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Initial post buckling of toroidal shell segments pressurized by external load

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Abstract

The static buckling pressure of imperfect toroidal shell segments under external load is here determined asymptotically by assuming that the stress-free imperfection can be represented as a two-term double Fourier series expansion. The buckling modes are taken strictly in the shape of the imperfection and simply-supported boundary conditions are assumed. Non-linear Karman-Donnell equations relevant to toroidal shell segments are used and the result clearly shows ,among other things , that the buckling load depends on all the Fourier coefficients that are admitted in the imperfection representation. The result is particularized to that of imperfect cylindrical shell segments. The load degradation is found to be of order two-thirds of the imperfection amplitude.

1.0 Introduction

The stability of imperfection elastic structures, under various loading conditions, is an important loading condition normally sought for, for purposes of practical applications and practical assimilation of engineering structural materials. In this investigation, we consider imperfect toroidal shell segments loaded by an external static pressure and aimed at determining the static buckling load, assuming that the imperfection of the structures can be adequately represented as a two-term double Fourier series expansion. Non-linear Karman-Donnell theory is assumed on the basis of asymptotically exact solution found for the initial post–buckling behaviour of the structures.

2.0 Formulation

The original derivation, as it concerns toroidal shell segments, was carefully formulated by Stein and McElman [1], while Hutchinson [2] later studied the buckling behaviour of such structures under three loading conditions, namely, lateral pressure, external pressures and axial tension. Relatively recent studies of the structures were given by Oyesanya [3-5]. As in [6], we shall , in the non-linear Karman-Donnell shell theory that follows, let the components of the generalized stress couple be M_x, M_y and M_{xy} while the generalized components of the stress resultant are represented by N_x, N_y and N_{xy}, where all these are functions of the spatial variables x and y. We shall in the same token let $\in_x, \in_y$ and $\in_{xy}$ be the strains while K_x, K_y and K_{xy} are the components of the bending strains. The outward normal displacement is $W(X,Y)$ while $U(X,Y)$ and $V(X,Y)$ are the in-plane tangential displacements. The strains are given [1,2] by

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$$\begin{pmatrix} \in_x \\ \in_y \\ \in_{xy} \\ K_x \\ K_y \\ K_{xy} \end{pmatrix} = \begin{pmatrix} U,_x + \dfrac{W}{r_x} \\ V,_y + \dfrac{W}{r_y} \\ \dfrac{1}{2}(U,_y + V,_x) \\ -W,_{xx} \\ -W,_{yy} \\ -W,_{xy} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} W,_x^2 + 2\overline{W},_x W,_x \\ W,_y^2 + 2\overline{W},_y W,_y \\ W,_x W,_y + \overline{W},_x W,_y + W,_x \overline{W},_y \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.1}$$

while the stress-strain relationship is given as

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{pmatrix} C & \nu C & 0 & 0 & 0 \\ \nu C & C & 0 & 0 & 0 \\ 0 & 0 & (1-\nu)C & 0 & 0 \\ 0 & 0 & 0 & D & \nu D \\ 0 & 0 & 0 & \nu D & D \\ 0 & 0 & 0 & 0 & (1-\nu)D \end{pmatrix} \begin{pmatrix} \in_x \\ \in_y \\ \in_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix} \tag{2.2}$$

where $C = \dfrac{Eh}{1-\nu^2}$ and $D = \dfrac{Eh^3}{12(1-\nu^2)}$, E is the Young's modulus while ν is the Poisson's ratio and h is the shell thickness. The equations of equilibrium are formulated on the basis of variational principle of virtual work for non-linear Donnell theory in the form

$$\iint_S \left[N_x \delta \in_x + N_y \delta \in_y + 2N_{xy} \delta \in_{xy} + M_x \delta \kappa_x + M_y \delta \kappa_y + 2M_{xy} \delta \kappa_{xy}\right] dx\, dy$$
$$+ \iint_S \overline{P} \delta W \, dx\, dy - \iint_C \overline{N} \delta U \, dx\, dy = 0 \tag{2.3}$$

where $\overline{P}$ is the applied pressure and $\overline{N}$ is the stress resultant at the ends of the shell segments while

$$\delta \in_x = \delta U,_x + \frac{\delta W}{r_x} + W,_x \delta W,_x + \overline{W},_x \delta W,_x \quad \text{and} \quad \delta \kappa_x = -\delta W,_{xx}, \text{ etc} \tag{2.4}$$

Here, δ is the variational operator and a subscript following a comma indicates partial differentiation .Variational calculus would normally lead to three differential equations in *U,V* and *W*. However, by introducing the stress function F(X,Y) , and using $N_x = F,_{yy}, N_y = F,_{xx}, N_{xy} = -F,_{xy}$, two of the equations are satisfied automatically. Finally, the compatibility equation and equilibrium equation are respectively given [2-4] in terms of the outward normal *W*(*X*,*Y*) and Airy stress function *F*(*X*,*Y*) as

$$\frac{1}{Eh}\nabla^4 F - \frac{1}{r_y} W,_{XX} - \frac{1}{r_x} W,_{YY} = -S\left(W, \frac{1}{2}W + \overline{W}\right) \tag{2.5}$$

$$D\nabla^4 W + \frac{1}{r_y} F,_{X\,X} + \frac{1}{r_x} F,_{Y\,Y} + \overline{P}\left[\frac{1}{2}\left(W+\overline{W}\right),_{X\,X} + \left(1-\frac{r_y}{2r_x}\right)\left(W+\overline{W}\right),_{Y\,Y}\right] = S\left(W+\overline{W}, F\right) \quad (2.6)$$

where X and Y are the usual spatial variables in the axial and circumferential directions respectively and

$\overline{W}(X,Y)$ is a stress-free initial imperfection, ∇^4 is the usual biharmonic operator, namely $\nabla^4 \equiv \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)^2$, while $S(P,Q)$, is the bilinear operator given by

$$S(P,Q) = P,_{X\,X}\, Q,_{Y\,Y} + P,_{Y\,Y}\, Q,_{X\,X} - 2P,_{X\,Y}\, Q,_{XY} \quad (2.7)$$

Toroidal shell segments are characterized by two radii of curvature, namely r_x and r_y in such a way if $\frac{r_y}{r_x} = r < 0$, the structures are said to be bowed-in, if $\frac{r_y}{r_x} = r > 0$, they are said to be bowed-out, where as if $\frac{r_y}{r_x} = r = 0$,the structures are said to correspond to cylindrical shell segments. However, if $\frac{r_y}{r_x} = r = 1$, the shell segments are said to be locally spherical on each point on the surface. In this investigation, we shall pay attention to toroidal shell segments that are shallow with respect to the axial coordinate and this is characterized by $\frac{L}{r_x} < 1$, where L is the shell length. We w introduce the following non-dimensional quantities (slightly different from those in [3,4]):

$$x = \frac{\pi X}{L},\ \mathrm{y} = \frac{\mathrm{Y}}{\mathrm{r}},\ \xi = \left(\frac{\mathrm{L}}{\pi\,\mathrm{r}}\right)^2,\ \mathrm{H} = \frac{\mathrm{h\,r_x}}{\mathrm{r}},\ \mathrm{K}(\xi) = -\frac{Eh}{D(1+\xi)^2}\left\{\frac{1}{r_y}\left(\frac{L}{\pi}\right)\right\}^2 \quad (2.8a)$$

$$w = \frac{W}{h},\ \in \overline{\mathrm{w}} = \frac{\overline{\mathrm{W}}}{\mathrm{h}},\ \ \overline{\lambda} = \frac{\overline{\mathrm{P}}\mathrm{r}_y L^2}{D\pi^2} \quad (2.8b)$$

Normally, the condition $|\in| < 1$ holds, however in this analysis, we consider $0 < \in < 1$, and now assume F and W in the following form

$$F = \frac{1}{2}\overline{\mathrm{P}}\mathrm{r}_y\left(X^2 + + \frac{\alpha Y^2}{2}\right) + \frac{Eh^2L^2}{\pi^2 r_y (1+\xi)^2} f\ ;\ \mathrm{W} = \frac{\overline{\mathrm{P}}\mathrm{r}_y\left(1-\frac{\alpha\nu}{2}\right)}{Eh} + hw \quad (2.9)$$

where the first terms on each of F and W above are the pre-buckling approximations while α takes the value α =1, if pressure contributes to axial stress through end plates, otherwise α =0 if pressure acts laterally. On substituting (2.9) into (2.8a,b),we have the following equations

$$\overline{\nabla}^4 f - (1+\xi)^2\left(w,_{x\,x} + \xi\,\mathrm{r}w,_{y\,y}\right) = -H(1+\xi)^2\,\overline{S}\left(w, \frac{w}{2} + \in \overline{w}\right) \quad (2.10)$$

$$\overline{\nabla}^4 w - K(\xi)\left(f,_{xx} + \xi\, \mathrm{rf},_{yy}\right) + \overline{\lambda}\left[\frac{\alpha}{2}\left(w + \in \overline{w}\right),_{xx} + \xi\left(1 - \frac{\alpha r}{2}\right)\left(w + \in \overline{w}\right),_{yy}\right] = -K(\xi) H \overline{S}\left(f, w + \in \overline{w}\right) \tag{2.11a}$$

$$w = w,_{xx} = f = f,_{xx} = 0 \;\; at \;\; x = 0,\ \pi. \tag{2.11b}$$

where

$$\overline{\nabla}^4 \equiv \left(\frac{\partial^2}{\partial x^2} + \xi \frac{\partial^2}{\partial y^2}\right)^2, \; \overline{\mathrm{S}}(\mathrm{P},\mathrm{Q}) = P,_{xx}\, Q,_{yy} + P,_{yy}\, Q,_{xx} - 2P,_{xy}\, Q,_{xy} \tag{2.12}$$

Here , $\overline{\lambda}$ is the load amplitude parameter ,whose value at buckling we are to determine.

3.0 Classical buckling load

The classical buckling load λ_C is defined as the minimum value of $\overline{\lambda}$ for there to exist a nontrivial solution to the corresponding linear problem for the case where the structures are deemed perfect . The associated equation , from (2.10) and (2,11a,b) ,are

$$\overline{\nabla}^4 f - (1+\xi)^2\left(w,_{xx} + \xi\, \mathrm{rw},_{yy}\right) = 0 \tag{3.1}$$

$$\overline{\nabla}^4 w - K(\xi)\left(f,_{xx} + \xi\, \mathrm{rf},_{yy}\right) + \overline{\lambda}\left[\frac{\alpha}{2} w,_{xx} + \xi\left(1 - \frac{\alpha r}{2}\right)\mathrm{w}_{,yy}\right] = 0 \tag{3.2}$$

$$w = w,_{xx} = f = f,_{xx} = 0 \;\; at \;\; x = 0,\ \pi. \tag{3.3}$$

To solve (3.1-3.3), we let

$$\begin{pmatrix} w \\ f \end{pmatrix} = \left(\begin{bmatrix} a_{m\mathrm{k}}^{(1)} \\ a_{m\mathrm{k}}^{(2)} \end{bmatrix} \cos \mathrm{mx} + \begin{bmatrix} b_{m\mathrm{k}}^{(1)} \\ b_{m\mathrm{k}}^{(2)} \end{bmatrix} \sin \mathrm{mx}\right) \sin\left(\mathrm{ky} + \phi_{\mathrm{mk}}\right) \sin \mathrm{ny} \tag{3.4}$$

On substituting (3.4) into (3.1), we have

$$b_{m\mathrm{k}}^{(\mathrm{i})} = -\frac{(1+\xi)^2\left(m^2 + r\,\xi\,\mathrm{k}^2\right)\mathrm{a}_{\mathrm{m\,k}}^{(\mathrm{i})}}{\left(m^2 + \xi\,\mathrm{k}^2\right)^2}, \quad \mathrm{i} = 1,2. \tag{3.5}$$

If we now substitute (3.5) into (3.2), we get

$$\overline{\lambda} = \frac{\left(m^2 + \xi\, k^2\right)^2 - K(\xi)\left(m^2 + r\xi\, k^2\right)^2 (1+\xi)^2 \left(m^2 + k^2 \xi\right)^{-2}}{\frac{\alpha\,\mathrm{m}^2}{2} + k^2 \xi\left(1 - \frac{\alpha\,\mathrm{r}}{2}\right)} \tag{3.6}$$

The minimum value [1-5] of $\overline{\lambda}$, namely λ_C , is obtained by treating k as a continuous variable , assuming that k is sufficiently large ,and setting $m = 1$ to get

$$\lambda_C = \frac{\left(1 + \xi\, n^2\right)^2 - K(\xi)\left(1 + r\xi\, n^2\right)^2 (1+\xi)^2 \left(1 + n^2 \xi\right)^{-2}}{\frac{\alpha}{2} + n^2 \xi\left(1 - \frac{\alpha\,\mathrm{r}}{2}\right)} \tag{3.7}$$

where n is the integer value of k that minimizes $\overline{\lambda}$ .The corresponding values of the normal displacement and Airy stress function are

$$\begin{pmatrix} w \\ f \end{pmatrix} = \left(a_{1n}^{(i)} \left[\begin{matrix} 1 \\ -\dfrac{(1+\xi)^2(1+n^2 r\xi)}{(1+n^2\xi)^2} \end{matrix} \right] \cos x + b_{1n}^{(i)} \left[\begin{matrix} 1 \\ -\dfrac{(1+\xi)^2(1+n^2 r\xi)}{(1+n^2\xi)^2} \end{matrix} \right] \sin x \right) \sin ny \qquad (3.8)$$

4.0 Non-linear theory

We shall now write $\overline{\lambda} = \lambda\lambda_C$, $0 < \lambda < \lambda_C$, and henceforth, aim at determining the buckling load λ_S ,(i.e. the value of $\lambda = \lambda_S$ at buckling) by solving (2.10) and (2.11a, b). We define the buckling load λ_S, for $0 < \lambda_S < \lambda_C$, as the maximum load parameter which the structures can support prior to buckling .We now let

$$\begin{pmatrix} f(x,y) \\ w(x,y) \end{pmatrix} = \sum_{i=1}^{\infty} \begin{pmatrix} f^{(i)}(x,y) \\ w^{(i)}(x,y) \end{pmatrix} \in^{i} \qquad (4.1)$$

On substituting (4.1) into (2.10) and (2.11a,b), we have the following equations

$$M\left(f^{(1)}, w^{(1)}\right) = 0 \qquad (4.2a)$$

$$N\left(f^{(1)}, w^{(1)}\right) + \lambda\lambda_C \left[\frac{\alpha}{2} \overline{w}_{,xx} + \xi\left(1 - \frac{\alpha r}{2}\right) \overline{w}_{,yy} \right] = 0 \qquad (4.2b)$$

$$M\left(f^{(2)}, w^{(2)}\right) = -H(1+\xi)^2 \left[\frac{1}{2} \overline{S}\left(w^{(1)}, w^{(1)}\right) + \overline{S}\left(w^{(1)}, \overline{w}\right) \right] \qquad (4.3a)$$

$$N\left(f^{(2)}, w^{(2)}\right) = -HK(\xi)\left[\overline{S}\left(f^{(1)}, w^{(1)}\right) + \overline{S}\left(f^{(1)}, \overline{w}\right)\right] \qquad (4.3b)$$

$$M\left(f^{(3)}, w^{(3)}\right) = -H(1+\xi)^2 \left[\overline{S}\left(w^{(1)}, w^{(2)}\right) + \overline{S}\left(w^{(2)}, \overline{w}\right)\right] \qquad (4.3a)$$

$$N\left(f^{(3)}, w^{(3)}\right) = -HK(\xi)\left[\overline{S}\left(f^{(1)}, w^{(2)}\right) + \overline{S}\left(f^{(2)}, w^{(1)}\right) + \overline{S}\left(f^{(2)}, \overline{w}\right)\right] \qquad (4.4b)$$

$$w^{(i)} = w^{(i)}_{,xx} = f^{(i)} = f^{(i)}_{,xx} = 0 \text{ at } x = 0, \pi., i = 1,2,3,\Lambda \qquad (4.5)$$

where

$$M\left(f^{(i)}, w^{(i)}\right) \equiv \overline{\nabla}^4 f^{(i)} - (1+\xi)^2 \left(w^{(i)}_{,xx} + r\,\xi\, w^{(i)}_{,yy}\right) \qquad (4.6a)$$

$$N\left(f^{(i)}, w^{(i)}\right) \equiv \overline{\nabla}^4 w^{(i)} - K(\xi)\left(f^{(i)}_{,xx} + r\,\xi\, f^{(i)}_{,yy}\right) + \lambda\lambda_C \left[\frac{\alpha}{2} w^{(i)}_{,xx} + \xi\left(1 - \frac{\alpha r}{2}\right) w^{(i)}_{,yy} \right], \; i = 1,2,3, \qquad (4.6b)$$

In line with the boundary conditions (4.5), we let the imperfection $\overline{w}(x,y)$ take the form

$$\overline{w}(x,y) = (a\cos ny + b\sin ny)\sin x \qquad (4.7)$$

and now let

$$\begin{pmatrix} f^{(i)}(x,y) \\ w^{(i)}(x,y) \end{pmatrix} = \sum_{p,q}^{\infty} \left[\begin{pmatrix} f_1^{(i)} \\ w_1^{(i)} \end{pmatrix} \cos py + \begin{pmatrix} f_2^{(i)} \\ w_2^{(i)} \end{pmatrix} \sin py \right] \sin qx \qquad (4.8)$$

We now substitute (4.8) into (4.2a) and simplify to get

$$f_1^{(1)} = -\frac{(1+\xi)^2\left(q^2 + \xi\, rp^2\right) w_1^{(1)}}{\left(q^2 + \xi\, p^2\right)^2}, \quad f_2^{(1)} = -\frac{(1+\xi)^2\left(q^2 + \xi\, rp^2\right) w_2^{(1)}}{\left(q^2 + \xi\, p^2\right)^2} \qquad (4.9a)$$

If $p = m$, $q = n$,both integers, then we have

$$f_j^{(1)} = -\frac{(1+\xi)^2\left(m^2+\xi\,\mathrm{n}^2 r\right)\mathrm{w}_\mathrm{j}^{(1)}}{\left(\mathrm{m}^2+\xi\,\mathrm{n}^2\right)^2},\ \mathrm{j}=1,2 \tag{4.9b}$$

We shall particularly need the case $m=1$ (this value is associated with buckling) in which case, we have

$$f_j^{(1)} = l_0 w_j^{(1)}\ ,\ \mathrm{l}_0 = -\frac{(1+\xi)^2\left(1+\xi\,\mathrm{n}^2 r\right)}{\left(1+\xi\,\mathrm{n}^2\right)^2}\ \ \mathrm{j}=1,2. \tag{4.9c}$$

We next substitute (4.8) into (4.2b), using (4.7) as well as (4.9c) to get

$$w_1^{(1)} = \beta\,\mathrm{a} = \frac{\lambda\lambda_\mathrm{C} l_1\,\mathrm{a}}{\mathrm{l}_2},\ w_2^{(1)} = \beta\,\mathrm{b} = \frac{\lambda\lambda_\mathrm{C} l_1\,\mathrm{b}}{\mathrm{l}_2}, \tag{4.10a}$$

where

$$l_1 = \frac{\alpha}{2} + n^2\xi\left(1-\frac{\alpha\,\mathrm{r}}{2}\right), \mathrm{l}_2 = \left[\left(1+n^2\xi\right)^2 - \lambda\lambda_C l_1 - K(\xi)\left(1+n^2 r\xi\right)^2(1+\xi)^2\left(1+n^2\xi\right)^{-2}\right],$$

$$\beta = \frac{\lambda\lambda_\mathrm{C} l_1}{l_2} \tag{4.10b}$$

We next substitute for terms on the right hand sides of (4.3a,b) and get

$$M\left(f^{(2)}, w^{(2)}\right) = -n^2 H(1+\xi)^2\left[Q_1 \sin 2\mathrm{ny} + \mathrm{Q}_2 \cos 2\mathrm{ny} + \mathrm{Q}_3 \cos 2\mathrm{x}\right] \tag{4.11}$$

$$N\left(f^{(2)}, w^{(2)}\right) = -n^2 HK(\xi)\left[Q_4 \sin 2\mathrm{ny} + \mathrm{Q}_5 \cos 2\mathrm{ny} + \mathrm{Q}_6 \cos 2\mathrm{x}\right] \tag{4.12}$$

$$w^{(2)} = w_{,x\,\mathrm{x}}^{(2)} = f^{(2)} = f_{,x\,\mathrm{x}}^{(2)} \tag{4.13}$$

$$Q_1 = \frac{1}{2} w_1^{(1)} w_2^{(1)} + a\, w_2^{(1)} + \mathrm{b}\ \mathrm{w}_1^{(1)}\ ,\ \mathrm{Q}_2 = \frac{1}{2}\left(w_1^{(1)\,2} - w_2^{(1)2}\right) + \left(a w_1^{(1)\,2} - b w_2^{(1)\,2}\right) \tag{4.14}$$

$$Q_3 = -\left\{\frac{1}{2}\left(w_1^{(1)\,2} + w_2^{(1)2}\right) + \left(a\, w_1^{(1)\,2} + b\, w_2^{(1)\,2}\right)\right\} \tag{4.15a}$$

$$Q_4 = \left\{f_1^{(1)} w_2^{(1)} + f_2^{(1)} w_1^{(1)} + b\ f_1^{(1)} + a f_2^{(1)}\right\} = l_0\left(2 w_1^{(1)} w_2^{(1)} + a\, w_2^{(1)} + \mathrm{b}\ \mathrm{w}_1^{(1)}\right) \tag{4.15b}$$

$$Q_5 = \left\{f_1^{(1)} w_1^{(1)} - f_2^{(1)} w_2^{(1)} - b\ f_2^{(1)} + a f_1^{(1)}\right\} = l_0\left(w_1^{(1)2} - w_2^{(1)2} + a\, w_1^{(1)} - b\, w_2^{(1)}\right) \tag{4.15c}$$

$$Q_6 = -\left\{f_1^{(1)} w_1^{(1)} + f_2^{(1)} w_2^{(1)} + b\ f_2^{(1)} + a f_1^{(1)}\right\} = -l_0\left(w_1^{(1)2} + w_2^{(1)2} + a\, w_1^{(1)} + b\, w_2^{(1)}\right) \tag{4.15d}$$

We now substitute (4.8) ,for $i=2$, into (4.12), multiply the resultant equation through, first by *cosuny sin*mx and next by *sinunysin*mx (for , *u* and *m* to be determined) and note that for *u* = 2, *p = 2n*, and *q = m*, we have, in the first and second cases (and for *m* odd)

$$f_j^{(2)} = \frac{-(1+\xi)^2\left\{\dfrac{4H\,n^2 Q_2}{m\,\pi} + \left(m^2 + 4n^2 r\xi\right)\right\}\mathrm{w}_\mathrm{j}^{(2)}}{\left(m^2+4n^2\xi\right)^2},\ \mathrm{j}=1,2 \tag{4.16}$$

We next substitute (4.8) ,for $i=2$, into (4.12), multiply the resultant equation through, first by *cosuny sin*mx and next by s*inunysin*mx (for, u, *p,q* and *m* as determined before) and get, for the first and second respective cases and *m* odd

$$w_1^{(2)} = -\frac{R_1}{R_2},\ w_2^{(2)} = -\frac{R_3}{R_2}, \mathrm{R}_1 = 4HK(\xi)n^2\left[Q_5 + Q_2\left(m^2+4n^2 r\,\xi\right)(1+\xi)^2\left(m^2+4n^2\,\xi\right)^{-2}\right] \tag{4.17a}$$

$$R_2 = m\pi\left[\begin{array}{l}\left(m^2+4n^2\,\xi\right)^2-\lambda\lambda_C\left\{\dfrac{m^2\alpha}{2}+4n^2\xi\left(1-\dfrac{\alpha\,\mathrm{r}}{2}\right)\right\}\\ \qquad\qquad -K(\xi)\left(m^2+4n^2\mathrm{r}\,\xi\right)^2(1+\xi)^2\left(m^2+4n^2\,\xi\right)^{-2}\end{array}\right] \tag{4.17b}$$

$$R_3 = 4HK(\xi)\left[Q_4+Q_1\left(m^2+4n^2 r\,\xi\right)(1+\xi)^2\left(m^2+4n^2\,\xi\right)^{-2}\right] \tag{4.17c}$$

and

$$w_2^{(2)} = \frac{-4HK(\xi)n^2\left[Q_4+Q_1\left(m^2+4n^2 r\xi\right)(1+\xi)^2\left(m^2+4n^2\xi\right)^{-2}\right]}{R_2} \tag{4.17d}$$

Henceforth, the value of any function of m, say $f_j^{(i)}(m)$,evaluated at $m=1$, shall be denoted simply as $\tilde{f}_j^{(i)}$. Thus from (4.17a-c), we have

$$\tilde{w}_1^{(2)} = -\frac{4HK(\xi)}{\tilde{R}_2}\left(Q_5+l_{16}Q_2\right),\ \mathrm{l}_{16} = \left[\left(1+4n^2 r\,\xi\right)(1+\xi)^2\left(1+4n^2\xi\right)^{-2}\right] \tag{4.18a}$$

$$\tilde{R}_2 = \pi\left[\left(1+4n^2\xi\right)^2-\lambda\lambda_C\left\{\frac{\alpha}{2}+4n^2\xi\left(1-\frac{\alpha\,\mathrm{r}}{2}\right)\right\}-K(\xi)\left(1+4n^2\mathrm{r}\xi\right)^2(1+\xi)^2\left(1+4n^2\xi\right)^{-2}\right] \tag{4.18b}$$

On substituting for Q_5 and Q_2 into (4.18a) from (4.15c) and (4.14), we get

$$\tilde{w}_1^{(2)} = -\left[l_{17}\left(w_1^{(1)2}-w_2^{(1)2}\right)+l_{18}\left(aw_1^{(1)}-\mathrm{b}\,w_2^{(1)}\right)\right],\ \mathrm{l}_{17} = \frac{4HKn^2}{\tilde{R}_2}\left(\frac{1}{2}l_{16}+l_0\right),\ \ \mathrm{l}_{18} = \frac{4HKn^2}{\tilde{R}_2}\left(l_{16}+l_0\right) \tag{4.19}$$

similarly, for $w_2^{(2)}$ in (4.17d),evaluated at m = 1, we get

$$\tilde{w}_2^{(2)} = -\left[l_{19}w_1^{(1)}w_2^{(1)}+l_{20}\left(bw_1^{(1)}+aw_1^{(1)}\right)\right],\ \ l_{19} = \left(2l_0+l_{16}\right),\ \mathrm{l}_{20} = \left(l_0+l_{16}\right) \tag{4.20}$$

It similarly follows from (4.16) (for j = 1), that

$$\tilde{f}_1^{(2)} = -\left(l_{13}Q_2+l_{14}\tilde{w}_1^{(2)}\right);\ l_{13} = \frac{4(1+\xi)^2 n^2 H}{\pi\left(1+4n^2\xi\right)^2} \tag{4.21}$$

$$\mathrm{l}_{14} = \frac{4(1+\xi)^2\left(1+4n^2 r\xi\right)}{\pi\left(1+4n^2\xi\right)^2} \tag{4.22}$$

If we substitute for Q_2 in (4.21) from (4.14), we get

$$\begin{aligned}\tilde{f}_1^{(2)} &= l_{21}w_1^{(1)2}+l_{22}w_2^{(1)2}+al_{23}w_1^{(1)}+bl_{24}w_2^{(1)}\ ;\ \mathrm{l}_{21} = \left(l_{14}l_{17}-\frac{1}{2}l_{13}\right)\\ &= -l_{22}\ ;\mathrm{l}_{23} = \left(l_{14}l_{18}-l_{13}\right) = -l_{24}\end{aligned} \tag{4.23}$$

Similarly, from (4.16) and for j=2, evaluated at m = 1, we get

$$\tilde{f}_2^{(2)} = -\left(l_{13}Q_1+l_{14}\tilde{w}_2^{(2)}\right) \tag{4.24a}$$

On substituting in (4.24a) for Q_1 from (4.14), and for $\tilde{w}_2^{(2)}$ from (4.20), we have

$$\tilde{f}_2^{(2)} = l_{25}w_1^{(1)}w_2^{(1)}+al_{26}w_2^{(1)}+bl_{27}w_1^{(1)},\ \mathrm{l}_{25} = \left(l_{14}l_{19}-\frac{1}{2}l_{13}\right),\ \mathrm{l}_{26} = \left(l_{20}l_{14}-l_{13}\right) = l_{27} \tag{4.24b}$$

Thus far, we infer that

$$w^{(2)}(x,y)=\sum_{m=1,3,5,\Lambda}^{\infty}\left(w_1^{(2)}\cos 2ny+w_2^{(2)}\sin 2ny\right)\sin \mathrm{mx},$$
$$w_1^{(1)}(x,y)=\left(w_1^{(1)}\cos \mathrm{ny}+\mathrm{w}_2^{(1)}\sin \mathrm{ny}\right)\sin \mathrm{x} \tag{4.25}$$

We now substitute on the right hand sides of (4.4a,b) and simplify to get

$$M\left(f^{(3)},w^{(3)}\right)=-(1+\xi)^2 H\sum_{m=1,3,5,\Lambda}^{\infty}\Big[\left(4n^2+m^2n^2\right)\Big\{\left(\mathrm{w}_1^{(1)}w_1^{(2)}+aw_1^{(2)}\right)\cos\mathrm{ny}\cos 2\mathrm{ny}\sin\mathrm{x}\sin\mathrm{mx}$$
$$+\left(\mathrm{w}_1^{(1)}w_2^{(2)}+aw_2^{(2)}\right)\sin\mathrm{ny}\cos 2\mathrm{ny}\cos\mathrm{x}\cos\mathrm{mx}+\left(\mathrm{w}_2^{(1)}w_1^{(2)}+bw_1^{(2)}\right)\sin\mathrm{ny}\cos 2\mathrm{ny}\sin\mathrm{x}\sin\mathrm{mx}$$
$$+\left(\mathrm{w}_2^{(1)}w_2^{(1)}+bw_2^{(2)}\right)\sin\mathrm{ny}\sin 2\mathrm{ny}\sin\mathrm{x}\sin\mathrm{mx}\Big\}+4n^2m\Big\{\left(\mathrm{w}_1^{(1)}w_2^{(2)}+aw_2^{(2)}\right)\sin\mathrm{ny}\cos 2\mathrm{ny}\cos\mathrm{x}\cos\mathrm{mx}$$
$$+\left(\mathrm{w}_2^{(1)}w_1^{(2)}+bw_1^{(1)}\right)\cos\mathrm{ny}\sin 2\mathrm{ny}\cos\mathrm{x}\cos\mathrm{mx}-\left(\mathrm{w}_1^{(1)}w_1^{(2)}+aw_1^{(2)}\right)\sin\mathrm{ny}\sin 2\mathrm{ny}\cos\mathrm{x}\cos\mathrm{mx}$$
$$-\left(\mathrm{w}_2^{(1)}w_2^{(2)}+bw_2^{(2)}\right)\cos\mathrm{ny}\cos 2\mathrm{ny}\cos\mathrm{x}\cos\mathrm{mx}\Big\}\Big] \tag{4.26}$$

$$N\left(f^{(3)},w^{(3)}\right)=-HK(\xi)\sum_{m=1,3,5,\Lambda}^{\infty}\Big[\left(4n^2+m^2n^2\right)\Big\{\left(\mathrm{w}_1^{(1)}\mathrm{f}_1^{(2)}+f_1^{(1)}w_2^{(1)}+af_1^{(2)}\right)\cos\mathrm{ny}\cos 2\mathrm{ny}\sin\mathrm{x}\sin\mathrm{mx}$$
$$+\left(\mathrm{w}_1^{(1)}\mathrm{f}_2^{(2)}+f_1^{(1)}w_2^{(2)}+af_2^{(2)}\right)\sin\mathrm{ny}\cos 2\mathrm{ny}\cos\mathrm{x}\cos\mathrm{mx}+\left(\mathrm{w}_2^{(1)}\mathrm{f}_1^{(2)}+f_2^{(1)}w_1^{(2)}+bf_1^{(2)}\right)$$
$$\times\sin\mathrm{ny}\cos 2\mathrm{ny}\sin\mathrm{x}\sin\mathrm{mx}+\left(\mathrm{w}_2^{(1)}\mathrm{f}_2^{(2)}+f_2^{(1)}w_2^{(2)}+bf_2^{(2)}\right)\sin\mathrm{ny}\sin 2\mathrm{ny}\,\mathrm{ny}\sin\mathrm{x}\sin\mathrm{mx}$$
$$+4n^2m\Big\{\left(\mathrm{w}_1^{(1)}\mathrm{f}_2^{(2)}+f_1^{(1)}w_2^{(2)}+af_2^{(2)}\right)\sin\mathrm{ny}\cos\ 2\mathrm{ny}\cos\mathrm{x}\sin\mathrm{mx}$$
$$-\left(\mathrm{w}_1^{(1)}\mathrm{f}_1^{(2)}+f_1^{(1)}w_1^{(2)}+af_1^{(2)}\right)\sin\mathrm{ny}\sin\ 2\mathrm{ny}\cos\mathrm{x}\cos\mathrm{mx}$$
$$+\left(\mathrm{w}_1^{(2)}\mathrm{f}_2^{(1)}+f_1^{(2)}w_2^{(1)}+bf_1^{(2)}\right)\cos\mathrm{ny}\sin 2\mathrm{ny}\cos\mathrm{x}\cos\mathrm{mx}$$
$$-\left(\mathrm{w}_2^{(1)}\mathrm{f}_2^{(2)}+f_2^{(1)}w_2^{(2)}+bf_2^{(2)}\right)\cos\mathrm{ny}\cos 2\mathrm{ny}\cos\mathrm{x}\cos\mathrm{mx}\Big\}\Big] \tag{4.27}$$

We substitute (4.8) into (4.26), for $i=3$, multiply the resultant equation, first by $\cos\beta$ ny sinumx and next $\sin\beta ny$sinumx and, in each case, note that the following combinations of β and u are required to

give the necessary eigen Airy stress functions: $(i)\beta=1,\mathrm{u}=1$ (ii) $\beta=1$ $\mathrm{u}=3$,$(iii)\beta=3$ $\mathrm{u}=1$.
Based on these combinations, the eigen Airy stress functions in the first multiplication are in the shapes of (a) cosnysinmx ,(b) *cosny sin*3*mx* and (c) *cos*3*nysinx*. Similarly, the eigen Airy stress functions in the second multiplication are in the shapes of (d) sin ny sinmx, (e) sinnysin3mx and (*f*) sin3nysinx. Of these, it is only (a) and (d) , for the case m=1 ,that are likely to be in the shapes of the imperfection as in (4.7) and that will eventually [2] have a dominant role in the buckling process. Thus the associated Airy stress functions, for any m (odd) ,in these two cases are respectively evaluated from (4.26) as

$$f_1^{(3)}=-\frac{(1+\xi)^2\left(m^2+n^2\xi\,\mathrm{r}\right)w_1^{(3)}}{\left(m^2+n^2\xi\right)^2}+\frac{(1+\xi)^2 H}{2\pi\left(m^2+n^2\xi\right)^2}\sum_{m=1,3,5,\Lambda}^{\infty}\Big[\Big\{\left(w_1^{(1)}w_1^{(2)}+aw_1^{(2)}\right)$$
$$+\left(w_2^{(1)}w_2^{(2)}+bw_2^{(2)}\right)\Big\}\Big\{\omega_{\mathrm{m}}\left(4n^2+m^2n^2\right)+4n^2m\,\theta_{\mathrm{m}}\Big\}\Big] \tag{4.28a}$$

$$\omega_m=\left\{2-\left(\frac{1}{1+2m}-\frac{1}{1-2m}\right)\right\},\ \theta_{\mathrm{m}}=\left(\frac{1}{1+2m}+\frac{2}{2m-1}\right) \tag{4.28b}$$

and

$$f_2^{(3)} = -\frac{(1+\xi)^2\left(m^2+n^2\xi\,\mathrm{r}\right)w_2^{(3)}}{\left(m^2+n^2\xi\right)^2} + \frac{(1+\xi)^2 H}{2\pi\left(m^2+n^2\xi\right)^2}\sum_{m=1,3,5,\Lambda}^{\infty}\Big[\Big\{\left(w_1^{(1)}w_1^{(2)}+aw_1^{(2)}\right)$$
$$-\left(w_2^{(1)}w_1^{(2)}+bw_1^{(2)}\right)\Big\}\Big\{\omega_{\mathrm{m}}\left(4n^2+m^2n^2\right)+4n^2m\,\theta_{\mathrm{m}}\Big\}\Big] \tag{4.29}$$

Since the case m=1 in (4.28a,b) is essential in the buckling process ,we therefore evaluate $\tilde{f}_1^{(3)}$ and $\tilde{f}_2^{(3)}$ as

$$\tilde{f}_1^{(3)} = -\left(\frac{1+\xi}{1+n^2\xi}\right)^2\tilde{w}_1^{(3)} + \frac{n^2H}{2\pi}\left(\frac{1+\xi}{1+n^2\xi}\right)^2\left(5\tilde{\omega}_1+4\tilde{\theta}_1\right)$$
$$\times\left[w_1^{(2)}\tilde{w}_1^{(2)}+w_2^{(1)}\tilde{w}_2^{(2)}+a\tilde{w}_1^{(2)}+b\tilde{w}_2^{(2)}\right] \tag{4.30a}$$

$$\tilde{f}_2^{(3)} = -\left(\frac{1+\xi}{1+n^2\xi}\right)^2\tilde{w}_2^{(3)} + \frac{n^2H}{2\pi}\left(\frac{1+\xi}{1+n^2\xi}\right)^2$$
$$\times\left(5\tilde{\omega}_1+4\tilde{\theta}_1\right)\left[w_1^{(1)}\tilde{w}_2^{(2)}-w_2^{(1)}\tilde{w}_1^{(2)}+a\tilde{w}_2^{(2)}-b\tilde{w}_1^{(2)}\right] \tag{4.30b}$$

where $\tilde{\omega}_1$ and $\tilde{\theta}_1$ are the values of ω_m and θ_m respectively at m=1,and which are easily evaluated from (4.28b).We next substitute (4.8) into (4.27) , for $i=3$, and in a similar analysis that led to the determination of (4.28a,b) and (4.29), determine the normal displacements $w_1^{(3)}$ and $w_2^{(3)}$ corresponding to (4.28a) and (4.29) respectively as

$$w_1^{(3)} = \frac{HD_1}{2\pi^2\psi_m}\Bigg[\pi\sum_{m=1,3,5,\Lambda}^{\infty}\Big\{\Big\{\Big\{\left(4n^2+m^2n^2\right)\omega_m+4n^2m\theta_m\Big\}\Big\{\left(\mathrm{w}_1^{(1)}f_1^{(2)}+f_1^{(1)}w_2^{(1)}+af_1^{(2)}\right)$$
$$+\left(\mathrm{w}_2^{(1)}f_2^{(2)}+f_2^{(1)}w_2^{(2)}+bf_2^{(2)}\right)\Big\}\Big\}\Big\}+\frac{4\left(m^2+n^2r\xi\right)(1+\xi)^2}{\left(m^2+n^2\xi\right)^2} \tag{4.31a}$$
$$\times\sum_{m=1,3,5,\Lambda}^{\infty}\Big\{\Big\{\Big\{\left(4n^2+m^2n^2\right)\omega_m+4n^2m\theta_m\Big\}\Big\{\left(\mathrm{w}_1^{(1)}\mathrm{w}_1^{(2)}+aw_1^{(2)}\right)+\left(\mathrm{w}_2^{(1)}\mathrm{w}_2^{(2)}+bw_2^{(2)}\right)\Big\}\Big\}\Big\}\Bigg]$$

where

$$\psi_m = \left[\left(m^2+n^2r\xi\right)^2-\lambda\lambda_{\mathrm{C}}\left\{\frac{\alpha\,\mathrm{m}^2}{2}+n^2\xi\left(1-\frac{\alpha r}{2}\right)\right\}-K(\xi)\left(m^2+n^2r\xi\right)^2(1+\xi)^2\left(m^2+n^2\xi\right)^{-2}\right] \tag{4.31b}$$

$$D_1 = \frac{Eh}{D(1+\xi)^2}\left\{\frac{1}{r_y}\left(\frac{L}{\pi}\right)^2\right\}^2 \tag{4.31c}$$

$$w_2^{(3)} = \frac{HD_1}{2\pi^2\psi_m}\Bigg[\pi\sum_{m=1,3,5,\Lambda}^{\infty}\Big\{\Big\{\Big\{\left(4n^2+m^2n^2\right)\omega_m+4n^2m\theta_m\Big\}\Big\{\left(\mathrm{w}_1^{(1)}f_2^{(2)}+f_1^{(1)}w_2^{(2)}+af_2^{(2)}\right)$$
$$-\left(\mathrm{w}_2^{(1)}f_1^{(2)}+f_2^{(1)}w_1^{(2)}+bf_1^{(2)}\right)\Big\}\Big\}\Big\}+\frac{4\left(m^2+n^2r\xi\right)(1+\xi)^2}{\left(m^2+n^2\xi\right)^2}\times$$

$$\sum_{m=1,3,5,\Lambda}^{\infty}\left\{\left\{\left\{\left(4n^2+m^2n^2\right)\omega_m+4n^2m\theta_m\right\}\left\{\left(w_1^{(1)}w_1^{(2)}+aw_1^{(2)}\right)-\left(w_2^{(1)}w_2^{(2)}+bw_1^{(2)}\right)\right\}\right\}\right\}\Bigg] \quad (4.32)$$

In particular, when m=1,we have , from (4.31a-c) and (4.32), after some simplification,

$$\tilde{w}_1^{(3)}=\frac{HD_1n^2\left(5\tilde{\omega}_1+4\tilde{\theta}_1\right)}{2\pi^2\tilde{\psi}_1}\Big[l_{28}w_1^{(1)3}+l_{29}w_1^{(1)\,2}w_2^{(1)}+l_{30}w_1^{(1)}w_2^{(1)\,2}+l_{31}w_2^{(1)\,3}+l_{32}w_1^{(1)2}$$
$$+l_{33}w_1^{(1)}w_2^{(1)}+l_{34}w_1^{(1)2}+l_{35}w_1^{(1)}+l_{36}w_2^{(1)}\Big] \quad (4.33)$$

$$\tilde{w}_2^{(3)}=\frac{HD_1n^2\left(5\tilde{\omega}_1+4\tilde{\theta}_1\right)}{2\pi^2\tilde{\psi}_1}\Big[l_{37}w_2^{(1)3}+l_{38}w_1^{(1)\,2}w_2^{(1)}+l_{39}w_1^{(1)\,2}$$
$$+l_{40}w_1^{(1)}w_2^{(1)}+l_{41}w_2^{(1)2}+l_{42}w_1^{(1)}+l_{43}w_2^{(1)}\Big] \quad (4.34)$$

where

$$l_{28}=-D_2l_{17}+D_1\left(l_{21}-l_0l_{17}\right),l_{29}=-D_2l_{17},\ \ l_{30}=D_2l_{17}+D_1\left(l_{22}+l_0l_{17}+l_{25}-l_0l_{19}\right) \quad (4.35a)$$

$$l_{31}=D_2l_{17}\ \ ,l_{32}=D_2\left\{a\left(l_{17}-l_{18}\right)+bl_{17}\right\}+D_1a\left(l_{23}-l_0l_{18}\right) \quad (4.35b)$$

$$l_{33}=D_2l_{18}\left(b-a\right)+bD_1\left(l_{24}+l_0l_{18}+l_{26}-l_0l_{20}+l_{25}\right),l_{34}$$
$$=D_2\left(bl_{18}-al_{17}\right)+D_1\left(a_{26}-bl_0l_{20}+al_{22}\right) \quad (4.35c)$$

$$l_{35}=D_2l_{18}\left(a^2+ab\right)+D_1\left(a^2l_{23}+b^2l_{27}\right),\ \ l_{36}=-D_2l_{18}\left(ab+b^2\right)+D_1ab\left(l_{24}+l_{26}\right) \quad (4.35d)$$

$$I_{37}=-D_2l_{17}-D_1\left(l_{22}+l_0l_{17}\right),\ l_{38}=D_2\left(l_{17}-l_{19}\right)+D_1\left(l_{25}-l_0l_{19}-l_{21}+l_ol_{17}\right) \quad (4.35e)$$

$$l_{39}=-D_2bl_{20}+D_1b\left(l_{27}-l_0l_{20}-l_{21}\right) \quad (4.35f)$$

$$l_{40}=D_2\left\{a\left(l_{18}-l_{20}-l_{19}\right)+bl_{19}\right\}+aD_1\left\{l_{26}-l_0l_{20}+l_{25}-l_{23}+l_0l_{18}\right\} \quad (4.35g)$$

$$l_{41}=-D_2bl_{18}-bD_1\left(l_{24}+l_0l_{18}+l_{22}\right)\ ,\ l_{42}=l_{20}D_2\left(b^2-ab\right)+abD_1\left(l_{27}-l_{23}\right) \quad (4.35h)$$

$$l_{43}=D_2l_{20}\left(ab-a^2\right)+D_1\left(a^2l_{26}-b^2l_{24}\right),\mathrm{D}_2=\left(\frac{1+\xi}{1+n^2\xi}\right)^2\left(1+n^2r\xi\right),\tilde{\psi}_1=l_2 \quad (4.35i)$$

Here ,$\tilde{\psi}_1$ is the value of ψ_m evaluated at m=1 and it takes the same value as l_2 in (4.10b). Thus we now write

$$w^{(3)}\left(x,y\right)=\sum_{m=1,3,5,\Lambda}^{\infty}\left(w_1^{(3)}\cos\mathrm{ny}+\mathrm{w}_2^{(3)}\sin\mathrm{ny}\right)\sin\mathrm{mx} \quad (4.36a)$$

so that the overall displacement w(x,y) now becomes

$$w\left(x,y\right)=\in\left(w_1^{(1)}\cos\mathrm{ny}+\mathrm{w}_2^{(1)}\sin\mathrm{ny}\right)\sin\mathrm{x}+\in^2\left(\sum_{m=1,3,5,\Lambda}^{\infty}\left(w_1^{(2)}\cos 2\mathrm{ny}+\mathrm{w}_2^{(2)}\sin 2ny\right)\sin\mathrm{mx}\right)$$

$$+\in^3\left(\sum_{m=1,3,5,\Lambda}^{\infty}\left(w_1^{(3)}\cos\mathrm{ny}+\mathrm{w}_2^{(3)}\sin ny\right)\sin\mathrm{mx}\right)+O\left(\in^4\right) \quad (4.36b)$$

However, it is only the displacement components that are in the shape of imperfection that have a dominant role in the buckling process and so, henceforth, we neglect terms of $O\left(\in^2\right)$ in (4.36b) and now evaluate the remaining displacement at $m=1$ to get

$$w(x,y) = \in \left(w_1^{(1)} \cos ny + w_2^{(1)} \sin ny\right) \sin x + \in^3 \left(\tilde{w}_1^{(3)} \cos ny + \tilde{w}_2^{(3)} \sin ny\right) \sin x + O\left(\in^4\right) \tag{4.37}$$

5.0 Buckling load, λ_S

According to Budiansky and Amazigo [6] and Ette [7-9], the buckling load λ_S is obtained from the maximization

$$\frac{d\lambda}{dw} = 0 \tag{5.1}$$

However in order to eliminate the spatial dependence in (4.37),we first determine the same equation at critical values x_a , y_a of x and y respectively. The conditions for this are

$$w,_x \left(x_a, y_a\right) = w,_y \left(x_a, y_a\right) = 0 \tag{5.2}$$

We let

$$y_a = y_0 + \in^2 y_2 + \Lambda \tag{5.3}$$

From the first of (5.2), using (4.37), we have

$$x_a = \frac{\pi}{2} \tag{5.4}$$

On substituting (5.4) in the second of (5.2), using 4.37) and (5.3) and equating the coefficients of $\in$ and $\in^3$, we obtain respectively

$$y_0 = \frac{1}{n} \tan^{-1}\left(\frac{w_2^{(1)}}{w_1^1}\right) = \frac{1}{n} \tan^{-1}\left(\frac{b}{a}\right) \tag{5.5}$$

and

$$-n^2 y_2 \left(w_1^{(1)} \cos ny_0 + w_2^{(1)} \sin ny_0\right) + n\left(-\tilde{w}_1^{(3)} \sin ny_0 + \tilde{w}_2^{(3)} \cos ny_0\right) = 0 \tag{5.6}$$

From (5.6) we get

$$y_2 = \frac{1}{n}\left(\frac{\tilde{w}_2^{(3)} w_1^{(1)} - \tilde{w}_1^{(3)} w_2^{(1)}}{w_1^{(1)2} + w_2^{(1)2}}\right) \tag{5.7}$$

By now evaluating (4.37) at x_a , y_a , using (5.4)-(5.7), we have

$$w = C_1 \in + C_3 \in^3 + \Lambda \tag{5.8a}$$

$$C_1 = \sqrt{\left(w_1^{(1)2} + w_2^{(1)2}\right)}\ , C_3 = \frac{\left(\tilde{w}_1^{(3)} w_1^{(1)} + \tilde{w}_2^{(3)} w_2^{(1)}\right)}{\sqrt{\left(w_1^{(1)2} + w_2^{(1)2}\right)}} \tag{5.8b}$$

As in [6-8], we first reverse the series (5.8a,b) and obtain

$$\in = d_1 w + d_3 w^3 + \Lambda \tag{5.9a}$$

$$d_1 = \frac{1}{C_1}\ ,\ d_3 = -\frac{C_3}{C_1^4} \tag{5.9b}$$

By substituting into (5.9a) for w from (5.8a) and equating the coefficients of $\in$ and $\in^3$, we have

The maximization (5.1) easily follows direct from (5.9a) to yield

$$\in = \frac{2}{3}\sqrt{\frac{C_1}{3C_3}} = \frac{2}{3}\sqrt{\frac{a w_1^{(1)} + b w_2^{(1)}}{3\left(a\tilde{w}_1^{(3)} + b\tilde{w}_2^{(3)}\right)}} \tag{5.10}$$

where all functions of λ in (5.10) are now evaluated at λ_S .On substituting for all the terms in (5.10), we have

$$\left[\left(1+n^2\xi^2\right)^2-\lambda_S\lambda_C\left\{\frac{\alpha}{2}+n^2\xi\left(1-\frac{\alpha\,\mathrm{r}}{2}\right)\right\}-K(\xi)\left(1+n^2r\xi^2\right)^2(1+\xi)^2\left(1+n^2\xi^2\right)^{-2}\right]^{\frac{3}{2}}$$

$$=\frac{3\pi\sqrt{6}}{2}\lambda_S\lambda_C n\,a\in\left\{\frac{\alpha}{2}+n^2\xi\left(1-\frac{\alpha\,\mathrm{r}}{2}\right)\right\}\sqrt{\frac{Pl_{28}}{\left\{1+\left(\frac{b}{a}\right)^2\right\}}}\quad\sqrt{HD_1\left(5\tilde{\omega}_1+4\tilde{\theta}_1\right)} \tag{5.11}$$

where

$$P=\left[1+\frac{a^2}{l_{28}}\left(\frac{l_2}{\lambda_S\lambda_C l_1}\right)^3\left\{\left(al_{31}+bl_{37}\right)w_2^{(1)3}+\left(al_{27}+bl_{28}\right)w_1^{(1)2}w_2^{(1)}+al_{30}w_1^{(1)}w_2^{(1)2}+\left(al_{32}+bl_{39}\right)w_1^{(1)2}\right.\right.$$

$$\left.+\left(al_{33}+bl_{40}\right)w_1^{(1)}w_2^{(1)}+\left(al_{34}+bl_{41}\right)w_1^{(1)2}+\left(al_{35}+bl_{42}\right)w_1^{(1)}+\left(al_{36}+bl_{43}\right)w_2^{(1)}\right] \tag{5.12}$$

6.0 Analysis of results

The results (5.11) and (5.12) are asymptotically valid for n>5 as well as for the case where the imperfection amplitude is less than one half of the imperfection amplitude (i.e. $\in<\frac{h}{2}$). Notwithstanding their seeming lengthy nature , the results are simple and straightforward formulae that determine the static buckling load λ_S in a simple manner because all other terms appearing there are either specified or already derived. By setting $r=0$, we automatically obtain equivalents results valid for imperfect cylindrical shell segments under the same loading conditions. By respectively setting the Fourier coefficients $a=0$ (in the first instance) and $b=0$ (in the second instance), we automatically and respectively obtain equivalent results valid for imperfections in the forms $\overline{w}=b\sin\mathrm{ny}\sin\mathrm{x}$ and $\overline{w}=a\cos\mathrm{ny}\sin\mathrm{x}$ .An approximate result of (5.11) can be obtained by maintaining, in the expression $\tilde{w}_1^{(3)}$ and $\tilde{w}_2^{(3)}$,only the terms multiplying $w_1^{(1)3}$ and $w_2^{(1)3}$ and so, simplify to get

$$\left[\left(1+n^2\xi^2\right)^2-\lambda_S\lambda_C\left\{\frac{\alpha}{2}+n^2\xi\left(1-\frac{\alpha\,\mathrm{r}}{2}\right)\right\}-K(\xi)\left(1+n^2r\xi^2\right)^2(1+\xi)^2\left(1+n^2\xi^2\right)^{-2}\right]^{\frac{3}{2}}$$

$$\cong\frac{3\pi\sqrt{6}}{2}\lambda_S\lambda_C n\in\left\{\frac{\alpha}{2}+n^2\xi\left(1-\frac{\alpha\,\mathrm{r}}{2}\right)\right\}\quad\sqrt{HD_1\left(5\tilde{\omega}_1+4\tilde{\theta}_1\right)}\left[\frac{\mathrm{a}^4l_{28}+b^3\left(al_{31}+bl_{37}\right)}{a^2+b^2}\right]^{\frac{1}{2}} \tag{6.1}$$

On substituting for λ_C only the left hand sides of (5.11) and (5.13) from (3.7), we get respectively

$$\left(1-\lambda_S\right)^{\frac{3}{2}}=\frac{3\pi\sqrt{6}}{2}na\in\left[\lambda_C\left\{\frac{\alpha}{2}+n^2\xi\left(1-\frac{\alpha\,\mathrm{r}}{2}\right)\right\}\right]^{-\frac{1}{2}}\sqrt{HD_1\left(5\tilde{\omega}_1+4\tilde{\theta}_1\right)}\sqrt{\frac{\mathrm{P}\left(\lambda_S\right)l_{28}}{\left\{1+\left(\frac{b}{a}\right)^2\right\}}} \tag{6.2}$$

and

$$\left(1-\lambda_S\right)^{\frac{3}{2}} \cong \frac{3\pi\sqrt{6}}{2} n \in \left[\lambda_C\left\{\frac{\alpha}{2}+n^2\xi\left(1-\frac{\alpha \mathrm{r}}{2}\right)\right\}\right]^{-\frac{1}{2}} \sqrt{HD_1\left(5\tilde{\omega}_1+4\tilde{\theta}_1\right)}\left[\frac{a^4 l_{28}+b^3\left(a l_{31}+b l_{37}\right)}{a^2+b^2}\right]^{\frac{1}{2}} \quad (6.3)$$

We clearly observe from (5.14) and (5.15) that the load degradation is of order $\in^{\frac{2}{3}}$. We equally observe from all the results that the buckling load λ_S depends on the two Fourier coefficients *a* and *b* as well as on the multiplicative coupling of these coefficients. In general, the results would normally depend on all the Fourier coefficients admitted in the formulation but the analysis becomes increasingly prohibitive if the number of these coefficients is greater than two. By assigning various values to the parameter *r*, we can explore the variation of λ_S with various values of *r*.

References

[1] M. Stein and J. A. McElman(1965): Buckling of segments of toroidal shells, AIAA J, 3, 1704.

[2] J. W. Hutchinson (1967): Initial post-buckling behaviour of toroidal shell segments, Int. J. Solids Struct. 3, 97-115.

[3] M. O. Oyesanya (2002): Asymptotic analysis of imperfection sensitive toroidal shell segments with modal imperfection, J. of Nigerian Assoc. of Maths .Physics, 6,197-206.

[4] M. O. Oyesanya (2002): Secondary Bifurcation states and stability of the toroidal shell segment. ABACUS-The journal of the Mathematical Association of Nigeria, 29(2), 34-46.

[5] M. O. Oyesanya (2003): Analysis of imperfection sensitivity of toroidal shell segment with random imperfection, J. of Nigerian Assoc. Of Maths. Physics, 7, 207-214.

[6] B. Budiansky and J.C. Amazigo (1968): Initial post buckling behaviour of cylindrical shells under external pressure, J. of Maths. Physics, 47, 3, 223-235.

[7] A. M. Ette (2006): On the Buckling of lightly damped cylindrical shells modulated by a periodic load, J. Nigerian Assoc. Maths. Physics, 10, 327-344.

[8] A. M. Ette (2006): On a two-parameter buckling of a lightly damped spherical cap trapped by a step load, [9] J. of Nigerian Maths. Soc. 24, 7-26.

[10] A.M.Ette (2007): On a two –small- parameter dynamic stability of a lightly damped spherical shell pressurized by a harmonic excitation, J .Nigerian Assoc. Maths. Physics, 11,333-362.