

**Perturbation technique on the dynamic stability of a lightly damped cylindrical shell axially stressed by an impulse**

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**Abstract**

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*In this investigation, we examine the dynamic stability of a viscously damped finite imperfect circular cylindrical shell stressed by an axially impacted dynamic impulse. The formulation contains two small but mathematically independent parameters upon which asymptotic expansions are initiated using regular perturbation procedures. Simply-supported boundary conditions are assumed and the ensuing imperfection is taken strictly in the shape of the first term in the Fourier sine series expansion. The buckling modes, which are assumed strictly in the shape of imperfection, are obtained and the dynamic buckling impulse is evaluated nontrivially. It is observed that a viscously damped imperfect cylindrical shell buckles at a relatively higher dynamic impulse than an undamped one. Hence, damped cylindrical shells are dynamically more stable than undamped ones in the sense that they can sustain higher dynamic pressures and become unstable only when the pressures are excessive.*

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**1.0 Introduction**

Investigations into the dynamic stability of thin-walled elastic structures have received global attention in the past forty years and various methods and techniques, both analytical and numerical (sometimes, a combination of both) have been devised by Mathematicians and designers of modern engineering structures. The first major investigations in this respect were given by Budiansky and Hutchinson [1-3] in a series of awe-inspired publications based on approximate theory of dynamic buckling of imperfection-sensitive structures, which in themselves, were a direct extension to the dynamic setting of Koiter's [4] original static theory on post buckling behaviour of elastic structures. In this investigation, we are exploring the extent of the dynamic stability, under an impulse driven dynamic load, of a lightly and viscously damped imperfect circular cylindrical shell in the hope of highlighting the role or effects (if any) of light damping in the buckling process. The analysis is primarily predicated on the use of regular perturbation technique in asymptotic expansions of the variables in which a two-small-parameter nonlinear coupled elastic system is analyzed using a two-time method analysis.

**2.0 Formulation**

Assuming that the normal displacement at a point on the shell surface is  $W(X, T, T)$  and Airy stress function is  $F(X, Y, T)$ , where  $X$  and  $Y$  are the spatial variables and  $T$  is the time variable, the usual Karman-Donnell compatibility equation and equilibrium equation for a damped circular cylindrical shell of length  $L$ , radius  $R$  and thickness  $h$  are respectively given [5] by

$$\frac{1}{Eh} \nabla^4 F - \frac{1}{R} W_{,xx} = S\left(W, \frac{W}{2} + \bar{W}\right) \quad (2.1)$$

$$\rho W_{,TT} + C_0 W_{,T} + D \nabla^4 W + \frac{1}{R} F_{,xx} = S(W + \bar{W}, F) \quad (2.2)$$

where  $E$  is the Young's modulus,  $\rho$  is the mass per unit area, and the cylindrical shell is subjected to a light viscous damping force  $C_0$  per unit area per velocity, as well as an axially applied impulse force

$\Lambda(T)$  per unit area. The bending stiffness is  $D = \frac{Eh^3}{12(1-\nu^2)}$ , where  $\nu$  is the Poisson's ratio and

$\bar{W}(X, Y)$  is the time-independent stress-free initial normal displacement while a subscript following a comma denotes partial differentiation with respect to the independent variable indicated by the subscript, and  $S$  is a bilinear operator defined by

$$S(P, Q) = P_{,xx} Q_{,yy} + P_{,yy} Q_{,xx} - 2P_{,xy} Q_{,xy} \quad (2.3)$$

Here  $\nabla^4$  is the usual biharmonic operator namely  $\nabla^4 = \left( \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \right)^2$ .

For convenience, we introduce the following non-dimensional quantities

$$x = \frac{\pi X}{L}, \quad y = \frac{Y}{h}, \quad \in = \frac{\bar{W}}{h}, \quad w = \frac{W}{h}, \quad \bar{t} = \frac{T \pi^2 \left( \frac{D}{\rho} \right)^{\frac{1}{2}}}{L^2}, \quad 2\eta = \frac{LC_0}{\rho \pi}, \quad (2.4a)$$

$$I \delta(\bar{t}) = \frac{L^2 R \Lambda(T)}{\pi^2 D}, \quad A = \frac{L^2 \sqrt{12(1-\nu^2)}}{\pi^2 R h}, \quad \xi = \frac{L^2}{\pi^2 R^2}, \quad K(\xi) = -\left( \frac{A}{1+\xi} \right)^2, \quad H = \frac{h}{R} \quad (2.4b)$$

where  $\in$  is a small parameter which measures the amplitude of imperfection while  $\eta$  is small parameter which measures the amplitude of the light viscous damping. Though both  $\in$  and  $\eta$  satisfy the inequalities  $0 < \in << 1$  and  $0 < \eta << 1$ , yet they are mathematically independent. We consider the introduction of light damping a revolutionary departure from most of the existing dynamic buckling investigations to date, some which include Wang and Tien [6-8], Schenk and Schueller [9], Danielson [10], Ette [11], Wei et al [12], Batra and Wei [13] and Zhang et al [14], among others. Our intention to investigate the effects of damping is borne out of the observation that most physical systems in dynamical settings are never devoid of some form of damping and dynamic buckling phenomena are not exceptions.

We consider homogeneous normal displacement and velocity and neglect both axial and circumferential inertia. We assume simply-supported boundary conditions and neglect boundary layer effects by assuming that the pre-buckling deflection is constant and so we write

$$F = -\frac{1}{2} IR \left( X^2 + \frac{\alpha Y^2}{2} \right) + \frac{Eh^2 L^2}{\pi^2 R (1+\xi^2)} f \quad (2.5a)$$

$$W = \frac{IR^2 \left( 1 - \frac{\alpha \nu}{2} \right)}{Eh} + h w \quad (2.5b)$$

The first terms on the right hand sides of (2.5a,b) indicate the pre-buckling approximation while  $\alpha$  takes the value  $\alpha=1$  if the pressure contributes to axial stress through end plates, otherwise  $\alpha=0$  if pressure acts laterally. On substituting (2.4a,b) and (2.5a,b) into (2.1) and (2.2), we have respectively.

$$\bar{\nabla}^4 f - (1 + \xi)^2 w_{xx} = -(1 + \xi)^2 H \bar{S} \left( w, \frac{w}{2} + \in \bar{w} \right) \quad (2.6)$$

$$w_{tt} + 2\eta w_t + \bar{\nabla}^4 w - K(\xi) f_{xx} + I \delta(t) \left[ \frac{\alpha}{2} (w + \in \bar{w})_{xx} + \xi (w + \in \bar{w})_{yy} \right] = H K(\xi) \bar{S} [(w + \in \bar{w}), f] \quad (2.7)$$

where  $\bar{\nabla}^4 \equiv \left( \frac{\partial^2}{\partial x^2} + \xi \frac{\partial^2}{\partial y^2} \right)^2$ ,  $\bar{S}(P, Q) = P_{xx} Q_{yy} + P_{yy} Q_{xx} - 2P_{xy} Q_{xy}$

Simply-supported boundary conditions imply

$$w = w_{xx} = f = f_{xx} = 0 \text{ at } x = 0, \pi \quad (2.8)$$

and homogeneous initial conditions imply

$$w(x, y, 0^-) = w_t(x, y, 0^-) = 0, 0 < x < \pi, 0 < y < 2\pi \quad (2.9)$$

We note that  $\delta(\bar{t})$  is the Dirac -delta function of time and  $I$  is its amplitude. Our main aim is to determine a particular value of  $I$  at buckling, namely  $I_D$ , called the dynamic buckling impulse, at which the structure becomes unstable dynamically . We define  $I_D$  as the maximum value of the load amplitude  $I$  for the solution of the problem (2.6) - (2.9) to be bounded for all time  $t > 0^-$  .

### 3.0 Classical buckling load

To determine the classical buckling load  $\lambda_C$ , we substitute 1 for  $\delta(t)$  and rename  $I$  as  $\lambda$ .We ignore the inertia term and define  $\lambda_C$  as the minimum load parameter for which the solution of the associated linear problem of the perfect cylindrical shell has a nontrivial solution .The ensuing differential equations, from (2.6) and (2.7), are respectively

$$\bar{\nabla}^4 f - (1 + \xi)^2 w_{xx} \quad (3.1a)$$

$$\bar{\nabla}^4 w - K(\xi) f_{xx} + \lambda_C \left[ \frac{\alpha}{2} (w + \in \bar{w})_{xx} + \xi (w + \in \bar{w})_{yy} \right] = 0 \quad (3.1b)$$

with homogeneous boundary conditions  $w = w_{xx} = f = f_{xx} = 0$  at  $x = 0, \pi$  (3.1c)

For solution of (3.1a,b) , we assume  $(w, f) = (a_{mk}, b_{mk}) \sin(ky + \varphi_{mk}) \sin mx$  (3.2)

Such a solution was given in [5] as

$$\lambda_C = \frac{\left[ (1 + n^2 \xi)^2 - (1 + \xi)^2 K(\xi) (1 + n^2 \xi)^{-2} \right]}{\frac{\alpha}{2} + n^2 \xi} \quad (3.3a)$$

where  $n$ ,in (3.3a), is the critical integer value of  $k$  that minimizes  $\lambda$  .The corresponding displacement and stress function are

$$(w, f) = \left( 1, -\frac{(1 + \xi)^2}{(1 + n^2 \xi)^2} \right) a_{1n} \sin(ny + \varphi_{1n}) \sin x \quad (3.3b)$$

where have substituted for  $K(\xi)$  in (3.3a) in order to get (3.3b).

#### 4.0 Dynamic theory

We now solve equations (2.6)-(2.9) in full. The procedure for obtaining the dynamic buckling impulse is as follows:

(a) We shall first determine uniformly valid asymptotic expression of  $w(x, y, \bar{t})$ , and hence that of  $f(x, y, \bar{t})$  by using regular perturbation

- (b) We shall next determine the maximum value of the displacement  $w(x, y, \bar{t})$  and lastly
- (c) determine the dynamic buckling impulse  $I_D$  [5,10,11] using the maximization

$$\frac{dI}{dw_a} = 0 \quad (4.1)$$

where  $w_a$  is the maximum of  $w(x, y, \bar{t})$ .

We assume that the impulse acts, as is customary, within a very small time interval namely  $0^- < \bar{t} < 0^+$  and that the displacement,  $w(x, y, \bar{t})$ , is continuous within and across that interval. We integrate (7) from  $0^-$  to  $0^+$  and get

$$w_{,\bar{t}\bar{t}} + 2\eta w_{,\bar{t}} + \bar{\nabla}^4 w - K(\xi)f_{,xx} = -HK(\xi)[\bar{S}(w + \bar{w}, f)] , \bar{t} > 0^+ \quad (4.2a)$$

$$w(x, y, 0^+) = 0 , w_{,\bar{t}}(x, y, 0^+) = -I \left[ \frac{\alpha}{2} (w + \bar{w})_{,xx} + \xi (w + \bar{w})_{,yy} \right] \quad (4.2b)$$

$$0 < x < \pi , 0 < y < 2\pi ; w = w_{,xx} = f = f_{,xx} = 0 \text{ at } x = 0, \pi \quad (4.2c)$$

Equation (2.6) still suffices, though, is now defined in the interval  $t > 0^+$ . The sole essence of the impulse is to impact to the structure a non-vanishing initial velocity immediately after its action so that, a considerable quantum of initial kinetic energy which accelerates the structure, is thereafter unleashed on the structure after the impulse's action. This is characteristic of impulses, like Earthquakes, whose initial kinetic energy could cause considerable catastrophes and wide spread damages far and wide- all depending on the initial magnitude of  $I$  measured in Richter scale. We let

$$\tau = \eta \bar{t} ; t = \bar{t} + \frac{(\omega_2 \in^2 + \omega_3 \in^3 + \omega_4 \in^4 + \Lambda)}{\eta} , \omega_i = \omega_i(\tau), \omega_i(0) = 0, i = 2, 3, 4, \Lambda \quad (4.3a)$$

and also let  $w(x, y, \bar{t}) = U(x, y, t, \tau)$  so that

$$w_{,\bar{t}} = (1 + \omega'_2 \in^2 + \omega'_3 \in^3 + \Lambda) U_{,\tau} + \eta U_{,\tau\tau} \quad (4.3b)$$

$$w_{,\bar{t}\bar{t}} = (1 + \omega'_2 \in^2 + \omega'_3 \in^3 + \Lambda)^2 U_{,\tau\tau} + 2\eta (1 + \omega'_2 \in^2 + \omega'_3 \in^3 + \Lambda) U_{,\tau\tau} \quad (4.3c)$$

$$+ \eta^2 U_{,\tau\tau\tau} + \eta (1 + \omega''_2 \in^2 + \omega''_3 \in^3 + \Lambda) U_{,\tau\tau} \quad (4.3d)$$

where  $\frac{d(\ )}{d\tau} = (\ )'$ . Henceforth any evaluation at  $t = 0^+$  shall simply be said to be at  $t=0$ . We now let

$$\begin{pmatrix} f(x, y, t, \tau) \\ w(x, y, t, \tau) \end{pmatrix} = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \begin{pmatrix} f^{ij}(x, y, t, \tau) \\ w^{ij}(x, y, t, \tau) \end{pmatrix} \in^i \eta^j \quad (4.4)$$

where the  $i$   $j$  as in  $f^{ij}$  and  $w^{ij}$  are superscripts and not powers. We substitute (4.3a-d) and (4.4) into (2.6) and (4.2a-c) and equate the coefficients of  $\in^i \eta^j$ ,  $i=1,2,3,\Lambda$   $j=0,1,2,\Lambda$  and obtain the following equations:

$$L^{(1)}(U^{10}, f^{10}) \equiv \bar{\nabla}^4 f^{10} - (1+\xi)^2 U_{,xx}^{10} = 0 \quad (4.5a)$$

$$L^{(2)}(U^{10}, f^{10}) \equiv U_{,tt}^{10} + \bar{\nabla}^4 U^{10} - K(\xi) f_{,xx}^{10} = 0 \quad (4.5b)$$

$$L^{(1)}(U^{11}, f^{11}) = 0 \quad (4.6a)$$

$$L^{(2)}(U^{11}, f^{11}) = -2U_{,tt}^{10} - 2U_{,t}^{10} \quad (4.6b)$$

$$L^{(1)}(U^{20}, f^{20}) = -(1+\xi)^2 H \left[ \frac{1}{2} \bar{S}(U^{10}, U^{10}) + \bar{S}(U^{10}, \bar{w}) \right] \quad (4.7a)$$

$$L^{(2)}(U^{20}, f^{20}) = -HK(\xi) [\bar{S}(U^{10}, f^{10}) + \bar{S}(f^{10}, \bar{w})] \quad (4.7b)$$

$$L^{(1)}(U^{21}, f^{21}) = -(1+\xi)^2 H \left[ \frac{1}{2} \bar{S}(U^{10}, U^{11}) + \bar{S}(U^{11}, \bar{w}) \right] \quad (4.8a)$$

$$L^{(2)}(U^{21}, f^{21}) = -HK(\xi) [\bar{S}(U^{10}, U^{11}) + \bar{S}(U^{11}, f^{10}) + \bar{S}(f^{11}, \bar{w})] - 2U_{,tt}^{20} - 2U_{,t}^{20} \quad (4.8b)$$

$$L^{(1)}(U^{30}, f^{30}) = -(1+\xi)^2 H [\bar{S}(U^{10}, U^{20}) + \bar{S}(U^{20}, \bar{w})] \quad (4.9a)$$

$$L^{(2)}(U^{30}, f^{30}) = -HK(\xi) [\bar{S}(U^{10}, f^{20}) + \bar{S}(U^{20}, f^{10}) + \bar{S}(f^{20}, \bar{w})] - 2\omega'_2 U_{,tt}^{10} \quad (4.9b)$$

$$L^{(1)}(U^{31}, f^{31}) = -(1+\xi)^2 H \left[ \frac{1}{2} \bar{S}(U^{11}, U^{20}) + \frac{1}{2} \bar{S}(U^{10}, U^{21}) + \bar{S}(U^{21}, \bar{w}) \right] \quad (4.10a)$$

$$L^{(2)}(U^{31}, f^{31}) = -HK(\xi) [\bar{S}(U^{21}, f^{10}) + \bar{S}(U^{20}, f^{11}) + \bar{S}(U^{10}, f^{21}) + \bar{S}(U^{11}, f^{20}) + \bar{S}(f^{21}, \bar{w})] - 2(\omega'_2 U_{,tt}^{11} + \omega'_2 U_{,t}^{10} + \omega'_2 U_{,t}^{10} + \omega'_2 U_{,tt}^{10} + U_{,t}^{30} + U_{,tt}^{30}) \quad (4.10b)$$

The boundary conditions are  $U^{ij} = U_{,xx}^{ij} = f^{ij} = f_{,xx}^{ij} = 0$   $i=1,2,3,\Lambda$ ;  $j=0,1,2,3$  (4.11a)

The initial conditions are evaluated at  $t=\tau=0$  and are given as follows

$$U^{10} = 0 ; U_{,t}^{10} = -I \left[ \frac{\alpha}{2} (U^{10} + \bar{w})_{,xx} + \xi (U^{10} + \bar{w})_{,yy} \right] \quad (4.11b)$$

$$U_{,t}^{1r} + U_{,t}^{1k} = -I \left[ \frac{\alpha}{2} U_{,xx}^{1r} + \xi U_{,yy}^{1r} \right] ; U_{,t}^{20} + U_{,t}^{20} = -I \left[ \frac{\alpha}{2} U_{,xx}^{20} + \xi U_{,yy}^{20} \right] \quad (4.11c)$$

$$U_{,t}^{2r} + U_{,t}^{2k} = -I \left[ \frac{\alpha}{2} U_{,xx}^{2r} + \xi U_{,yy}^{2r} \right] ; U_{,t}^{30} + \omega'_2(0) U_{,t}^{10} = -I \left[ \frac{\alpha}{2} U_{,xx}^{30} + \xi U_{,yy}^{30} \right] \quad (4.11d)$$

$$U_{,t}^{3r} + U_{,t}^{3k} + \omega'_2(0) U_{,t}^{1r} = -I \left[ \frac{\alpha}{2} U_{,xx}^{3r} + \xi U_{,yy}^{3r} \right] \quad (4.11e)$$

$$k = r-1 ; r = 1, 2, 3, \Lambda$$

For solutions of (4.5a) - (4.11e), we let

$$\begin{pmatrix} f^{ij} \\ w^{ij} \end{pmatrix} = \sum_{p,q=1,2,3,\Lambda}^{\infty} \left\{ \begin{pmatrix} f_1^{ij}(t, \tau) \\ w_1^{ij}(t, \tau) \end{pmatrix} \cos qy + \begin{pmatrix} f_2^{ij}(t, \tau) \\ w_2^{ij}(t, \tau) \end{pmatrix} s \sin qy \right\} \sin px \quad (4.12)$$

We now let

$$\bar{w}(x, y) = \bar{a} \sin x \sin ny \quad (4.13)$$

We substitute (4.12) into (4.5a) for  $i = 1, j = 0$  and easily get

$$f_1^{10} = -\frac{(1+\xi)^2 p^2 U_1^{10}}{(p^2 + \xi q^2)^2}, \quad f_2^{10} = -\frac{(1+\xi)^2 p^2 U_2^{10}}{(p^2 + \xi q^2)^2} \quad (4.14)$$

In particular if  $p=1$  and  $q=n$ , we have

$$f_1^{10} = -\frac{(1+\xi)^2 U_1^{10}}{(1+\xi n^2)^2}, \quad f_2^{10} = -\frac{(1+\xi)^2 U_2^{10}}{(1+\xi n^2)^2} \quad (4.15)$$

We substitute (4.12) into (4.5b), multiply the resultant equation by  $\sin mx \sin ny$ , bearing in mind the initial conditions as in (4.11b) and get, for  $m=1$ , and using (4.14b) and (4.13)

$$U_{2,t}^{10} + \psi^2 U_2^{10} = 0, \quad U_2^{10}(0,0) = 0, \quad U_{2,t}^{10}(0,0) = I \bar{a} \left( \frac{\alpha}{2} + n^2 \xi \right), \quad \psi^2 = \left[ (1+n^2 \xi)^2 + \left( \frac{A}{1+n^2 \xi} \right)^2 \right] \quad (4.16)$$

If we however multiply the above substitution into (4.5b) by  $\cos mx \sin ny$  and simplify, using the first of (4.15), we get

$$U_{1,t}^{10} + \psi_k^2 U_1^{10} = 0; \quad U_1^{10}(0,0) = 0, \quad U_{1,t}^{10}(0,0) = 0, \quad \psi_k^2 = \left\{ (m^2 + (kn^2)^2 \xi)^2 + \left( \frac{Am^2}{(m^2 + (kn^2)^2 \xi)} \right)^2 \right\} \quad \forall k, m \quad (4.17)$$

On solving (4.16), we get

$$U_2^{10}(t, \tau) = \beta_2(\tau) \cos \psi t + \zeta_2(\tau) \sin \psi t, \quad \beta_2(0) = 0, \quad \zeta_2(0) = \bar{a} B, \quad B = \frac{I \left( \frac{\alpha}{2} + n^2 \xi \right)}{\psi} \quad (4.18)$$

On solving (4.17), we get

$$U_1^{10}(t, \tau) = \beta_1(\tau) \cos \psi_k t + \zeta_1(\tau) \sin \psi_k t; \quad \beta_1(0) = 0; \quad \zeta_1(0) = 0 \quad (4.19)$$

We now substitute the relevant terms on the right hand sides of (4.6a,b) and get

$$L^{(1)}(U^{11}, f^{11}) = 0 \quad (4.20a)$$

$$L^{(2)}(U^{11}, f^{11}) = -2 \left[ (U_{1,t}^{10} + U_{1,t}^{10}) \cos x + (U_{2,t}^{10} + U_{2,t}^{10}) \sin x \right] \sin ny \quad (4.20b)$$

$$U^{11}(x, y, 0, 0) = 0; \quad U_{,t}^{11}(x, y, 0, 0) + U_{,t}^{10}(x, y, 0, 0) = 0 \quad (4.20c)$$

We substitute (4.12) into (4.20a), for  $i = 1, j = 1$ , and get

$$f_1^{11} = -\frac{(1+\xi)^2 U_1^{11}}{(1+\xi n^2)^2}, \quad f_2^{11} = -\frac{(1+\xi)^2 U_2^{11}}{(1+\xi n^2)^2} \quad (4.21a)$$

In particular, for  $p = m, q = n$  we have

$$f_1^{11} = -\frac{(1+\xi)^2 m^2 U_1^{11}}{(m^2 + \xi n^2)^2}, \quad f_2^{11} = -\frac{(1+\xi)^2 m^2 U_2^{11}}{(m^2 + \xi n^2)^2} \quad (4.21b)$$

We now substitute the relevant terms into (4.20b), multiply through by  $\sin mx \sin ny$ , simplify and to ensure a uniform valid solution in the time scale  $t$  equate to zero the coefficients of  $\cos \psi_k t$  and  $\sin \psi_k t$  and get

$$\xi'_2 + \zeta'_2 = 0; \quad \beta'_2 + \beta'_2 = 0 \quad (4.22a)$$

The solutions (4.22a), subject to (4.18) are

$$\xi_2(\tau) = \zeta_2(0)e^{-\tau}; \beta_2(\tau) = 0 \quad (4.22b)$$

If, instead, we multiply (4.20b) by  $\cos mx \sin ny$  and equate to zero coefficient of  $\cos \psi_k t$  and  $\sin \psi_k t$  in order to maintain a uniformly valid solution in  $t$ , we have

$$\xi'_1 + \zeta'_1 = 0; \beta'_1 + \beta'_1 = 0 \quad (4.23a)$$

On solving (4.23a) subject to (4.19), we have

$$\zeta_1(\tau) = \beta_1(\tau) \equiv 0 \quad (4.23b)$$

We therefore conclude that

$$U_2^{10}(t, \tau) = \zeta_2(0)e^{-\tau} \sin \psi t; U_1^{10}(t, \tau) = 0, \quad U^{10} = U_2^{10}(t, \tau) \sin x \sin ny \quad (4.24)$$

The remaining equations in (4.20b) are

$$U_{2,tt}^{11} + \psi^2 U_2^{11} = 0; U_{1,tt}^{11} + \psi^2 U_1^{11} = 0; \quad U_r^{11}(0,0) = 0; \quad U_{r,t}^{11}(0,0) = 0, \quad r = 1, 2. \quad (4.25)$$

On solving (4.24), we get

$$U_2^{11}(t, \tau) = \gamma_2(\tau) \cos \psi t + \theta_2(\tau) \sin \psi t; U_1^{11}(t, \tau) = \gamma_1(\tau) \cos \psi_k t + \theta_1(\tau) \sin \psi_k t \quad (4.26a)$$

$$\gamma_2(0) = \theta_2(0) = \gamma_1(0) = \theta_1(0) = 0 \quad (4.26b)$$

Solutions beyond this level reveal that  $U_1^{11}$  and  $U_2^{11}$  depend multiplicatively on the initial conditions (4.26b) so that, we expect  $U_1^{11} = 0, U_2^{11} = 0$  (4.26c)

We now substitute on the right hand sides of (4.7a,b) and get

$$L^{(1)}(U^{20}, f^{20}) = -n^2(1+\xi)^2 H \left[ \frac{1}{2} U_2^{10}{}^2 + \bar{a} U_2^{10} \right] (\cos 2x + \cos 2ny) \quad (4.27a)$$

$$L^{(2)}(U^{20}, f^{20}) = -n^2 H K(\xi) [U_2^{10} f_2^{10} + \bar{a} f_2^{10}] (\cos 2x + \cos 2ny) \quad (4.27b)$$

$$U^{20}(x, y, 0, 0) = 0; \quad U_t^{20}(x, y, 0, 0) = 0 \quad (4.27c)$$

We substitute (4.12) into (4.27a), for  $i = 2, j = 0$ , multiply thereafter by  $\cos kny \sin mx$  and for

$k = 2, m$  odd, we have

$$f_1^{20} = -\frac{1}{(m^2 + 4n^2\xi)^2} \left[ \frac{4}{m\pi} (1+\xi)^2 n^2 H \left\{ \frac{1}{2} (U_2^{10})^2 + \bar{a} U_2^{10} \right\} \right] \quad (4.28a)$$

If we however multiply (4.29a) by  $\sin kny \sin mx$ , we have

$$f_2^{20} = -\frac{(1+\xi)^2 m^2 U_2^{20}}{\{m^2 + (kn)^2 \xi\}^2}, \quad \forall m \quad (m \text{ positive integer}) \quad (4.28b)$$

Next, we substitute (4.12) into (4.27b), for  $i = 2, j = 0$ ; multiply thereafter by  $\cos kny \sin mx$  and for  $k = 2, m$  odd, we get, using (4.28a)

$$U_{1,tt}^{20} + \psi_2^2 U_1^{20} = \bar{a} Q_1 U_2^{10} + Q_2 (U_2^{10})^2; \quad U_2^{10}(0,0) = 0, \quad U_{1,t}^{20}(0,0) = 0 \quad (4.29a)$$

where

$$\psi_2^2 = \left[ (m^2 + 4n^2\xi)^2 + \left( \frac{Am^2}{m^2 + 4n^2\xi} \right)^2 \right], \quad Q_1 = -\frac{4A^2 m n^2}{\pi} \left[ \frac{1}{(m^2 + 4n^2\xi)^2} + \frac{1}{(m^2 + n^2\xi)^2} \right] \quad (4.29b)$$

$$Q_2 = -\frac{4A^2 m n^2}{\pi} \left[ \frac{1}{2(m^2 + 4n^2 \xi)^2} + \frac{1}{(m^2 + n^2 \xi)^2} \right] \quad (4.29c)$$

If we however multiply (4.27b) by  $\sin kny \sin mx$ , we get, using (4.30b)

$$U_{2,t,t}^{20} + \psi_k^2 U_2^{20} = 0 \quad U_2^{20}(0,0) = 0, \quad U_{2,t}^{20}(0,0) = 0 \quad (4.30)$$

On solving (4.29a-c), we have

$$U_1^{20} = \beta_3(\tau) \cos \psi t + \zeta_3(\tau) \sin \psi t + \frac{Q_2 \zeta_2^2}{2\psi_2^2} + \frac{\bar{a} Q_1 \zeta_2 \sin \psi t}{\psi_2^2 - \psi^2} - \frac{Q_1 \zeta_2^2 \cos 2\psi t}{2(\psi_2^2 - 4\psi^2)} \quad (4.31a)$$

$$\beta_3(0) = (\bar{a}B)^2 R_2; \quad R_2 = \frac{Q_2}{2} \left\{ \frac{1}{(\psi_2^2 - 4\psi^2)} - \frac{1}{\psi_2^2} \right\}; \quad \zeta_3(0) = -\bar{a}^2 R_1 B; \quad R_1 = \frac{Q_1 \psi}{\psi_2(\psi_2^2 - \psi^2)} \quad (4.31b)$$

On solving (4.30), we have

$$U_2^{20} = \gamma_3(\tau) \cos \psi_k t + \theta_3(\tau) \cos \psi_k t; \quad \gamma_3(0) = 0, \quad \theta_3(0) = 0 \quad (4.32)$$

We next substitute the relevant terms on the right sides of (4.8a,b) and get

$$L^{(1)}(U^{21}, f^{21}) = 0 \quad (3.33a)$$

$$L^{(2)}(U^{21}, f^{21}) = -2 \sum_{m=1,3,5,\Lambda}^{\infty} (U_{1,t}^{20} + U_{t,\tau}^{20}) \cos 2nysin mx - 2 \sum_{m=1,3,5,\Lambda}^{\infty} (U_{1,t}^{20} + U_{t,\tau}^{20}) \sin 2nysin mx \quad (4.33b)$$

$$U^{21}(x, y, 0, 0) = 0; \quad U_{t,t}^{21}(x, y, 0, 0) + U_{t,\tau}^{20}(x, y, 0, 0) = 0 \quad (4.33c)$$

We now substitute (4.12) into (4.33a) for  $i = 2, j = 1$ ; multiply the resultant equation, first by  $\cos kny \sin mx$  and next by  $\sin kny \sin mx$  and, in the first and second multiplications we get respectively

$$f_1^{21} = -\frac{m^2(1+\xi)^2 U_1^{21}}{\{m^2 + 4n^2 \xi\}^2}, \quad f_2^{21} = -\frac{m^2(1+\xi)^2 U_2^{21}}{\{m^2 + (kn)^2 \xi\}^2} \quad (4.34)$$

where the first of (4.34) is valid for  $k = 2$  and  $m$  odd, and the second is valid for all  $k$  and all  $m$ . Next, we substitute for  $U_1^{20}$  and  $U_2^{20}$  from (4.31a) and (4.32) respectively, into (4.33b), then use (4.12) for  $i = 2, j = 1$ ; multiply by  $\cos kny \sin mx$  and to ensure a uniformly solution in  $t$ , equate to zero the coefficients of  $\cos 2\psi t$  and  $\sin 2\psi t$  for the case  $k = 2$  and get respectively

$$\zeta'_3 + \zeta_3 = 0; \quad \beta'_3 + \beta_3 = 0 \quad (4.35a)$$

However, if we multiply (4.33b) by  $\sin kny \sin mx$  and equate to zero the coefficients of  $\cos \psi_k t$  and  $\sin \psi_k t$ , we get

$$\theta'_3 + \theta_3 = 0; \quad \gamma'_3 + \gamma_3 = 0 \quad (4.35b)$$

The solutions of (4.35a,b) are

$$\zeta_3 = \zeta_3(0) e^{-\tau}, \quad \beta_3 = \beta_3(0) e^{-\tau}, \quad \theta_3(\tau) = \gamma_3(\tau) = 0 \quad (4.35c)$$

We thus conclude that

$$U^{20} = \sum_{m=1,3,5,\Lambda}^{\infty} U_1^{20} \cos 2nysin mx, \quad U_2^{20} = 0 \quad (4.36)$$

The remaining equation in (4.33b) now takes the form

$$U_{1,t,t}^{21} + \psi^2 U_1^{21} = Q_3(\tau) \cos \psi t + Q_4(\tau) \sin 2\psi t; \quad U_1^{21}(0,0) = 0, \quad U_{1,t}^{21}(0,0) + U_{1,\tau}^{20}(0,0) = 0 \quad (4.37a)$$

$$Q_3 = -\frac{2Q_1(\zeta'_2 + \zeta_2)}{\psi_2^2 - \psi^2}; Q_4 = -\frac{2\psi Q_2}{\psi_2^2 - 4\psi^2} \left\{ \zeta_2^2 + (\zeta_2')' \right\}; Q_3(0) = 0; Q_4(0) = -\frac{2\psi Q_2(\bar{a}B)^2}{\psi_2^2 - 4\psi^2} \quad (4.37b)$$

$$U_{2,t,t}^{21} + \psi_k^2 U_2^{21} = 0, U_2^{21}(0,0) = 0, U_{2,t}^{21}(0,0) + U_{2,\tau}^{20}(0,0) = 0 \quad (4.37c)$$

On solving (4.37a,b), we have

$$\begin{aligned} U_1^{21} &= \beta_4(\tau) \cos \psi_2 t + \zeta_4(\tau) \sin \psi_2 t + \frac{Q_3 \cos \psi t}{\psi_2^2 - \psi^2} + \frac{Q_4 \sin 2\psi t}{\psi_2^2 - 4\psi^2}; \\ \beta_4(0) &= 0, \zeta_4(0) = -\frac{4(\bar{a}B)^2 Q_4 \psi^2}{(\psi_2^2 - 4\psi^2)^2} \end{aligned} \quad (4.38)$$

On solving (4.37c), we have

$$U_2^{21}(t, \tau) = \gamma_4(\tau) \cos \psi_k t + \theta_4(\tau) \sin \psi_k t; \gamma_4(0) = \theta_4(0) = 0 \quad (4.39)$$

We now substitute the relevant terms on the right hand sides of (4.9a,b) and simplify to get

$$\begin{aligned} L^{(1)}(U^{30}, f^{30}) &= -(1+\xi)^2 H \sum_{m=1,3,5,\Lambda}^{\infty} [(U_2^{10} U_1^{20} + \bar{a} U_1^{20}) \{ (4n^2 + m^2 n^2) \sin ny \cos 2ny \sin x \sin mx \\ &+ 4n^2 m \cos ny \sin 2ny \cos x \cos mx \}] \end{aligned} \quad (4.40a)$$

$$L^{(2)}(U^{30}, f^{30}) = -HK(\xi) \sum_{m=1,3,5,\Lambda}^{\infty} [(U_2^{10} f_1^{20} + U_1^{20} f_2^{10} + \bar{a} f_1^{20}) \{ (4n^2 + m^2 n^2) \sin ny \cos 2ny \sin x \sin mx \}] \quad (4.40b)$$

$$+ 4n^2 m \cos ny \sin 2ny \cos x \cos mx] - 2\omega'_2 U_{2,t,t}^{10} \sin ny \sin mx \quad (4.40c)$$

We substitute (4.12) into (4.40a), for  $i = 3, j = 0$ , then, multiply through by  $\sin kny \sin \beta mx$  and get,

$$\text{for } k = \beta = 1, \quad f_2^{30} = \left( \frac{1 + \xi}{1 + n^2 \xi} \right)^2 \left[ \frac{n^2 H}{2\pi} (5\varphi_1 - 4\Omega_1) (U_2^{10} U_1^{20} + \bar{a} U_1^{20}) - U_2^{30} \right] \quad (4.41a)$$

where  $\varphi_1$  and  $\Omega_1$  are respectively the values of  $\varphi_m$  and  $\Omega_m$  at  $m = 1$  and

$$\varphi_m = 2 + \left\{ \frac{1}{1 - 2m} - \frac{1}{2m + 1} \right\}, \quad \Omega_m = \left[ \frac{1}{2m + 1} + \frac{1}{2m - 1} \right] \quad (4.41b)$$

The other combinations of  $k$  and  $\beta$  are  $k = 3, \beta = 1$  and  $k = 1, \beta = 3$ . These two will respectively give rise to stress functions here denoted by  $f_{23}^{30}$  and  $f_{24}^{30}$ , and to corresponding eigen stress functions in the shapes of  $w_{23}^{30} \sin 3ny \sin mx$  and  $w_{24}^{30} \sin ny \sin 3mx$ . Since these are not in the shape of imperfection, they are henceforth neglected. On multiplying (4.40a) by  $\cos kny \sin \beta mx$ , we have

$$f_1^{30} = -\frac{(1 + \xi)^2 (\beta m)^2 U_1^{30}}{\{(\beta m)^2 + (k n)^2 \xi\}^2}, \quad \forall k, \beta \text{ (each, positive integer)} \quad (4.41c)$$

Next, we substitute (4.12), for  $i = 3, j = 0$ , into (4.40b), multiply through by  $\sin kny \sin \beta mx$  and for  $k = \beta = 1$ , we get, using (4.41a),

$$U_{2,t,t}^{30} + \psi^2 U_2^{30} = \frac{H n^2 (5\varphi_1 - 4\Omega_1)}{2\pi} \left[ \left( \frac{1+\xi}{1+n^2\xi} \right)^2 (U_2^{10} U_2^{20} + \bar{a} U_1^{20}) + K(\xi) (U_2^{10} f_1^{20} + U_1^{20} f_2^{10} + \bar{a} f_1^{20}) \right] - 2\omega'_2 U_{1,t,t}^{10} \quad (4.42a)$$

$$U_2^{30}(0,0) + \omega'_2(0) U_2^{10}(0,0) = 0 \quad (4.42b)$$

If we however multiply (4.40b) by  $\cos kny \sin \beta mx$  and simplify, using (4.43c), we get

$$U_{1,t,t}^{30} + \psi_k^2 U_1^{30} = 0 ; U_1^{30}(0,0) = U_{1,t}^{30}(0,0) = 0 \quad (4.42c)$$

To solve (4.42a,b), we simplify it further by substituting for all the relevant terms there and getting

$$U_{2,t,t}^{30} + \psi^2 U_2^{30} = A^2 l_0 [ Q_5 + Q_6 \{ \beta_3 \sin(\psi + \psi_2)t - \zeta_3 \cos(\psi + \psi_2)t + \beta_3 \sin(\psi - \psi_2)t + \zeta_3 \cos(\psi - \psi_2)t \} + Q_7 \cos 2\psi t + Q_8 \sin 3\psi t + l_2 \bar{a} \{ \beta_3 \cos \psi_2 t + \zeta_3 \sin \psi_2 t \} ] \quad (4.43a)$$

$$+ \left\{ \frac{1}{2} \zeta_2^3 Q_2 \left( \frac{1}{\psi_2^2} + \frac{1}{2(\psi_2^2 - 4\psi^2)} \right) + \frac{l_2 \bar{a}^2 Q_2 \zeta_2}{(\psi_2^2 - \psi^2)} + \bar{a}^2 \zeta_2 l_4 + \frac{3l_5 \zeta_2^3}{4} + \frac{2\omega'_2 \zeta_2 \psi^2}{l_0 A^2} \right\} \sin \psi t \\ U_2^{30}(0,0) = 0 , U_{2,t}^{30}(0,0) + \omega'_2(0) U_{2,t,t}^{10}(0,0) = 0 \quad (4.43b)$$

where

$$l_0 = \frac{(5\varphi_1 - 4\Omega_1) n^2 H}{2\pi} , Q_5 = \left[ \frac{l_1 \zeta_2^2 \bar{a}^2 Q_1}{2(\psi_2^2 - \psi^2)} + \frac{\bar{a} l_3 \zeta_2^2}{2} + \frac{\bar{a} l_2 Q_2 \zeta_2^2}{2\psi_2} \right] \quad (4.44a)$$

$$l_1 = A^2 \left[ 2 \left( \frac{1}{1+n^2\xi} \right)^2 + \frac{1}{(1+4n^2\xi)^2} \right] , l_2 = \left[ \left( \frac{1}{1+n^2\xi} \right)^2 + \frac{1}{(1+4n^2\xi)^2} \right] \quad (4.44b)$$

$$l_3 = \frac{2Hn^2}{\pi(1+4n^2\xi)^2} \{ A^2 - 2(1+\xi)^2 \} , l_4 = \frac{4n^2 H}{\pi(1+4n^2\xi)^2} , l_5 = \frac{2n^2 H}{\pi(1+4n^2\xi)^2} \quad (4.44c)$$

$$Q_6 = \frac{l_1 \zeta_2^2}{2} , Q_7 = -\bar{a} \zeta_2^2 \left[ \frac{Q_1 l_1}{(\psi_2^2 - \psi^2)} + \frac{l_2 Q_2}{2(\psi_2^2 - 4\psi^2)} + \frac{l_3}{2} \right] \quad (4.44d)$$

$$Q_8 = -\zeta_2^3 \left\{ \frac{l_1}{4} + \frac{Q_2 l_1}{2(\psi_2^2 - 4\psi^2)} \right\} , Q_5(0) = \bar{a}^3 B^3 Q_{10} , Q_{10} = \left[ \frac{l_1 Q_1}{(\psi_2^2 - \psi^2)} + l_3 + \frac{l_2 Q_2}{\psi_2^2} \right] \quad (4.44e)$$

$$Q_6(0) = \frac{\bar{a} B l_1}{2} , Q_7(0) = (\bar{a} B)^2 Q_{11} , Q_{11} = \left[ \frac{l_1 Q_1}{2(\psi_2^2 - \psi^2)} + \frac{l_2 Q_2}{2(\psi_2^2 - 4\psi^2)} + \frac{l_3}{2} \right] \quad (4.44f)$$

$$Q_8(0) = -(\bar{a} B)^3 Q_{13} , Q_{13} = \left[ \frac{l_5}{4} + \frac{Q_2 l_1}{2(\psi_2^2 - 4\psi^2)} \right] \quad (4.44g)$$

To ensure a uniformly valid solution in  $t$ , we equate to zero in (4.45a) the coefficient of  $\sin \psi t$  and get

$$\omega'_2(\tau) = -\frac{l_0}{2\psi^2} \left\{ \frac{l_1 Q_2^2 \zeta_2}{2} \left( \frac{1}{\psi_2^2} + \frac{1}{2(\psi_2^2 - 4\psi^2)} \right) + \frac{l_2 \bar{a}^2 Q_1}{(\psi_2^2 - \psi^2)} + \bar{a}^2 l_4 + \frac{3l_5 \zeta_2^2}{4} \right\} \quad (4.45a)$$

$$\omega'_2(0) = -\frac{A^2 l_0 \bar{a}^2}{2\psi^2} (B^2 Q_{14} + Q_{15}), Q_{14} = \left\{ \frac{l_1 Q_2^2}{2} \left( \frac{1}{\psi_2^2} + \frac{1}{2(\psi_2^2 - 4\psi^2)} \right) + \frac{3l_5}{4} \right\}, \quad (4.45b)$$

$$Q_{15} = l_4 + \frac{l_2 Q_1}{(\psi_2^2 - \psi^2)} \quad (4.45c)$$

The remaining equation in (4.45a,b) is now solved to get

$$\begin{aligned} U_2^{30} &= \beta_5(\tau) \cos \psi t + \zeta_5(\tau) \sin \psi t + l_0 \left[ \frac{Q_5}{\psi^2} + Q_6 \left\{ \left\{ \frac{1}{\psi_2(2\psi + \psi_2)} \{ \zeta_3 \cos(\psi + \psi_2)t - \beta_3 \sin(\psi + \psi_2)t \} \right. \right. \right. \right. \\ &\quad \left. \left. \left. \left. + \frac{1}{\psi_2(2\psi - \psi_2)} \{ \beta_3 \sin(\psi - \psi_2)t + \zeta_3 \cos(\psi - \psi_2)t \} \right\} \right\} - \frac{Q_7 \cos 2\psi t}{3\psi^2} - \frac{Q_8 \sin 3\psi t}{8\psi^2} \right] \\ &\quad \left. + \frac{l_2 \bar{a}}{\psi^2 - \psi_2^2} \{ \beta_3 \cos \psi_2 t + \zeta_3 \sin \psi_2 t \} \right] \end{aligned} \quad (4.46a)$$

$$\beta_5(0) = -\frac{\bar{a}^3 B^2 Q_{12} l_0}{2\psi^2}, \zeta_5(0) = \frac{l_0 (\bar{a} B)^3 Q_{16}}{\psi} + \frac{l_0 \bar{a}^3 B Q_{18}}{\psi} \quad (4.46b)$$

$$Q_{12} = \left[ Q_{10} - \frac{Q_1 \psi^3 l_1}{\psi_2(\psi^2 - \psi_2^2)} \left\{ \frac{1}{(2\psi - \psi_2)} + \frac{1}{(2\psi + \psi_2)} \right\} + \frac{2Q_{11}}{3} + \frac{l_2 \psi^2 Q_2 R_2}{(\psi^2 - \psi_2^2)^2} \right] \quad (4.46c)$$

$$Q_{16} = \left[ \frac{3Q_{13}}{8\psi} - \frac{Q_{14}}{2\psi} + \frac{R_2}{2\psi_2} \left\{ \left( \frac{\psi + \psi_2}{2\psi + \psi_2} \right) - \left( \frac{\psi - \psi_2}{2\psi - \psi_2} \right) \right\} \right], Q_{18} = -\left[ \frac{l_2 R_1 \psi_2}{\psi^2 - \psi_2^2} + \frac{l_0 Q_{15}}{2\psi^2} \right] \quad (4.46d)$$

On solving (4.42c), we have

$$U_1^{30}(t, \tau) = \gamma_5(\tau) \cos \psi_2 t + \theta_5(\tau) \sin \psi_2 t, \quad \gamma_5(0) = \theta_5(0) = 0 \quad (4.47)$$

We now substitute on the right hand sides of (4.10a,b) and simplify to get

$$\begin{aligned} L^{(1)}(U^{31}, f^{31}) &= -(1 + \xi)^2 H \sum_{m=1,3,5,\Lambda}^{\infty} \left[ \left\{ \frac{1}{2} U_2^{11} U_1^{20} + \frac{1}{2} U_1^{21} U_2^{10} + \bar{a} U_1^{21} \right\} \times \right. \\ &\quad \left. \{ (4n^2 + m^2 n^2) \sin ny \cos 2ny \sin x \sin mx + 4n^2 m \cos ny \sin 2ny \cos x \cos mx \} \right] \end{aligned} \quad (4.48a)$$

$$L^{(2)}(U^{31}, f^{31}) = -Hk(\xi) \sum_{1,3,5,\Lambda}^{\infty} \left[ \{ U_2^{10} f_1^{21} + U_1^{11} f_1^{20} + U_1^{20} f_2^{11} + U_1^{21} f_2^{10} - \bar{a} f_1^{21} \} \times \right.$$

$$\left. \{ (4n^2 + m^2 n^2) \sin ny \cos 2ny \sin x \sin mx + 4n^2 m \cos ny \sin 2ny \cos x \cos mx \} \right] - 2 \left\{ \omega_2'' U_{2,t}^{10} + \omega_2' U_{2,t}^{10} + \omega_2' U_{2,t\tau}^{10} + U_{2,t}^{30} + U_{2,t\tau}^{10} \right\} \sin x \sin ny \quad (4.48b)$$

We substitute (4.2) into (4.50a), for  $i = 3, j = 1$ , multiply the resultant equation by  $\sin kny \sin \beta mx$  and for  $k = \beta = 1$ , we have

$$f_2^{31} = \frac{(1+\xi)^2 n^2}{2\pi(1+n^2\xi)^2} \left[ H \left( \frac{1}{2} U_1^{21} U_2^{10} + \bar{a} U_1^{21} \right) - 2\pi U_2^{31} \right] \quad (4.48c)$$

where we have taken  $U_2^{11} \equiv 0$ . Other combinations of  $k$  and  $\beta$  are  $k=1, \beta=3$ ;  $k=3, \beta=1$ . These other combinations will eventually lead to buckling modes that are not in the shape of imperfection and they are henceforth neglected. If we multiply (4.50a) by  $\cos kny \sin \beta mx$ , we have

$$f_1^{31} = -\frac{(1+\xi)^2 (\beta m)^2 U_1^{31}}{\{(\beta m)^2 + (k n)^2 \xi\}^2}, \quad \forall k, \beta \text{ (each, positive integer)} \quad (4.48d)$$

We next substitute (4.13) into (4.50b), for  $i=3, j=1$ ; multiply through by  $\sin kny \sin \beta mx$  and get, using (4.50c)

$$U_{2,tt}^{31} + \psi^2 U_2^{31} = A^2 l_0 [Q_{19} U_1^{21} U_2^{10} + Q_{20} \bar{a} U_{1,t\tau}^{21}] - 2 \left( \begin{array}{l} \omega''_2 U_{2,t}^{10} + \omega'_2 U_{2,t}^{10} + \omega'_2 U_{2,t\tau}^{10} \\ + U_{2,t}^{30} + U_{2,t\tau}^{30} \end{array} \right) \quad (4.49a)$$

$$U_2^{31}(0,0) = 0, \quad U_{2,t}^{31}(0,0) + U_{2,t\tau}^{30}(0,0) = 0 \quad (4.49b)$$

$$\text{where } Q_{19} = \left[ 1 - \left( \frac{1+n^2\xi}{n} \right)^2 \left\{ \left( \frac{1}{1+n^2\xi} \right) + \frac{1}{(1+4n^2\xi)^2} \right\} \right], \quad Q_{20} = \left( 1 - \frac{1}{n(1+4n^2\xi)^2} \right) \quad (4.49c)$$

On further simplifying (61a), we have

$$U_{2,tt}^{31} + \psi^2 U_2^{31} = A^2 l_0 [Q_{22} \cos \psi_2 t + Q_{23} \sin \psi_2 t + Q_{24} \sin 2\psi t + Q_{25} \cos(\psi + \psi_2) t + Q_{26} \sin(\psi + \psi_2) t + Q_{27} \cos(\psi - \psi_2) t + Q_{28} \sin(\psi - \psi_2) t + Q_{29}(0) \cos 3\psi t] \quad (4.50a)$$

$$+ R_3 \cos \psi t + R_4 \sin \psi t$$

$$U_2^{31}(0,0) = 0, \quad U_{2,t}^{31}(0,0) + \omega'_2(0) U_{2,t}^{11} + U_{2,t\tau}^{30}(0,0) = 0 \quad (4.50b)$$

$$Q_{22} = \bar{a} Q_{20} \zeta_4 - \frac{2\zeta'_3 \psi l_2 \bar{a}}{\psi^2 - \psi_2^2} - \frac{2\zeta_3 \psi l_2 \bar{a}}{\psi^2 - \psi_2^2}, \quad Q_{23} = \bar{a} Q_{20} \beta_4 + \frac{2\beta'_3 \psi l_2 \bar{a}}{\psi^2 - \psi_2^2} \quad (4.50c)$$

$$Q_{24} = \frac{Q_{19} \zeta_2 Q_3}{2(\psi^2 - \psi_2^2)} + \frac{Q_{20} \bar{a} Q_4}{(\psi^2 - 4\psi_2^2)} - \frac{4Q_7}{3\psi} - \frac{4Q'_7}{3\psi} \quad (4.50d)$$

$$Q_{25} = \left[ -\frac{Q_{19} \zeta_2 \zeta_4}{2} + \frac{2l_0 (Q_6 \beta_3)' (\psi + \psi_2)}{\psi_2 (2\psi + \psi_2)} + \frac{2l_0 (Q_6 \beta_3) (\psi + \psi_2)}{\psi_2 (2\psi + \psi_2)} \right] \quad (4.50e)$$

$$Q_{26} = \left[ \frac{Q_{19} \zeta_2 \beta_4}{2} + \frac{2l_0 (Q_6 \beta_3)' (\psi + \psi_2)}{\psi_2 (2\psi + \psi_2)} + \frac{2l_0 (Q_6 \beta_3) (\psi + \psi_2)}{\psi_2 (2\psi + \psi_2)} \right] \quad (4.50f)$$

$$Q_{27} = \left[ \frac{Q_{19} \zeta_2 \zeta_4}{2} - \frac{2(Q_6 \beta_3)' (\psi - \psi_2)}{\psi_2 (2\psi - \psi_2)} - \frac{2(Q_6 \beta_3) (\psi - \psi_2)}{\psi_2 (2\psi - \psi_2)} \right] \quad (4.51a)$$

$$Q_{28} = \left[ \frac{Q_{19}\zeta_2\beta_4}{2} + \frac{2(Q_6\beta_3)'(\psi - \psi_2)}{\psi_2(2\psi - \psi_2)} + \frac{2(Q_6\beta_3)(\psi - \psi_2)}{\psi_2(2\psi - \psi_2)} \right], Q_{29}(0) = (\bar{a}B)^3 Q_{42} \quad (4.51b)$$

$$Q_{42} = \left[ \frac{3Q_{33}}{4\psi} + \frac{3Q_{13}}{4\psi} - \frac{2Q_{19}Q_2\psi}{(\psi_2^2 - 4\psi^2)^2} \right], Q_{33} = \frac{l_5}{4} + \frac{Q_2l_1}{2(\psi_2^2 - 4\psi^2)} = Q_{13} \quad (4.51c)$$

$$R_3 = \left\{ -2\omega_2''\psi\zeta_2 - 2\omega_2'\psi\zeta_2 + \frac{A^2l_0Q_{19}\zeta_2Q_4}{2(\psi_2^2 - 4\psi^2)} - 2\psi\omega_2'\zeta_2' - 2\psi\zeta_5' - 2\psi\zeta_5 \right\} \quad (4.51d)$$

$$R_4 = \left\{ \frac{Q_{20}\bar{a}Q_3}{(\psi_2^2 - \psi^2)} + 2\psi\beta_5' + 2\psi\beta_5 \right\} \quad (4.51e)$$

Because of the lengthy nature of  $Q_{29}(\tau)$ , we have approximated it by its exact value at  $\tau = 0$  which will be used later. The following simplifications are necessary in the next round of analysis.

$$\omega_2''(0) = \frac{(\bar{a}B)l_0A^2Q_{30}}{2\psi^2}, Q_{30} = \left[ l_1Q_2^2 \left\{ \frac{1}{\psi_2^2} + \frac{1}{2(\psi_2^2 + \psi^2)} \right\} + \frac{3l_5}{2} \right]; \quad (4.52a)$$

$$Q_3'(0) = 0, Q_4'(0) = -\frac{4\psi Q_2(\bar{a}B)^2}{\psi_2^2 - 4\psi^2} \quad (4.52b)$$

$$Q_5'(0) = -\bar{a}^3B^2Q_{31}, Q_{31} = \left[ \frac{l_1Q_1}{\psi_2^2 - \psi^2} + l_3 + \frac{l_2Q_2}{\psi_2^2} \right], Q_6'(0) = -\frac{\bar{a}Bl_1}{2}, Q_7'(0) = 2\bar{a}^3B^2Q_{32} \quad (4.52c)$$

$$Q_{32} = \left[ \frac{l_1Q_7}{\psi_2^2 - \psi^2} + \frac{l_3}{2} + \frac{l_2Q_2}{2(\psi_2^2 - 4\psi^2)} \right], Q_8'(0) = (\bar{a}B)^3 Q_{33}; \quad (4.52d)$$

$$Q_{22}(0) = -\bar{a}^3B^2Q_{34}, Q_{34} = \frac{4Q_{20}Q_2}{(\psi_2^2 - 4\psi^2)^2} \quad (4.52e)$$

$$Q_{23}(0) = 0, Q_{24}(0) = 2\bar{a}^3B^2Q_{37}, Q_{37} = \left[ \frac{4Q_{20}Q_2}{(\psi_2^2 - 4\psi^2)^2} - \frac{4Q_{32}}{3\psi} + \frac{2Q_{11}}{3\psi} \right], Q_{25}(0) = (\bar{a}B)^3 Q_{38} \quad (4.52f)$$

$$Q_{38} = \left[ \frac{Q_2Q_{19}}{(\psi_2^2 - 4\psi^2)} - \frac{2R_2l_0(\psi + \psi_2)(1 + l_1)}{2\psi_2(2\psi + \psi_2)} + \frac{l_0l_1R_2(\psi + \psi_2)}{\psi_2(2\psi + \psi_2)} \right], Q_{26}(0) = (\bar{a}B)^3 Q_{39} \quad (4.52g)$$

$$Q_{39} = -\left[ \frac{3l_0l_1R_2(\psi + \psi_2)}{\psi_2(2\psi + \psi_2)} \right], Q_{27}(0) = (\bar{a}B)^3 Q_{40}, Q_{40} = \left[ -\frac{2\psi^2Q_{19}}{\psi_2(\psi_2^2 - 4\psi^2)^2} + \frac{2R_2l_1(\psi - \psi_2)}{\psi_2(2\psi - \psi_2)} \right] \quad (4.52h)$$

$$Q_{28}(0) = (\bar{a}B)^3 Q_{41}, Q_{41} = -\frac{R_2l_1(\psi - \psi_2)}{\psi_2(2\psi - \psi_2)} \quad (4.52i)$$

To ensure a uniformly valid solution in  $t$ , we equate to zero in (4.52a) the coefficients of  $\cos \psi t$  and  $\sin \psi t$  and get respectively

$$R_3 = \left\{ -2\omega_2''\psi\zeta_2 - 2\omega_2'\psi\zeta_2 + \frac{A^2l_0Q_{19}\zeta_2Q_4}{2(\psi_2^2 - 4\psi^2)} - 2\psi\omega_2'\zeta_2' - 2\psi\zeta_5' - 2\psi\zeta_5 \right\} = 0 \quad (4.53a)$$

$$R_4 = \left\{ \frac{Q_{20}\bar{a}Q_3}{(\psi_2^2 - \psi^2)} + 2\psi\beta_5' + 2\psi\beta_5 \right\} = 0 \quad (4.53b)$$

The solutions of (4.53a,b) are respectively

$$\zeta_5(\tau) = e^{-\tau} \left[ \frac{1}{2\psi} \int_0^t \left\{ \zeta_2 \left( \frac{Q_{19}Q_4 A^2 l_0}{2(\psi_2^2 - 4\psi^2)} - 2\omega_2''\psi (\zeta_2' + \zeta_2) \right) - 2\omega_2'\psi (\zeta_2' + \zeta_2) \right\} e^s ds + \zeta_5(0) \right] \quad (4.53c)$$

and

$$\beta_5(\tau) = -e^{-\tau} \left[ \frac{1}{2\psi} \int_0^t \left( \frac{\bar{a}Q_{20}Q_4 Q_3}{\psi_2^2 - \psi^2} \right) e^s ds - \beta_5(0) \right] \quad (4.53d)$$

$$\zeta_5'(0) = l_0 (\bar{a}B)^3 Q_{17} - \frac{l_0 \bar{a}^3 B Q_{18}}{\psi}, Q_{17} = \left[ \frac{A^2 Q_2 Q_{19}}{2(\psi_2^2 - 4\psi^2)^2} - \frac{Q_{30}}{\psi^2} - \frac{Q_{16}}{\psi} \right] Q_{18} \quad (4.53e)$$

$$\beta_5'(0) = -\beta_5(0) = \frac{\bar{a}^3 B^2 Q_{12} l_0}{2\psi^2} \quad (4.53f)$$

is as in (4.48d). On solving the remaining equation in (4.52a,b), we get

$$U_2^{31}(t, \tau) = \beta_6(\tau) \cos \psi t + \zeta_6(\tau) \sin \psi t + A^2 l_0 \left[ \frac{1}{\psi_2^2 - \psi^2} \{ Q_{22} \cos \psi_2 t + Q_{23} \sin \psi_2 t \} \right. \\ \left. - \frac{Q_{24} \sin 2\psi t}{3\psi^2} - \frac{1}{\psi_2(2\psi + \psi_2)} \{ Q_{25} \cos(\psi + \psi_2) t + Q_{26} \sin(\psi + \psi_2) t \} \right. \\ \left. + \frac{1}{\psi_2(2\psi - \psi_2)} \{ Q_{27} \cos(\psi - \psi_2) t + Q_{28} \sin(\psi - \psi_2) t \} - \frac{Q_{29}(0) \cos 3\psi t}{8\psi^2} \right] \quad (4.54a)$$

$$\beta_6(0) = -A^2 l_0 \left[ \frac{Q_{22}}{\psi_2^2 - \psi^2} - \frac{Q_{25}}{\psi_2(2\psi + \psi_2)} + \frac{Q_{27}}{\psi_2(2\psi - \psi_2)} - \frac{Q_{29}}{8\psi^2} \right] \Big|_{\tau=0} \quad (4.54b)$$

$$\zeta_6(0) = (\bar{a}B)^3 Q_{47} + \bar{a}^3 B^2 Q_{48} + \bar{a}^3 B Q_{49}, Q_{47} = -\frac{A^2 l_0}{\psi \psi_2} \left[ \left( \frac{\psi + \psi_2}{2\psi + \psi_2} \right) Q_{39} - \left( \frac{\psi - \psi_2}{2\psi - \psi_2} \right) Q_{41} \right] \quad (4.54c)$$

$$Q_{48} = \frac{1}{\psi} \left[ \frac{A^2 l_0 Q_{37}}{3\psi} - \frac{l_0 Q_{12}}{2\psi^2} - l_0 \left\{ \frac{l_0 R_1}{\psi_2} \left( \frac{1}{2\psi - \psi_2} + \frac{1}{2\psi + \psi_2} \right) - \frac{Q_{31}}{\psi^2} - \frac{2Q_{32}}{3\psi^2} \right\} \right] \quad (4.54d)$$

$$Q_{49} = \frac{l_0 l_2 R_2}{\psi(\psi^2 - \psi_2^2)} \quad (4.54e)$$

If we multiply (4.50b) by  $\cos kny \sin \beta mx$  and maintain a uniformly valid solution in  $t$ , we get

$$\theta'_5 + \theta_5 = 0; \quad \gamma'_5 + \gamma_5 = 0 \quad (4.55a)$$

On solving (4.57a) we get

$$\theta(\tau) = \gamma(\tau) \equiv 0 \quad (4.55b)$$

The remaining equation in (4.50b), after multiplying by  $\cos kny \sin \beta mx$ , are

$$U_{1,t,t}^{31} + \psi_k^2 U_1^{31} = 0, \quad U_1^{31}(0,0) = U_{1,t}^{31}(0,0) = 0 \quad (4.56a)$$

On solving (4.58a), we get

$$U_1^{31}(t, \tau) = \gamma_6(\tau) \cos \psi_k t + \theta_6(\tau) \sin \psi_k t, \quad \gamma_6(0) = \theta_6(0) = 0 \quad (4.56b)$$

So far, we give the normal displacement,  $U(x, y, t, \tau)$  as

$$U(x, y, t, \tau) = U_2^{10} \sin ny \sin x + \epsilon^3 (U_2^{30} + \eta U_2^{31}) \sin ny \sin x + O(\epsilon^3 \eta^2) + O(\epsilon^3 \eta^2) \quad (4.57)$$

where we have neglected  $U_1^{20}$  and  $U_1^{21}$  because they are not in the shape of (4.13).

## 5.0 Maximum displacement

As a function of space and time variables, the conditions for maximum displacement are

$$U_{,x} = 0, \quad U_{,y} = 0; \quad (1 + \omega'_2 \epsilon^2 + \Lambda) U_{,t} + \eta U_{,\tau} = 0 \quad (5.1a)$$

From the first two in (5.1a), we have

$$x_a = \frac{\pi}{2}, \quad y_a = \frac{\pi}{2n} \quad (5.1b)$$

where  $x_a$  and  $y_a$  are the values of  $x$  and  $y$  respectively at maximum displacement. We let  $t_a, \bar{t}_a$  and  $\tau_a$  be the values of  $t, \bar{t}$  and  $\tau$  respectively at maximum displacement and now assume the following asymptotic series

$$t_a = t_0 + \eta t_{01} + \epsilon^2 (t_{20} + \eta t_{21}) + \Lambda; \quad \bar{t}_a = \bar{t}_0 + \eta \bar{t}_{01} + \epsilon^2 (\bar{t}_{20} + \eta \bar{t}_{21}) + \Lambda \quad (5.2a)$$

$$\tau_a = \eta t_a = \eta \{ t_0 + \eta t_{01} + \epsilon^2 (t_{20} + \eta t_{21}) + \Lambda \} \quad (5.2b)$$

We substitute (5.2a,b) into the third equation in (5.1a), and equate the coefficients of  $\epsilon^i \eta^j, i=1,2,3, \Lambda; j=0,1,2, \Lambda$ . From the coefficients of  $\epsilon, \epsilon \eta$  and  $\epsilon^3$ , we have the following respective equations which are evaluated at  $(t_a, \tau_a) = (t_0, 0)$ :

$$\begin{aligned} U_{2,t}^{10} &= 0; \quad t_{01} U_{2,t,t}^{10} + t_0 U_{2,t,\tau}^{10} + U_{2,\tau}^{10} = 0; \\ t_{20} U_{2,t,\tau}^{10} + U_{2,t}^{30} + \omega'_2(0) U_{2,t}^{10} &= 0 \end{aligned} \quad (5.3)$$

From the first and second of (5.3), we have respectively

$$t_0 = \frac{\pi}{\psi} \quad \text{and} \quad t_{01} = \frac{1}{\psi^2} \quad (5.4a)$$

From the third of (5.3) we have

$$t_{20} = \frac{A^2 l_0 \bar{a}^2}{\psi^2} [B^2 Q_{50} + B Q_{51} + Q_{52}] \quad (5.4b)$$

$$Q_{50} = \left[ -\frac{l_1 R_2 (\psi + \psi_2) \cos(\psi + \psi_2) t}{2\psi_2 (2\psi + \psi_2)} + \frac{l_1 R_2 (\psi - \psi_2) \cos(\psi - \psi_2) t}{2\psi_2 (2\psi - \psi_2)} + \frac{3Q_{33}}{8\psi} \right] \Big|_{t=t_0} \quad (5.5a)$$

$$Q_{51} = \left[ \frac{\psi l_2 R_2 \sin \omega_2 t}{\psi^2 - \psi_2^2} - \frac{l_1 R_1 (\psi + \psi_2) \sin(\psi - \psi_2) t}{2\psi_2 (2\psi + \psi_2)} + \frac{3Q_{12}}{2\psi^2} + R_1 l_1 \sin(\psi - \psi_2) t \right] \Big|_{t=t_0} \quad (5.5b)$$

$$Q_{52} = -\frac{\psi l_2 R_1 \sin \psi_2 t}{\psi^2 - \psi_2^2} \quad (5.5c)$$

By evaluating the second equation of (4.3a) at the critical values of the time variables, using (5.2a,b) and equating the coefficients of (3.2) ( $1 \eta$ ) and ( $\epsilon^2 1$ ), we have respectively

$$\bar{t}_0 = t_0 = \frac{\pi}{\psi}, \bar{t}_{01} = t_{01} = \frac{1}{\psi^2}, \text{ and } \bar{t}_{20} = t_{20} - \omega'_2(0)t_0 \quad (5.5d)$$

The maximum displacement  $U(x_a, y_a, t_a, \tau_a; \in, \eta) = U_a$  is obtained by evaluating (4.57) at the critical values of the variables to get

$$U_a = \left. \left( U_2^{10} + \eta t_0 U_{2,\tau}^{10} \right) \right|_{(t_0,0)} + \left. \in^3 \left[ U_2^{30} + \eta \left( t_{20} U_{2,\tau}^{10} + t_{01} U_{2,t}^{30} + t_0 U_{2,\tau}^{30} + U_2^{31} \right) \right] \right|_{(t_0,0)} + O(\in \eta^2) + O(\in^3 \eta^2) \quad (5.6)$$

To simplify (5.5), we note the following evaluations:

$$U_{2,t}^{30}(t_0,0) = A^2 l_0 [(\bar{a}B)^3 Q_{50} + \bar{a}^3 B^2 Q_{51} + \bar{a}^3 B Q_{52}] \quad (5.7)$$

$$Q_{53} = \left. \left[ \frac{Q_{17}}{\psi} + \frac{l_1 R_2 \sin(\psi + \psi_2) t}{\psi_2 (2\psi + \psi_2)} - \frac{l_1 R_2 \sin(\psi - \psi_2) t}{\psi_2 (2\psi - \psi_2)} + \frac{Q_{33}}{8\psi^2} \right] \right|_{t=t_0} \quad (5.8a)$$

$$Q_{54} = \left. \left[ -\frac{Q_{31}}{\psi^2} - \frac{l_1 R_2 \cos(\psi + \psi_2) t}{\psi_2 (2\psi + \psi_2)} + \frac{l_1 R_2 \cos(\psi - \psi_2) t}{\psi_2 (2\psi - \psi_2)} + \frac{2Q_{32}}{3\psi^2} + \frac{l_2 R_2 \cos \psi_2 t}{\psi^2 - \psi_2^2} \right] \right|_{(t_0,0)} \quad (5.8b)$$

$$Q_{55} = \left. \left[ \frac{Q_{18}}{\psi} + \frac{l_2 R_1 \sin \psi_2 t}{\psi^2 - \psi_2^2} \right] \right|_{(t_0,0)}, U_2^{30}(t_0,0) = \frac{A^2 l_0}{\psi} [(\bar{a}B)^3 Q_{56} + \bar{a}^3 B^2 Q_{57} + \bar{a}^3 B Q_{58}] \quad (5.9a)$$

$$Q_{56} = \left. \left[ Q_{16} + \frac{l_1 R_2 \psi \sin(\psi + \psi_2) t}{2\psi_2 (2\psi + \psi_2)} + \frac{l_1 R_2 \psi \sin(\psi - \psi_2) t}{2\psi_2 (2\psi - \psi_2)} - \frac{Q_{13}}{8\psi} \right] \right|_{(t_0,0)} \quad (5.9b)$$

$$Q_{57} = \left. \left[ \frac{Q_{10}}{2\psi} + \frac{R_1 \psi \cos(\psi + \psi_2) t}{2\psi_2 (2\psi + \psi_2)} - \frac{l_1 R_1 \psi \cos(\psi - \psi_2) t}{2\psi_2 (2\psi - \psi_2)} + \frac{Q_{11}}{3\psi} + \frac{2l_2 R_2 \psi \cos \psi_2 t}{\psi^2 - \psi_2^2} \right] \right|_{(t_0,0)} \quad (5.9c)$$

$$Q_{58} = \left. \left[ Q_{18} - \frac{l_2 R_1 \sin \psi_2 t}{\psi^2 - \psi_2^2} \right] \right|_{(t_0,0)} \quad (5.9d)$$

$$U_2^{31}(t_0,0) = A^2 l_0 [(\bar{a}B)^3 Q_{59} + \bar{a}^3 B^2 Q_{60} + \bar{a}^3 B Q_{61}] \quad (5.10a)$$

$$Q_{59} = \left. \left[ \frac{Q_{47}}{A^2 l_0} - \frac{1}{\psi_2 (2\psi + \psi_2)} \{ Q_{38} \cos(\psi + \psi_2) t + Q_{39} \sin(\psi + \psi_2) t \} + \frac{Q_{40} \cos(\psi - \psi_2) t}{\psi_2 (2\psi - \psi_2)} + \frac{Q_{41} \sin(\psi - \psi_2) t}{\psi_2 (2\psi - \psi_2)} + \frac{Q_{42}}{8\psi^2} \right] \right|_{(t_0,0)}, Q_{60} = \left. \left[ Q_{48} - \frac{Q_{34} \cos \psi_2 t}{\psi^2 - \psi_2^2} \right] \right|_{(t_0,0)}, Q_{61} = \frac{Q_{49}}{A^2 l_0} \quad (5.10b)$$

$$U_2^{30}(t_0,0) = A^2 l_0 \frac{(\bar{a}B)^3}{\psi} [(\bar{a}B)^3 Q_{56} + \bar{a}^3 B^2 Q_{57} + \bar{a}^3 B Q_{58}] \quad (5.10c)$$

On substituting into (5.6), using (5.7) - (5.10c) and simplifying, we get

$$U_a = C_1 + \epsilon^3 C_3 + \Lambda \quad (5.11a)$$

$$C_1 = \left(1 - \frac{\eta\pi}{\psi}\right), C_3 = \frac{A^2 l_0 (\bar{a}B)^3 Q_{56} \tilde{S}}{\psi} \quad (5.11b)$$

$$\begin{aligned} \tilde{S} = & \left[ 1 + \frac{\psi}{Q_{56}} (B^{-1}Q_{57} + B^{-2}Q_{58}) + \frac{\eta}{Q_{56}} \{(\pi Q_{53} + \psi Q_{59}) + (\pi Q_{54} + \psi Q_{60})B^{-1} \right. \\ & \left. + B^{-2}(\pi Q_{55} + \psi Q_{61}) \} \right] \end{aligned} \quad (5.11c)$$

## 6.0 Dynamic buckling impulse $I_D$

The dynamic buckling impulse  $I_D$  is determined from the condition (4.1) where we now substitute  $U_a$  for  $w_a$ . The usual procedure [11] is to first reverse the series (5.11a) in the form

$$\epsilon = d_1 U_a + d_3 U_a^3 + \Lambda \quad (6.1a)$$

By substituting into (6.1a) for  $U_a$  from (5.11a) and equating the coefficients of  $\epsilon$  and  $\epsilon^3$ , we have

$$d_1 = \frac{1}{C_1}, \quad d_3 = -\frac{C_3}{C_1^4} \quad (6.1b)$$

The maximization (4.1) easily follows from (6.1a) to yield, after some simplification,

$$\epsilon = \frac{2}{3} \sqrt{\frac{C_1}{3C_3}} \quad (6.2)$$

where (6.2) is evaluated at  $I = I_D$ . On substituting into (6.2) for  $C_1$  and  $C_3$ , and simplifying ,we have

$$\frac{I_D \bar{a} \epsilon \left( \frac{\alpha}{2} + n^2 \xi \right)}{(1+n^2 \xi)^2 + \left( \frac{A}{1+n^2 \xi} \right)^2} = \frac{I_D \bar{a} \epsilon}{\lambda_C} = \frac{2}{3An} \sqrt{\frac{2\pi}{3}} \left\{ \frac{\psi \left( 1 - \frac{\eta\pi}{\psi} \right)}{H(5\varphi_1 - 4\Omega_1)Q_{56}\tilde{S}} \right\}^{\frac{1}{2}} \quad (6.3)$$

## 7.0 Analysis of result

The result (6.3) is asymptotically valid and guided by Koiter's [4] observation , its applicability is limited to any imperfection whose amplitude is less than one half of the shell thickness ( i.e.  $\epsilon < \frac{h}{2}$  ).

Similarly, Donnell's condition [5] demands that  $n > 5$ . A careful examination of the result reveals that the dynamic buckling impulse  $I_D$  is higher when damping is present compared to the case where there is no damping. The value of  $I_D$  however increases with increased damping, provided that the condition  $0 < \eta < 1$  is satisfied. Thus, with damping, the structure is dynamically more stable in the sense that it withstands higher impulse pressure and may eventually buckle at excessive pressure. We observe, from (6.3), that the result is such that we are able to relate the dynamic buckling impulse  $I_D$  to the classical buckling load  $\lambda_C$ . By setting  $\eta = 0$ , we automatically obtain the equivalent result for the no-damping

situation. We readily observe that  $I_D$  increases as  $\bar{a} \in$  decreases. This is expected. We have limited our investigation to the case where the buckling mode is in the shape of imperfection. According to Koiter [4], this limitation has the greatest effect on the buckling process. The novelty, however in this analysis is that our method allows us a lee-way to any possible consideration of the effects of cases where the buckling modes may not strictly be in the shape of imperfection, but could be a combination of buckling modes that are partly in the shape of imperfection and partly in any other geometric or trigonometric shapes such as  $\cos 2ny \sin x$  and  $\sin ny \sin 3mx$ . If we substitute from (4.18) into  $\tilde{S}$  for  $B$ , evaluated at  $I_D$ , we

have  $B(I_D) = \frac{I_D}{\lambda_c}$  so that

$$\tilde{S} = \left[ 1 + \frac{\psi}{Q_{56}} \left\{ \left( \frac{I_D}{\lambda_c} \right)^{-1} Q_{57} + \left( \frac{I_D}{\lambda_c} \right)^{-2} Q_{58} \right\} + \frac{\eta}{Q_{56}} \left\{ \left( \frac{I_D}{\lambda_c} \right)^{-1} (\pi Q_{54} + \psi Q_{60}) \right. \right.$$

$$\left. \left. + (\pi Q_{53} + \psi Q_{59}) + \left( \frac{I_D}{\lambda_c} \right)^{-2} (\pi Q_{55} + \psi Q_{61}) \right\} \right] \quad (7.1)$$

We readily observe from (6.3), with (7.1) substituted there for  $\tilde{S}$ , that (6.3) is an implicit formula for calculating  $I_D$ . An accurate, but approximate and straight forward formula for calculating  $I_D$  is to ignore the terms

$$\left( \frac{I_D}{\lambda_c} \right)^{-1} \text{ and } \left( \frac{I_D}{\lambda_c} \right)^{-2}$$

and the resultant formula, from (6.3), and (7.1) is

$$\frac{I_D \bar{a} \in}{\lambda_c} = \frac{2}{3A n} \sqrt{\frac{2\pi}{3}} \sqrt{\frac{\psi \left( 1 - \frac{\eta \pi}{\psi} \right)}{H(5\varphi_1 - 4\Omega_1) Q_{56} \left\{ 1 + \frac{\eta}{Q_{56}} (\pi Q_{53} + \psi Q_{59}) \right\}}} \quad (7.2)$$

The nature and restriction placed on viscous damping in this investigation must be appreciated. Here; the damping is not related to the amplitude of imperfection so that any variation of any of them does not necessarily affect the other.

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