u,u(v))$. As v increases P_1 moves first to the right (i.e. $u(v)$ increases) and then back to the left with $u(v)$ decreasing to 0 as $v \to \infty$ (see Figure 3.1).

When F^2 is large and v is small the behavior noted above still applies. But when v reaches a certain value v = v_1 there are two new points of intersection. We see that $P_1(v)$ and $P_2(v)$ move closer together and at a certain value $v = v_2$ merge and vanish leaving only $P_3(v)$ so that F^2 (=$f(u,v)$) has only one solution $u_2(v)$ for large v. (see Figure 3.2). We find that if we plot $|A|$ against $|\omega|$ a mode jumping phenomenon occur which has a very good resemblance to Zeeman's catastrophe machine (Zeeman 1986 [29]) signifying instability. (See Figure 3.3).

One significant observation is the fact that if we vary both F and ω (slowly) the surface that emerges is the same as that which describes the bending of a long thin strut and several other apparently

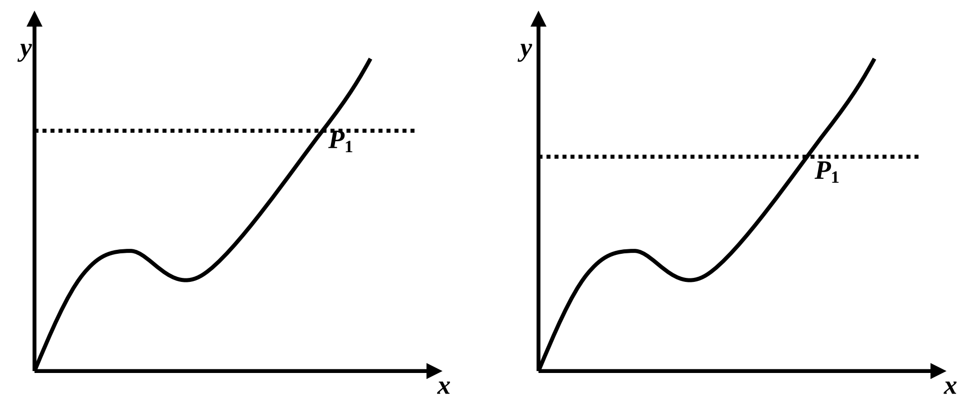

Figure 3.1: Roots of $f(u,v) = F^2$ for increasing values of v.

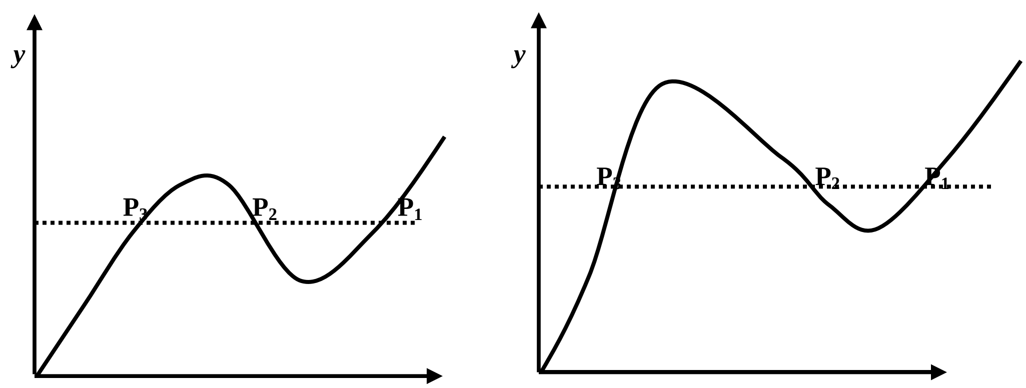

Figure 3.2: Roots of $f(u,v) = F^2$ for increasing values of v.

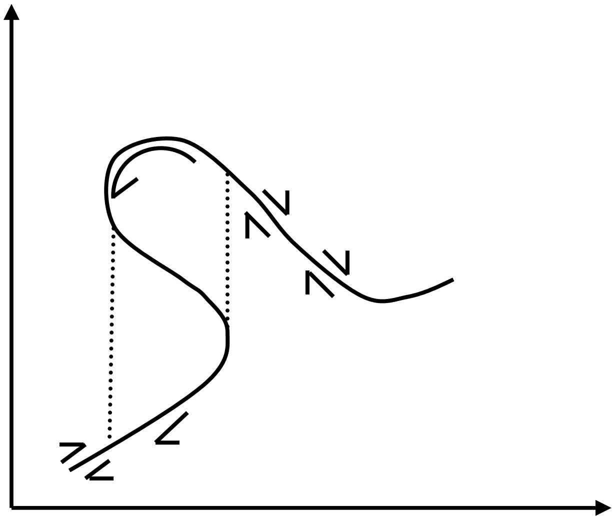

Figure 3.3: Effect of taking $\mu < 0$.

disparate physical and engineering phenomena. See Timoshenko et al (1974 [26]). We note that these same catastrophe-like results are found in the analysis of Ario (2004 [1]).

Our Duffing equation can be written as the system

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -\delta p + \alpha_0 q - \beta q^3 + \gamma \cos(\omega t) \end{aligned} \tag{3.11}$$

and the stability matrix given by

$$J = \begin{pmatrix} \dfrac{\partial \dot{q}}{\partial q} & \dfrac{\partial \dot{q}}{\partial p} \\ \dfrac{\partial \dot{p}}{\partial q} & \dfrac{\partial \dot{p}}{\partial p} \end{pmatrix} \tag{3.12}$$

is now

$$J = \begin{pmatrix} 0 & 1 \\ \alpha_0 - 3\beta q^2 & -\delta \end{pmatrix} \tag{3.13}$$

The eigenvalues for this matrix is given by

$$\lambda_{1,2} = \frac{1}{2}\left(-\delta \pm \sqrt{\delta^2 + 4\left(\alpha_0 - 3\beta q^2\right)}\right) \tag{3.14}$$

We note here also that

$$\det J = -\alpha_0 + 3\beta q^2 \quad \text{Trace J} = -\delta \tag{3.15}$$

Thus, for $\alpha_0 < 0$, $\beta > 0$ and $\delta < 0$, det$J > 0$, and Trace $J > 0$. For this case we have stability as shown in Dangelmayr and Kramer (1998 [5]), Oyesanya (2005 [21]), and Ario (2004 [1]).

For $\alpha_0 > 0,\ \beta < 0,\ \delta < 0$ we have det$J < 0$ and Trace $J > 0$ which is a case of instability.

For $\alpha_0 > 0,\ \beta < 0,\ \delta > 0$ we have det$J < 0$ and Trace $J < 0$ which also is a case of instability.

For $\alpha_0 > 0,\ \beta > 0,\ \delta > 0$ we have the following result

$$\det J = \begin{cases} <0 & for \quad \alpha_0 > 3\beta q^2 \\ 0 & for \quad \alpha_0 = 3\beta q^2 \\ >0 & for \quad \alpha_0 < 3\beta q^2 \end{cases} \quad \text{and Trace } J < 0 \tag{3.16}$$

We note of course that since we are considering a damped oscillation $\delta \neq 0$ which shows that for the damped oscillation case no Hopf bifurcation is possible since for this case Trace $J = 0$ condition is not met. The condition is met for $\delta = 0$ which is the undamped case.
The critical points occur for

$$q = \pm\sqrt{\frac{\alpha_0}{3\beta}} \tag{3.17}$$

It is now obvious from the above results that the critical points occur at $q = \pm 1$ and for $q = 1 - \varepsilon_1$ there is stability and instability occurs for $q = 1 + \varepsilon_1$ for some $\varepsilon_1 > 0$. Thus we have that

$$q \begin{cases} <1 & stability \\ =1 & critical \\ >1 & instability \end{cases} \tag{3.18}$$

We find that from equation (2.5) $p = 0$ for $q = \pm 1$ so that the critical points for this trajectories are (1, 0) and (-1, 0).

In continuing to discuss the stability of the Duffing oscillator we consider two results which we now state.

***Theorem* 1**. (Njoku and Omari 2003 [18])

Assume $\delta > 0$. Moreover, suppose that α is a strict lower and β is a strict upper solution of the equation $\ddot{q} + \delta\dot{q} + g(t,q) = h(t)$ which satisfy $\alpha \leq \beta$. Then, the equation has at least one unstable T-periodic solution $\hat{s}$, with $\alpha << \hat{s} << \beta$ provided that the number of the T-periodic solutions is finite.

***Theorem* 2**

Consider the equation (3.5) with $k = 0$ and b, ω, η fixed ($\eta > 0$). Provided that $\left(\frac{\omega}{3}\right)^2 > b$, there is a value F_0 of F depending on b, ω, and η such that the equation $\ddot{q} + b(q + \eta q^3) = F_0\cos(\omega t)$ has a solution $q(t) = A_0\cos\frac{\omega t}{3}, \quad A_0 \neq 0$.

***Proof*:**

Let $q(t) = A_0\cos\frac{\omega t}{3}, \quad A_0 \neq 0$. Then

$$\begin{aligned} \ddot{q} + b(q + \eta q^3) &= A\left(-\left(\frac{\omega}{3}\right)^2\cos\left(\frac{\omega t}{3}\right) + b\cos\left(\frac{\omega t}{3}\right) + b\eta A^2((\frac{3}{4}\cos\left(\frac{\omega t}{3}\right) + \frac{1}{4}\cos\omega t)\right) \\ &= A\left(\left(b - \left(\frac{\omega}{3}\right)^2\right) + \left(\frac{3b\eta}{4}\right)A^2\right)\cos\frac{\omega t}{3} + \left(\frac{b\eta A^3}{4}\right)\cos\omega t = F\cos\omega t \end{aligned} \tag{3.19}$$

Provided that

$$\left(b-\left(\frac{\omega}{3}\right)^2\right)+\left(\frac{3b\eta}{4}\right)A^2=0, \qquad (*) \tag{3.20}$$

and $\dfrac{b\eta A^3}{4}=F$. If $\left(\dfrac{\omega}{3}\right)^2>b$ then (*) has a real non-zero solution and we are done.

We note that under the conditions above $A\cos\left(\dfrac{\omega t+2\pi}{3}\right)$ *and* $A\cos\left(\dfrac{\omega t+4\pi}{3}\right)$ are also solutions. We note that we cannot have an exact solution q(t) = $A\cos\left(\dfrac{\omega t}{3}+\theta\right)$ with A $\neq$ 0 to the equation (3.5). But a solution close to $A_{1/3}\cos\left(\dfrac{\omega t}{3}+\theta_{1/3}\right)+A_1\cos(\omega t+\theta_1)$ exists if $\omega > \omega_0$ for some critical frequency ω_0 and if k, b and F take a certain range of values. The solution is stable. This is a stable solution containing an unexpected sub harmonic $A_{1/3}\cos\left(\dfrac{\omega t}{3}+\theta_{1/3}\right)$. Thus a lower and an upper solution can be found. We therefore now treat the damped case as a perturbation of the undamped case. Let us consider the total potential energy of the orbit for the undamped system.

$$U=\frac{(\dot q)^2}{2}-\kappa\omega_0^2\left(\frac{q^2}{2}-\frac{q^4}{4}\right) \tag{3.21}$$

A homoclinic orbit is obtained for U = 0 with the velocity of the orbit given by

$$|\dot q|=\sqrt{\kappa}\,\omega_0 q\sqrt{1-\frac{q^2}{2}} \tag{3.22}$$

If we now consider the damped case we can consider the total energy within the damping energy given by

$$U=\frac{(\dot q)^2}{2}+\delta\dot q q-\kappa\omega_0^2\left(\frac{q^2}{2}-\frac{q^4}{4}\right) \tag{3.23}$$

If we now set a limit near the orbit of the no damped case which preserves the total energy U = 0 we have

$$0=\dot q^2+2\delta\dot q q-\kappa\omega_0^2\left(q^2-\frac{q^4}{2}\right) \tag{3.24}$$

giving

$$\dot q=\delta q\left(-1\pm\sqrt{1+\frac{\kappa\omega_0^2}{\delta^2}\left(1-\frac{q^2}{2}\right)}\right) \tag{3.25}$$

This gives the orbit for the damped case.

A comparison of equations (3.22) and (3.25) reveals the displacement and velocity response of the damped case vis-à-vis the undamped case. This is shown in Figure 3.4. An analysis in Ario (2004 [1]) showed the existence of a snap-through phenomenon. As can be seen from Figure 3.5 the condition of the Theorem 1 is possible. Since the number of variables involved is small only a finite number of typical surfaces can exist. Theorem 1 can be interpreted in terms of seismic time series in this way: If the solution lies between the primary (maximum) peak and a lower (minimum) peak there is likely going to be at least one earthquake occurrence (which may be depicted as a secondary peak that may show up as a sub-harmonic) and we can then adduce that equation (3.5) has at least one asymptotically unstable *T*-periodic solution $\hat{q}$.

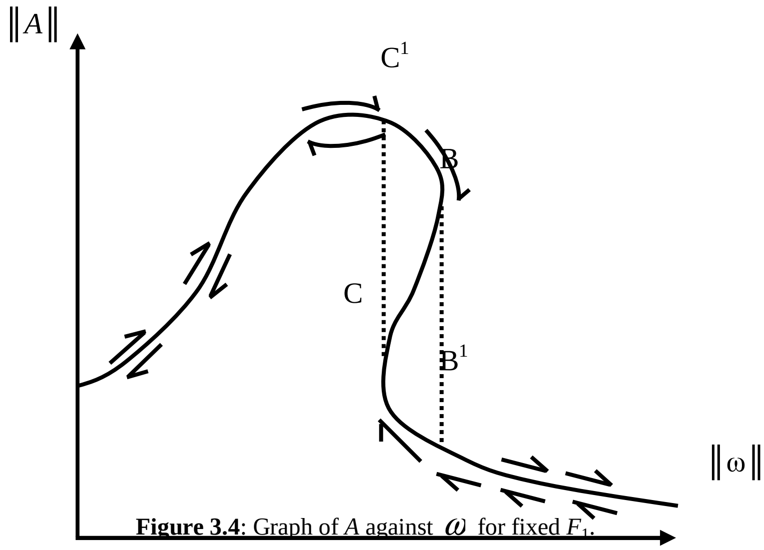

Figure 3.4: Graph of *A* against ω for fixed F_1.

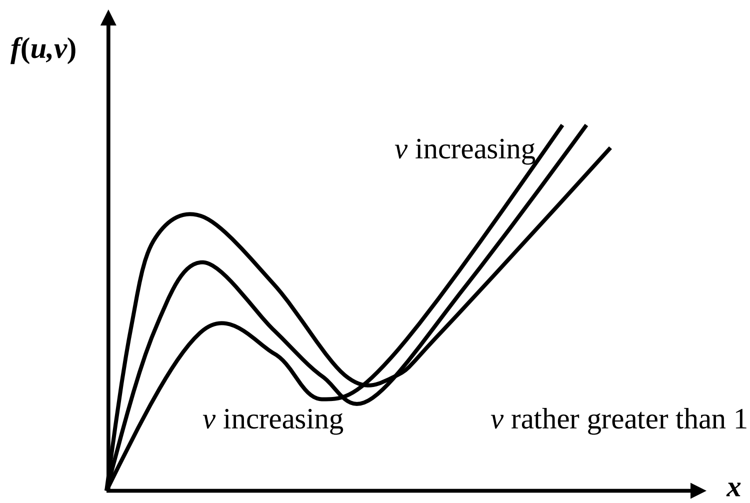

Figure 3.5: Snap-through phenomenon for damped oscillation.

4.0 Significance to Seismic analysis

The question we now want to address is: what relevance has this analysis of the Duffing oscillator to seismic analysis and prediction of earthquake occurrence? Consequent on our analysis of the Duffing oscillator the following become evident.

- The response of the Duffing oscillator exhibits snap-through phenomenon.
- A stable solution containing sub-harmonics exists for a Duffing oscillator.
- There exists at least one asymptotically unstable T-periodic solution.
- There is no room for existence of Hopf bifurcation.
- The primary peak of microseism spectra can be interpreted in terms of the resonant response of the Earth's crust and mantle shown by the snap-through phenomenon.
- Since sub-harmonic solutions exist secondary peaks should be expected in the seismic time series.

- Microseism activity, as a resonant (stochastic) response of the mantle lies between the high frequency local response of the medium |(random) and the (linear) free oscillations low frequency response of the whole Earth.

We should note that **o**ur analysis above is for equation (2.3). The other model typified by equation (2.1) needs to be analyzed. But we note that the additional term on the right hand side is to add a chaotic source function which may be attributable to meteorological, oceanic or aeronomy influences like wind gust. An example of this may be something like wind loading on buildings (Melbourne 1977 [14]) or on the continental shelf.

We note that for this type of loading the Weibull distribution (Newland 1984 [17], Norton 1989 [19]) is the most appropriate. In the limit when the shape parameter tends to infinity the Weibull distribution approaches the Dirac delta function which behaves like the Gaussian pulse. It can be shown that the Gaussian pulse behaves like the Fejer kernel which has a representation given by (see Oyesanya 2007 [22]).

$$F_n(x) = 1 + 2\sum_{k=1}^{n} \cos kx$$

which can be seen as an expansion of a series

$$\sum_{k=0}^{n} a_k \cos kx \quad \ni,\ a_0 = 1, a_k = 2 \ \ for\ \ k > 1.$$

Consequently the Weibull distribution can be given a Fejer kernel representation. This shows that equation (2.1) can be expressed as equation (2.3). This implicates that the above analysis qualitatively suffices for the model.

5.0 Conclusions

We have shown that the Duffing oscillator is a good model showing the features of microseismic time series. Our study shows that earthquake occurrence can be predicted from the interpretation of our results for the Duffing oscillator. We have also shown that the two models depicted by equations (2.1) and (2.3) can be treated as the same with the Duffing oscillator where we have used Fejer kernel representation for the chaotic source function.

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