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Dynamical analysis of a prestressed elastic beam with general boundary conditions under the action of uniform distributed masses.

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Abstract

The dynamic analysis of a uniform beam resting on an elastic foundation and subjected to uniformly distributed moving masses is investigated in this paper. The solution technique is based on generalized finite integral transforms, the use of the properties of the Heaviside function as the generalized derivative of the Dirac Delta function in the distributed sense and a modification of the asymptotic method of Struble. The analytical and numerical analysis show that increase in the axial force N and foundation stiffness K decrease the response amplitude of the uniform Bernoulli-Euler beams when under the actions of both moving distributed force or moving distributed mass. Also, for all illustrative examples, for the same natural frequency, the critical speed for the moving mass problem is smaller than that of the moving force problem. Hence, resonance is reached earlier in the former. Thus, it is tragic to rely on moving force solution as an approximation to the moving mass problem.

***Keyword*:** Response of structures**,** Uniform beam, concentrated loads, distributed masses, moving masses, Boundary conditions, inertia effect.

1.0 Introduction

In the structural dynamics, the moving-load-induced vibration problems have been the important research topic for over 100 years. Therefore, a great amount of work dealing with the dynamic analysis of structures due to moving loads can be found from existing literature [1-6].

Until early this century, machine and structure usually had very high mass and damping because heavy beams, castings and timbers were used in their constructions. The dynamic response of these structures and machines was low since the vibration excitation sources were often small in magnitude. However, with the development of strong lightweight materials, increased knowledge of material properties and structural loading, improved analysis and design techniques, the mass of machines and structures built to fulfill a particular function has decreased. Furthermore, the efficiency and speed of machinery have increased so that the vibration excitation with reducing machine mass and damping has continued at an increasing rate to the present day when few, if any, machine can be designed without carrying out the necessary vibration analysis.

In general, the moving load problems are mathematically complex when the inertia effect of the moving load is taken into consideration. Thus most of the research works available in the literature are those in which this effect has been neglected. This is due, at least in part, to the great amount of computational labour, which is required both to set up and to solve the necessary equation. One important problem that arises when the inertia effect of the masses are considered is the singularity which occurs in the inertia terms of the governing equation of motion.. A major breakthrough in this field is the work of Stanisic et al [7] who solved the problem of simply supported non-mindling plate under a multi-mass moving system by making use of an approximation of Dirac-delta function. Only the inertia term which measures the effect of local acceleration in the direction of the deflection was considered. More recently, Oni [8] and Gbadeyan and Oni [9] presented a theory for determining the response of a finite Rayleigh beam (thick beam) under an arbitrary number of moving concentrated masses. The theory advanced involves the development of an analytical versatile technique which is based on the modified generalized finite integral transform and the modified Struble's method. An important feature of this technique is that it is applicable to all classical end conditions, as well as both thin and thick beam moving load problems.

It is remarked at this juncture that most of the studies available in literatures are moving load problems where authors have modeled the moving load by concentrated moving mass or moving lumped mass. However, in practice, moving loads are in the form of moving distributed masses rather than that of moving lumped mass. For this reason, Esmailzadez and Ghorashi [10] further studied the moving-load-induced vibration problem using a moving uniform distributed mass model instead of the moving lumped mass model. They solved the problem by means of the conventional analytical approach, which is only suitable for the simple horizontal beam and will suffer much difficulty if the structures are complicated. Also, it is noted that Esmailzadeh and Gorashi [10] considered only the vertical inertia effects of the distributed mass moving along a horizontal pinned-pinned beam. This vertical inertia is called inertia force. He neglected both coriolis force and centrifugal force of the inertia term in the governing differential equation. Wu [11] on the other hand studied the vibration analysis of a pinned-pinned beam and that of partial frame under the action of a moving uniformly distributed mass using finite element method and Newmark integration method. Other recent work that used uniformly distributed moving mass model include Dada [12] who worked on the vibration analysis of elastic plates under uniform partially distributed loads and Adetunde [13] who studied the dynamic response of Rayleigh beam carrying an added mass and traversed by uniform partially distributed moving loads. However their methods of solution are only suitable for simply supported end conditions. Thus, this study sets at solving this class of dynamical problem for all variants of classical boundary conditions often encountered by practicing engineers in the field.

2.0 Mathematical model

The problem of the dynamic response to a distributed load moving at uniform speed on a uniform elastic beam resting on elastic foundation is considered in this paper. The governing equation is the fourth order partial differential equation given by

$$\frac{\partial^2}{\partial x^2}\left[EJ\,\frac{\partial^2 W(x,t)}{\partial x^2}\right]-N\,\frac{\partial^2 W(x,t)}{\partial x^2}+\mu\frac{\partial^2 W(x,t)}{\partial t^2}+K(x)W(x,t)=P(x,t) \tag{2.1}$$

where x is the spatial coordinate t is the time, $W(x,t)$ is the transverse displacement E is Young's Modulus, J is the Moment of inertia, μ is the mass per unit length of the beam, N is the axial force and K is the elastic foundation. The moving load on the beam under consideration has mass commensurable with the mass of the beam. Thus, the load $p(x,t)$ takes the form Fryba [14]

$$P(x,t) = P_f(x,t)\left[1 - \frac{1}{g}\frac{d^2W(x,t)}{dt^2}\right] \tag{2.2}$$

where $P_f(x,t)$ is the continuous moving force acting on the beam model g is the acceleration due to gravity and $\frac{d}{dx}$ is the convective acceleration operator defined as

$$\frac{d}{dx} = \frac{\partial^2}{\partial t} + 2c\frac{\partial^2}{\partial x \partial t} + c^2\frac{\partial^2}{\partial x^2} \tag{2.3}$$

Furthermore, the moving force acting on the beam model here is defined as

$$P_f(x,t) = \sum_{i=1}^{N} MgH(x-ct) \tag{2.4}$$

where the time *t* is assumed to be limited to that interval of time within which the mass μ is on the beam, that is

$$0 \le ct \le L \tag{2.5}$$

and $H(x-ct)$ is the Heaviside function defined as

$$H(x-ct) = \begin{cases} 0, & for x < 0 \\ 1, & for\ x > 0 \end{cases} \tag{2.6}$$

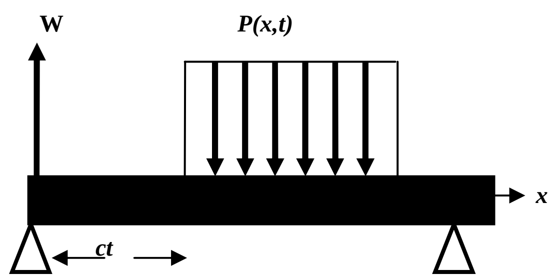

Figure 2.1: A distributed load on an elastic beam

with the properties,

(i) $$\frac{d}{dx}\{H(x-ct)\} = \delta(x-ct) \tag{2.7}$$

(ii) $$f(x)H(x-ct) = \begin{cases} 0, & for\ x < ct \\ f(x), & for\ x \ge ct \end{cases} \tag{2.8}$$

where $\delta(x-ct)$ represents the Dirac delta function and $H(x-ct)$ is a typical engineering function made to measure engineering applications which often involved functions that are either "off'" or "on".

In this paper, the Bernoulli-Euler beam under consideration is assumed to be uniform, that is, the beam properties, Young's modulus *E*, the moment of inertia *J* and the mass per unit length μ of the beam do not vary along the span of the beam. Substituting (2.2), (2.3), (2.4) and (2.5) into (2.1), one obtains,

$$EJ\frac{\partial^4 W(x,t)}{\partial x^4} - N\frac{\partial^2 W(x,t)}{\partial x^2} + \mu\frac{\partial^2 W(x,t)}{\partial t^2} + K(x)W(x,t)$$
$$+MH(x-ct)\left[\frac{\partial^2 W(x,t)}{\partial x^2} + 2c\frac{\partial^2 W(x,t)}{\partial x\partial t} + c^2\frac{\partial^2 W(x,t)}{\partial t^2}\right] = MH(x-ct) \tag{2.9}$$

The boundary conditions of the above problem are assumed to be arbitrary, that is, it can take any form of the classical boundary conditions. The initial conditions without any loss of generality is given by

$$W(x,t)\Big|_{t=0} = 0 = \frac{\partial W(x,t)}{\partial t}\Big|_{t=0} \tag{2.10}$$

3.0 Solution procedures

In this section, a general approach used in [2] is employed in order to solve the initial-value problem in equation (2.11).The approach involves in the first instance, the use of the generalized integral transformation technique to transform the governing fourth order partial differential equation . The resulting coupled second order ordinary differential equation is then treated using the modified asymptotic method of Struble and other integral transformation techniques. In order to solve equation (2.10) subject to the conditions (2.11), first, the generalized integral transformation technique is employed. This integral transformation technique is given by

$$\overline{W}(m,t) = \int_0^L W(x,t)U_m(x)dx \tag{3.1}$$

with the inverse

$$W(x,t) = \sum_{m=1}^{\infty}\frac{\mu}{W_m}\overline{W}(m,t)U_m(x) \tag{3.2}$$

where

$$W_m = \int_0^L \mu U_m^2(x)dx \tag{3.3}$$

and $U_m(x)$ is any function chosen such that the pertinent boundary conditions are satisfied. An appropriate selection of function for the beam problems are beam mode shapes. Thus, the mth normal mode of vibration of a uniform beam

$$U_m(x) = \sin\frac{\lambda_m x}{L} + A_m\cos\frac{\lambda_m x}{L} + B_m\sinh\frac{\lambda_m x}{L} + C_m\cosh\frac{\lambda_m x}{L} \tag{3.4}$$

is chosen as a suitable kernel of the integral transform (3.4) where, λ_m is the mode frequency, A_m, B_m, C_m are constants. The parameters λ_m , $A_m, B_m,$ and C_m are obtained by substituting (3.4) into the appropriate boundary conditions.

4.0 Transformation of equation

Applying the generalized integral transform (3.1), equation (2.10) can be written as

$$F_1G(0,L,t) + F_1G_A(t) - F_2G_B(t) + \overline{W}_{tt}(m,t) + F_3\overline{W}(m,t) + G_C(t) + G_D(t) + G_E(t) = MgH(x-ct) \tag{4.1}$$

where

$$F_1 = \frac{EJ}{\mu} \quad ,F_2 = \frac{N}{\mu} \quad ,F_3 = \frac{K}{\mu} \tag{4.2}$$

$$G(0,L,t) = \left[\frac{\partial^3 W(x,t)}{\partial x^3}U_m(x) - \frac{\partial^2 W(x,t)}{\partial x^2}\frac{d}{dx}U_m(x) + \frac{\partial W(x,t)}{\partial x}\frac{d^2}{dx^2}U_m(x) - W(x,t)\frac{d^3}{dx^3}U_m(x)\right]_0^L \tag{4.3}$$

$$G_A(t) = \int_0^L W(x,t)\frac{d^4U_m(x)}{dx^4}dx \tag{4.4}$$

$$G_B(t)=\int_0^L \frac{\partial^2 W(x,t)}{\partial x^2}U_m(x)dx \tag{4.5}$$

$$G_C(t)=\int_0^L MH(x-ct)\frac{\partial^2 W(x,t)}{\partial t^2}U_m(x)dx \tag{4.6}$$

$$G_D(t)=\int_0^L MH(x-ct)\frac{\partial^2 W(x,t)}{\partial x\partial t}U_m(x)dx \tag{4.7}$$

$$G_E(t)=\int_0^L MH(x-ct)\frac{\partial^2 W(x,t)}{\partial x^2}U_m(x)dx \tag{4.8}$$

It is generally known that the natural modes

$$U_m(x)=\sin\frac{\lambda_m x}{L}+A_m\cos\frac{\lambda_m x}{L}+B_m\sinh\frac{\lambda_m x}{L}+C_m\cosh\frac{\lambda_m x}{L} \tag{4.9}$$

satisfy the homogeneous differential equation

$$EJ\frac{d^4U_m(x)}{dx^4}-\mu\Omega_m^2(x)=0 \tag{4.10}$$

For the Euler beam, the parameter Ω_m is the natural circular frequency

defined by $$\Omega_m^2=\frac{\lambda_m^4}{L^4}\frac{EJ}{\mu} \tag{4.11}$$

From equation (4.10) we have $$\int_0^L W(x,t)\frac{d^4U_m(x)}{dx^4}dx=\frac{\mu}{EJ}\Omega_m^2\int_0^L W(x,t)U_m(x)dx \tag{4.12}$$

Thus, by (3.1) $$G_A(t)=\frac{\mu}{EJ}\overline{W}(m,t) \tag{4.13}$$

$$G_A(t)=\frac{\mu}{EJ}\overline{W}(m,t) \tag{4.13}$$

Noting that $\overline{W}(k,t)$ is just the co-efficient of the generalized integral. Transform,

$$\overline{W}(x,t)=\sum_{k=1}^{\infty}\frac{\mu}{W_k}\overline{W}(k,t)U_k(x) \tag{4.14}$$

thus, $$\frac{\partial^2}{\partial x^2}W(x,t)=\sum_{k=1}^{\infty}\frac{\mu}{W_k}\overline{W}(k,t)\frac{d^2}{dx^2}U_k(x) \tag{4.15}$$

so that integral (4.5) becomes, $$G_B(t)=\sum_{k=1}^{\infty}\overline{W}(k,t)\int_0^L\frac{d^2U_k(x)}{dx^2}U_m(x)dx \tag{4.16}$$

From [6], the Fourier cosine transform of $\delta(x-ct)$ is given by

$$\delta(x-ct)=\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty}Cos\frac{n\pi x}{L}Cos\frac{n\pi ct}{L} \tag{4.17}$$

When it is noted that $$H(x-ct)=\int\delta(x-ct)dx \tag{4.18}$$

It is straight forward to show that $$H(x-ct)=\frac{x}{L}+\frac{2}{n\pi}\sum_{n=1}^{\infty}Sin\frac{n\pi cx}{L}Cos\frac{n\pi ct}{L}+C^{o} \quad (4.19)$$

When use is made of equation (4.17) and (4.14), one obtains,

$$G_C(t)=\frac{1}{\alpha_k(x)}\sum_{k=1}^{\infty} M\overline{W}_{tt}(k,t)\left[\frac{x}{L}\int_0^L U_k(x)U_m(x)dx\right.$$

$$\left.+\frac{2}{n\pi}\sum_{k=1}^{\infty}Cos\frac{n\pi ct}{L}\int_0^L Cos\frac{n\pi x}{L}U_k(x)U_m(x)dx+C^{o}\int_0^L U_k(x)U_m(x)dx\right] \quad (4.20)$$

using similar argument in (4.14)and (4.17), it is straightforward to show that,

$$G_D(t)=\frac{1}{\alpha_k(x)}\sum_{k=1}^{\infty} MC\overline{W}_t(k,t)\left[\frac{2x}{L}\int_0^L \frac{dU_k(x)}{dx}U_m(x)dx\right.$$

$$\left.+\frac{2}{n\pi}\sum_{k=1}^{\infty}Cos\frac{n\pi ct}{L}\int_0^L Cos\frac{n\pi cx}{L}\frac{dU_k(x)}{dx}U_m(x)dx+2C^{o}\int_0^L \frac{dU_k(x)}{dx}U_m(x)d\right] \quad (4.21)$$

and $$G_E(t)=\frac{1}{\alpha_k(x)}\sum_{k=1}^{\infty} Mc^2\overline{W}(k,t)\left[\frac{x}{L}\int_0^L \frac{d^2U_k(x)}{dx^2}U_m(x)dx\right.$$

$$\left.+\frac{2}{n\pi}\sum_{k=1}^{\infty}Cos\frac{n\pi ct}{L}\int_0^L Cos\frac{n\pi cx}{L}\frac{d^2U_k(x)}{dx^2}U_m(x)dx+C^{o}\int_0^L \frac{d^2U_k(x)}{dx^2}U_m(x)dx\right] \quad (4.22)$$

Substituting (4.13), (4.16), (4.20), (4.21) and (4.22) into (4.1) after some simplifications, and rearrangement yields,

$$\overline{W}_{tt}(m,t)+\left[\Omega_m^2+\frac{K}{\mu}\right]\overline{W}(m,t)-\frac{N}{\mu}\sum_{k=1}^{\infty}\overline{W}(k,t)S_1(k,m)$$

$$+\varepsilon_0\left\{\sum_{k=1}^{\infty}\overline{W}_{tt}(k,t)S_2(k,m)+\frac{2}{n\pi m}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{W}_{tt}(k,t)Cos\frac{n\pi ct}{L}S_3(k,m,n)\right.$$

$$+LC^{o}\sum_{k=1}^{\infty}\overline{W}_{tt}(k,t)S_4(k,m)+2c\sum_{k=1}^{\infty}\overline{W}_t(k,t)S_5(k,m)$$

$$+\frac{4c}{n\pi}\frac{1}{\alpha_K(x)}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{W}_t(k,t)Cos\frac{n\pi ct}{L}S_6(k,m,n)+2cC^{o}\sum_{k=1}^{\infty}\overline{W}_t(k,t)S_7(k,m)$$

$$+\frac{Mc^2}{\mu L}\sum_{k=1}^{\infty}\overline{W}(k,t)S_8(k,m)+\frac{2c^2}{n\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{W}(k,t)Cos\frac{n\pi ct}{L}S_9(k,m,n)$$

$$+\frac{LMc^2C^{o}}{\mu L}\sum_{k=1}^{\infty}\overline{W}(k,t)S_{10}(k,m)$$

$$=\frac{PL}{\mu\lambda_m}\left[-Cos\,\lambda_m+A_m Sin\,\lambda_m+B_m Cosh\,\lambda_m+C_m Sinh\,\lambda_m\right.$$

$$+Cos\frac{\lambda_m ct}{L} - A_m Sin\frac{\lambda_m ct}{L} - B_m Cosh\frac{\lambda_m ct}{L} - C_m Sinh\frac{\lambda_m ct}{L}\ ] \qquad (4.23)$$

where, $$\varepsilon_0 = \frac{M}{\mu L} \qquad (4.24)$$

Equation (4.24) is the transformed equation governing the problem of a uniform Bernoulli-Euler beam on a constant elastic foundation when under the action of a traversing distributed load. This coupled non-homogeneous second order differential equation holds for all variants of the classical boundary conditions. In what follows, two special cases of the equation (4.24), namely, the moving force and the moving mass problems are discussed.

5.0 Solution of the transformed equation

5.1 Bernoulli-Euler beam traversed by moving force

An approximate model of the differential equation describing the response of a uniform Bernoulli-Euler beam resting on an elastic foundation and under the action of a moving distributed force may be obtained from (4.24) by setting $\varepsilon_0 = 0$. Thus, setting $\varepsilon_0 = 0$, equation (4.24) reduces to

$$\overline{W}_{tt}(m,t) = \left[\Omega_m^2 + \frac{K}{\mu}\right]\overline{W}(m,t) - \frac{N}{\mu}\sum_{k=1}^{\infty} \overline{W}(k,t)S_1\ (k,m)$$

$$= \frac{PL}{\mu\lambda_m}\left[- Cos\ \lambda_m + A_m Sin\ \lambda_m + B_m Cosh\ \lambda_m + C_m Sinh\ \lambda_m\right.$$

$$\left. +Cos\frac{\lambda_m ct}{L} - A_m Sin\frac{\lambda_m ct}{L} - B_m Cosh\frac{\lambda_m ct}{L} - C_m Sinh\frac{\lambda_m ct}{L}\ \right] \qquad (5.1)$$

Evidently, an exact solution to this equation is not possible. Though the equation yields readily to numerical technique, an analytical approximate method is desirable as the solution so obtained often shed light on the vital information about the vibrating system. Therefore, we are going to use a modification of the asymptotic method due to Struble often used in treating weakly homogeneous and non-homogeneous non-linear oscillatory system. To this end, equation (5.1) is rearranged to take the form

$$\overline{W}_{tt}(m,t) + \left[\gamma_{nf}^2 - \Gamma_0 S_1(m,m)\right]\overline{W}(m,t) - \Gamma_0 \sum_{\substack{k=1\\k\neq m}}^{\infty} \overline{W}(k,t)S_1\ (k,m)$$

$$= \frac{PL}{\mu\lambda_m}\left[- Cos\ \lambda_m + A_m Sin\ \lambda_m + B_m Cosh\ \lambda_m + C_m Sinh\ \lambda_m\right.$$

$$\left. +Cos\frac{\lambda_m ct}{L} - A_m Sin\frac{\lambda_m ct}{L} - B_m Cosh\frac{\lambda_m ct}{L} - C_m Sinh\frac{\lambda_m ct}{L}\ \right] \qquad (5.2)$$

where, $$\gamma_{nf} = \gamma_n^2 + \frac{K}{L} \quad \text{and} \quad \Gamma_0 = \frac{N}{\mu} \qquad (5.3)$$

By this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the effect of the axial force N. An equivalent free system operator defined by the modified frequency then replaces equation (5.2). First, the right hand side of equation (5.2) is set to zero, then we consider a parameter $\lambda < 1$ for any arbitrary ratio λ defined as

$$\lambda = \frac{\Gamma_0}{1+\Gamma_0} \qquad (5.4)$$

so that

$$\Gamma_0 = \lambda + 0(\lambda) \tag{5.5}$$

Substituting equation (5.5) into the homogeneous part of equation (5.2) one obtains

$$\frac{d^2}{dt^2}\overline{W}(m,t) + \left[\gamma_{nf}^2 - \lambda S_1(m,m)\right]\overline{W}(m,t) - \lambda\sum_{\substack{k=1\\k\neq m}}^{\infty}\overline{W}(k,t)S_1(k,m) = 0 \tag{5.6}$$

Setting λ to zero in equation (5.6) a situation corresponding to the case in which the axial force effect is regarded as negligible is obtained, then the solution of (5.6) becomes,

$$\overline{W}_{nf}(m,t) = C_{nf}Cos\left[\gamma_{nf}t - \phi_{nf}\right] \tag{5.7}$$

where $C_{nf}, \gamma_{nf}, and \phi_{nf}$ are constants

Furthermore as $\lambda < 1$, Struble's technique requires that the asymptotic solution of the homogeneous part of the equation (5.2) be the form

$$\overline{W}(m,t) = \beta(m,t)Cos\left[\gamma_{nf}t - \phi(m,t)\right] + \lambda\phi_1 + 0(\lambda^2) \tag{5.8}$$

where $\beta(m,t)$ and $\phi(m,t)$ are slowly varying functions of time.

To obtain the modified frequency, equation (5.8) and its derivatives are substituted into equation (5.6) and one obtains,

$$2\beta(m,t)\gamma_{nf}\dot{\phi}(m,t)Cos\left[\gamma_{nf}t - \phi(m,t)\right] - 2\dot{\beta}(m,t)\gamma_{nf}Sin\left[\gamma_{nf}t - \phi(m,t)\right]$$
$$+ \lambda S_1(m,m)\beta(m,t)Cos\left|\gamma_{nf}t - \phi(m,t)\right| = 0 \tag{5.9}$$

retaining terms to $0(\lambda)$ only. The variational equation are obtained by equating the coefficient of

$Sin\left|\gamma_{nf}t - \phi(m,t)\right|$ and $Cos\left|\gamma_{nf}t - \phi(m,t)\right|$

on both sides of the equation (5.9). Thus,

$$-2\dot{\beta}(m,t)\gamma_{nf} = 0 \tag{5.10}$$

and

$$2\beta(m,t)\gamma_{nf}\dot{\phi}(m,t)\ \lambda S_1(m,m)\beta(m,t) = 0 \tag{5.11}$$

Solving equations (5.10) and (5.11) respectively gives

$$\beta(m,t) = C_m^0 \tag{5.12}$$

$$\phi(m,t) = \frac{\lambda S_1(m,m)}{2\gamma_{nf}}t + \omega_{nf} \tag{5.13}$$

where C_m^0 and ω_{nf} are constants.

Therefore when the effect of the axial force is considered, the first approximation to the homogeneous system is

$$\overline{W}(m,t) = C_m^0 Cos\left[\theta_{am}t - \omega_{nf}\right] \tag{5.14}$$

where

$$\theta_{am} = \gamma_{nf}\left[1 - \frac{\lambda S_1(m,m)}{2\gamma_{nf}}\right] \tag{5.15}$$

represents the modified natural frequency due to the effect of axial force N. It is observed that when $\lambda = 0$, we recover the frequency of the moving force problem when the axial force effect of the beam is neglected. Thus to solve the non-homogeneous equation (5.2), the differential operator which act on $\overline{W}(m,t)$ and $\overline{W}(m,t)$ is replaced by the equivalent free system operator defined by modified frequency θ_{am}. Using equation (5.14) the homogeneous part of equation (5.2) can be written as

$$\frac{d^2}{dt^2}\overline{W}(m,t)+\theta_{am}^2\overline{W}(m,t)=0 \tag{5.16}$$

Hence, the entire equation (5.2) takes the form

$$\frac{d^2}{d^2}\overline{W}(m,t)+\theta_{am}\overline{W}(m,t)=\frac{PL}{\mu\lambda_m}\Big[-Cos\,\lambda_m+A_m Sin\,\lambda_m+B_m Cosh\,\lambda_m+C_m Sinh\,\lambda_m$$
$$+Cos\frac{\lambda_m ct}{L}-A_m Sin\frac{\lambda_m ct}{L}-B_m Cosh\frac{\lambda_m ct}{L}-C_m Sinh\frac{\lambda_m ct}{L}\Big] \tag{5.17}$$

To obtain the solution of (5.17), it is subjected to a Laplace transform and using convolution theory, expression for $\overline{W}(m,t)$ is obtained. Thus, on inversion, one obtains,

$$W(x,t)=\frac{1}{\alpha_m(x)}\sum_{m=1}^{\infty}\frac{PL}{\mu\lambda_m}\Bigg[\frac{H(m,t)(1-\mathrm{Cos}\,\theta_{am}t)}{\theta_{am}}-\frac{Cos\,\alpha_k t-Cos\,\theta_{am}t}{\theta_{am}^2-\alpha_k^2}$$
$$+\frac{A_m(\;Sin\alpha_k t+Sin\theta_{am}t\;)}{\theta_{am}^2-\alpha_k^2}+\frac{4B_m\theta_{am}\alpha_k Sin\theta_{am}tSinh\alpha_k t}{\alpha_k^4-\theta_{am}^4}$$
$$+\frac{2B_m\alpha_k^2 Cos\theta_{am}tCosh\alpha_k t}{\alpha_k^4-\theta_{am}^4}+\frac{B_m\theta_{am}Cosh\alpha_k t+B_m\left(\alpha_k^2+\theta_{am}^2\right)}{\alpha_k^4-\theta_{am}^4}$$
$$+\frac{4C_m\theta_{am}^2\alpha_k Sin\theta_{am}tCosh\alpha_k t}{\alpha_k^4-\theta_{am}^4}+\frac{2C_m\alpha_k^2 Cos\theta_{am}tSinh\alpha_k t}{\alpha_k^4-\theta_{am}^4}$$
$$+\frac{C_m\theta_{am}^2 Sin\,\alpha_k t}{\alpha_k^4-\theta_{am}^4}+\frac{C_m\alpha_k Sin\,\theta_{am}t\left(\alpha_k^2-\theta_{am}^2\right)}{\alpha_k^4-\theta_{am}^4}\Bigg]$$
$$\times\left(Sin\frac{\lambda_m x}{L}+A_m\cos\frac{\lambda_m x}{L}+B_m\sinh\frac{\lambda_m x}{L}+C_m\cosh\frac{\lambda_m x}{L}\right) \tag{5.18}$$

Equation (5.18) represents the transverse response to a moving force moving at constant velocity of a uniform Bernoulli-Euler beam resting on elastic foundation and having arbitrary support end conditions

5.2 Bernoulli-Euler beam traversed by a moving mass

In this section, the solution to the entire equation (4.24) is sought when no terms of the coupled differential equation is neglected. As in moving force problem, an exact analytical solution to equation (4.24) does not exist. Thus, the approximate analytical solution discussed earlier namely, modified Struble's asymptotic method is employed. Evidently, the homogenous part of equation (4.24) can be replaced by a free system operator defined by the modified frequency due to the presence of axial force N. Thus equation (4.24) can be rewritten in the form

$$\overline{W}_{tt}(m,t)+\theta_{am}^2\overline{W}(m,t)+\varepsilon_0\left\{\sum_{k=1}^{\infty}\overline{W}_{tt}(k,t)S_2(k,m)+\frac{2}{n\pi m}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{W}_{tt}(k,t)Cos\frac{n\pi ct}{L}S_3(k,m,n)\right.$$

$$+LC^{o}\sum_{k=1}^{\infty}\overline{W}_{tt}(k,t)S_4(k,m)+2c\sum_{k=1}^{\infty}\overline{W}_{t}(k,t)S_5(k,m)+\frac{4c}{n\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{W}_{t}(k,t)Cos\frac{n\pi ct}{L}S_6(k,m,n)$$

$$+2cC^{o}\sum_{k=1}^{\infty}\overline{W}_{t}(k,t)S_{7}(k,m)+\frac{Mc^{2}}{\mu L}\sum_{k=1}^{\infty}\overline{W}(k,t)S_{8}(k,m)+\frac{2c^{2}}{n\pi}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\overline{W}(k,t)Cos\frac{n\pi ct}{L}S_{9}(k,m,n)$$

$$+\frac{LMC^{2}C^{o}}{\mu L}\sum_{k=1}^{\infty}\overline{W}(k,t)S_{10}(k,m)=\frac{PL}{\mu\lambda_{m}}\left[-Cos\lambda_{m}+A_{m}Sin\lambda_{m}+B_{m}Cosh\lambda_{m}+C_{m}Sinh\lambda_{m}\right.$$

$$\left.+Cos\frac{\lambda_{m}ct}{L}-A_{m}Sin\frac{\lambda_{m}ct}{L}-B_{m}Cosh\frac{\lambda_{m}ct}{L}-C_{m}Sinh\frac{\lambda_{m}ct}{L}\right] \tag{5.19}$$

To tackle this problem, the same technique used in 5.2 is employed to obtain the modified frequency of the system due to the presence of the moving mass, namely

$$\theta_{bm}=\theta_{am}\left[1-\frac{\eta}{2}\left\{\left(S_{2}(m,m)+LC^{o}S_{4}(m,m)\right)-\frac{\left(Lc^{2}C^{o}S_{10}(m,m)+c^{2}S_{8}(m,m)\right)}{\theta_{am}^{2}}\right\}\right] \tag{5.20}$$

where

$$\eta=\frac{\varepsilon_{0}}{1+\varepsilon_{0}} \tag{5.21}$$

$$S_{2}(m,m)=S_{2}(k,m)\big|_{k=m}\ ,\ S_{8}(m,m)=S_{8}(k,m)\big|_{k=m} \tag{5.22}$$

$$S_{4}(m,m)=S_{4}(k,m)\big|_{k=m}\ ,\ S_{10}(m,m)=S_{10}(k,m)\big|_{k=m} \tag{5.23}$$

retaining $o(\eta)$ only. Hence the entire equation (5.2) takes the form,

$$\frac{d^{2}}{dt^{2}}\overline{W}(m,t)+\theta_{bm}^{2}\overline{W}(m,t)=\frac{\eta L^{2}g}{\lambda_{m}}\left[-\mathrm{H}(m,t)+Cos\frac{\lambda_{m}ct}{L}-A_{m}Sin\frac{\lambda_{m}ct}{L}-B_{m}Cosh\frac{\lambda_{m}ct}{L}-C_{m}Sinh\frac{\lambda_{m}ct}{L}\right] \tag{5.24}$$

This is analogous to equation (5.2).Thus, using similar arguments as in the previous section, $\overline{W}(m,t)$ can be obtained and on inversion gives,

$$W(x,t)=\frac{1}{\alpha_{m}(x)}\sum_{m=1}^{\infty}\frac{\varepsilon_{0}Lg}{\mu\lambda_{m}}\left[\frac{H(m,\mathrm{t})\left(1-\mathrm{Cos}\,\theta_{bi}t\right)}{\theta_{bi}}-\frac{Cos\alpha_{k}t-Cos\theta_{bi}t}{\theta_{bi}^{2}-\alpha_{k}^{2}}\right.$$

$$+\frac{A_{m}\left(Sin\alpha_{k}t+Sin\theta_{bm}t\right)}{\theta_{bm}^{2}-\alpha_{k}^{2}}+\frac{4B_{m}\theta_{bm}\alpha_{k}Sin\theta_{bm}tSinh\alpha_{k}t}{\alpha_{k}^{4}-\theta_{bm}^{4}}$$

$$+\frac{2B_{m}\alpha_{k}^{2}Cos\theta_{bm}tCosh\alpha_{k}t}{\alpha_{k}^{4}-\theta_{bm}^{4}}+\frac{B_{m}\theta_{bm}Cosh\alpha_{k}t+B_{m}\left(\alpha_{k}^{2}+\theta_{bm}^{2}\right)}{\alpha_{k}^{4}-\theta_{bm}^{4}}$$

$$+\frac{4C_{m}\theta_{bm}^{2}\alpha_{k}Sin\theta_{bm}tCosh\alpha_{k}t}{\alpha_{k}^{4}-\theta_{bm}^{4}}+\frac{2C_{m}\alpha_{k}^{2}Cos\theta_{bm}tSinh\alpha_{k}t}{\alpha_{k}^{4}-\theta_{bm}^{4}}$$

$$\left.+\frac{C_{m}\theta_{bm}^{2}Sin\alpha_{k}t}{\alpha_{k}^{4}-\theta_{bm}^{4}}+\frac{C_{m}\alpha_{k}Sin\theta_{bm}t\left(\alpha_{k}^{2}-\theta_{bm}^{2}\right)}{\alpha_{k}^{4}-\theta_{bm}^{4}}\right]$$

$$\times\left(Sin\frac{\lambda_{m}x}{L}+A_{m}\cos\frac{\lambda_{m}x}{L}+B_{m}\sinh\frac{\lambda_{m}x}{L}+C_{m}\cosh\frac{\lambda_{m}x}{L}\right) \tag{5.25}$$

Equation (5.25) represents the transverse response to a moving mass moving at constant velocity of a uniform Bernoulli-Euler beam resting on elastic foundation and having arbitrary support end conditions.

6.0 Applications

In this paper, some examples of classical boundary conditions are selected to illustrate the analyses presented. Such classical boundary conditions include; simply supported boundary conditions, Clamped-clamped ends conditions and, Clamped-free ends conditions (Cantilever beam).

6.1 Simply supported boundary conditions

In this case, the displacement and the bending moment vanish, Thus

$$W(0,t) = 0 = W(L,t)\,,\ \frac{\partial^2 W(0,t)}{\partial x^2} = 0 = \frac{\partial^2 W(L,t)}{\partial x^2} \tag{6.1}$$

Hence for normal modes

$$U_m(0) = 0 = U_m(L)\,,\ \frac{\partial^2 U_m(0)}{\partial x^2} = 0 = \frac{\partial^2 U_m(L)}{\partial x^2} \tag{6.2}$$

which implies that

$$U_k(0) = 0 = U_k(L)\,,\ \frac{\partial^2 U_k(0)}{\partial x^2} = 0 = \frac{\partial^2 U_k(L)}{\partial x^2} \tag{6.3}$$

Thus, making use the of boundary conditions above, it can be shown that

$$A_m = 0,\quad B_m = 0,\quad \mathrm{C_m} = 0 \quad \text{and} \quad \lambda_{\mathrm{m}} = m\pi \tag{6.4}$$

$$A_k = 0,\quad B_k = 0,\quad \mathrm{C_k} = 0 \quad \text{and} \quad \lambda_{\mathrm{k}} = k\pi \tag{6.5}$$

Thus, substituting equations (6.4) and (6.5) into equation (5.1) and rearranging, the moving force problem reduces to the non-homogeneous second order ordinary differential equation given by

$$\frac{d^2}{dt^2}\overline{W}(m,t) + \theta_{mf}^2\,\overline{W}(m,t) = \frac{PL}{\mu m\pi}\left[-(-1)^m + Cos\frac{m\pi ct}{L}\right] \tag{6.6}$$

where
$$\theta_{mf}^2 = \left\{\frac{EJ}{\mu}\left(\frac{m\pi}{L}\right)^4 + \frac{N}{\mu}\left(\frac{m\pi}{L}\right)^2 + \frac{K}{\mu}\right\} \tag{6.7}$$

Thus, solving equation (6.6) in conjunction with the initial conditions, one obtains an expression for $\overline{W}(m,t)$ which when inverted yields,

$$W(x,t) = \frac{2}{L}\sum_{m=1}^{\infty}\frac{PL}{\mu m\pi}\left[\frac{-(-1)^m}{\theta_{mf}^2} - \frac{Cos\,\omega_m t - Cos\,\theta_{mf}\,t}{\theta_{mf}^2 - \omega_m^2}\right] \tag{6.8}$$

Equation (6.8) above represents the transverse displacement response to a moving force moving at a constant velocity of a simply supported Bernoulli-Euler Beam resting on elastic foundation. Substituting equations (6.4) and (6.5) into equation (5.52) rearranging and following arguments similar to those in previous section, Struble technique is used to obtain

$$\theta_{mm} = \theta_{mf}\left[1 - \frac{\lambda}{2}\left\{\theta_{mf}\left(\frac{L}{2m^2\pi^2} + LC^o\right) - \sum_{n=1}^{\infty}\left(\frac{\theta_{mf}}{n\pi m} - \frac{c^2}{\theta_{mf}L}\right) + \frac{m^2\pi^2c^2}{\theta_{mf}L}\right\}\right] \tag{6.9}$$

Equation (6.9) is the modified frequency corresponding to the frequency of the simply supported system due to the presence of moving mass. Thus, the moving mass problem reduces to

$$\frac{d^2}{dt^2}\overline{W}(m,t) + \theta_{mm}^2\,\overline{W}(m,t) = \frac{PL}{\mu m\pi}\left[-(-1)^m + Cos\frac{m\pi ct}{L}\right] \tag{6.10}$$

solving equation (6.10) in conjunction with the initial conditions yields expression for $\overline{W}(m,t)$ and on inversion gives

$$W(x,t)=2\sum_{m=1}^{\infty}\frac{\lambda L^2 g}{m\pi}\left[\frac{(-1)^m(1-Cos\theta_{mm})}{\theta_{mm}}-\frac{(Cos\omega_k t-Cos\theta_{mm})}{\theta_{mm}^2-\omega_m^2}\right]\times Sin\frac{m\pi x}{L} \quad (6.11)$$

Equation (6.11) represents the transverse-displacement response to a distributed mass moving with constant speed of simply supported uniform Bernoulli-Euler beam resting on elastic foundation.

6.2 Clamped/fixed ends condition

At clamped-clamped ends, both deflection and slope vanish

$$W(0,t)=0=W(L,t) \quad \text{and} \quad \frac{\partial}{\partial x}W(0,t)=0=\frac{\partial}{\partial x}W(L,t) \quad (6.12)$$

And for normal modes, $U_m(0)=0=U_m(L)$ and $\frac{\partial}{\partial x}U_m(0)=0=\frac{\partial}{\partial x}U_m(L)$ (6.13)

which implies that,

$$U_k(0)=0=U_k(L) \text{ and } \frac{\partial}{\partial x}U_k(0)=0=\frac{\partial}{\partial x}U_k(L) \quad (6.14)$$

Thus it can be shown that

$$A_m=\frac{Sinh\lambda_m-Sin\lambda_m}{Cos\lambda_m-Cosh\lambda_m}=\frac{Cos\lambda_m-Cosh\lambda_m}{Sin\lambda_m+Sinh\lambda_m}=-C_m \text{ and } B_m=-1 \quad (6.15)$$

In view of (6.15), the frequency equation is given as

$$Cos\lambda_m Cosh\lambda_m=1 \quad (6.16)$$

It follows from equation (6.16) that

$$\lambda_1=4.73004,\lambda_2=7.85320,\lambda_3=10.99561, \quad (6.17)$$

By interchanging m and k in equation (6.14) and (6.15), an expression for A_k, B_k, C_k and corresponding frequency equations are obtained. Thus, the general solutions of the associated moving force and moving mass problems are obtained by substituting relevant results in (6.15) and (6.17) into (5.18) and (5.25)

6.3 One end clamped and one end free condition-cantilever beam

Next at $x=0$ the beam is taken to be clamped at the end $L=0$, the beam is free. Thus, the boundary conditions of the Bernoulli-Euler beam can be written as,

$$W(0,t)=0=\frac{\partial}{\partial x}W(0,t) \text{ and } \frac{\partial^2}{\partial x^2}W(L,t)=0=\frac{\partial^3}{\partial x^3}W(L,t) \quad (6.18)$$

And for normal modes, $U_m(0)=0=\frac{d}{dx}U_m(0)$ and $\frac{d^2}{\partial x^2}U_m(L)=0=\frac{d^3}{dx^3}U_m(L)$ (6.19)

which implies that, $U_k(0)=0=\frac{d}{dx}U_k(0)$ and $\frac{d^2}{dx^2}U_k(L)=0=\frac{d^3}{dx^3}U_k(L)$ (6.20)

Using (6.20), we can show that at $x=0$,

$$A_m=C_m \text{ and } B_m=-1 \quad (6.21)$$

$$A_m=-\frac{Sin\lambda_m-Sinh\lambda_m}{Cos\lambda_m+Cosh\lambda_m}=\frac{Cos\lambda_m-Cosh\lambda_m}{Sinh\lambda_m-Sin\lambda_m}=-C_m \text{ and } B_m=-1 \quad (6.22$$

and the frequency equation for the other end condition is $Cos\lambda_m Cosh\lambda_m = -1$ (6.23)

such that, $\lambda_1 = 1.875$, $\lambda_2 = 4.694$, $\lambda_3 = 7.855$, (6.24)

Using (6.22), (6.23) and (6.24) in equation (5.18) and (5.25), one obtains the displacement response respectively to a moving force and a moving mass of a uniform clamped-free ends of Bernoulli-Euler beam resting on elastic foundation.

7.0 Discussion of the analytical solutions

If the undamped system such as this is studied, it is desirable to examine the response amplitude of the dynamical system which may grow without bound. We call this resonance conditions. Equation (6.8) clearly shows that the simply supported elastic beam resting on elastic foundation and traversed by moving force reaches a state of resonance whenever

$$\theta_{mf} = \frac{m\pi c}{L} \tag{7.1}$$

while equation (5.48) indicates that the same beam under the action of a moving mass experiences resonance effect when

$$\theta_{mf} = \frac{m\pi c}{L} \tag{7.2}$$

From equation (6.9),

$$\theta_{mm} = \theta_{mf}\left\{1-\frac{\lambda}{2}\left\{\left[\left(\frac{L}{2m^2\pi^2}+LC^o\right)-\sum_{n=1}^{\infty}\left(\frac{\theta_{mf}}{n\pi m}-\frac{c^2}{\theta_{mf}L}\right)+\frac{m^2\pi^2c^2}{\theta_{mf}L}\right]\right\}\right\} \tag{7.3}$$

which implies

$$\theta_{mf} = \frac{m\pi c/L}{\left[1-\frac{\lambda}{2}\left\{\left[\left(\frac{L}{2m^2\pi^2}+LC^o\right)-\sum_{n=1}^{\infty}\left(\frac{\theta_{mf}}{n\pi m}-\frac{c^2}{\theta_{mf}L}\right)+\frac{m^2\pi^2c^2}{\theta_{mf}L}\right]\right\}\right]} \tag{7.4}$$

From equation (7.4) it is deduced that for the same natural frequency, the critical speed for the system consisting of a simply supported elastic beam resting on an elastic foundation and traversed by a force moving with a uniform speed is greater than that of the moving mass problem. Thus, for the same natural frequency of an elastic beam, resonance is reached earlier in the moving mass system than in the moving force system.

For the resonance conditions of other classical boundary conditions, equation (5.41) clearly shows that the uniform elastic beam resting on an elastic foundation and traversed by a force moving with a constant speed reaches a state of resonance whenever

$$\theta_{am} = \frac{m\pi c}{L} \tag{7.5}$$

while equation (5.25) shows that same beam under the action of a moving mass experiences resonance effect whenever

$$\theta_{bm} = \frac{m\pi c}{L} \tag{7.6}$$

From equation (5.18)

$$\theta_{bm} = \theta_{am}\left[1-\frac{\eta}{2}\left\{\left(S_2(m,m)+LC^oS_4(m,m)\right)-\frac{\left(Lc^2C^oS_{10}(m,m)+c^2S_8(m,m)\right)}{\theta_{am}^2}\right\}\right] \tag{7.7}$$

which implies

$$\theta_{am} = \frac{m\pi c/L}{1-\frac{\eta}{2}\left\{\left(S_2(m,m)+LC^oS_4(m,m)\right)-\frac{\left(Lc^2C^oS_{10}(m,m)+c^2S_8(m,m)\right)}{\theta_{am}^2}\right\}} \tag{7.8}$$

Evidently, from (7.7) and (7.8), the same results and analyses obtained in the case of a simply supported Bernoulli-Euler beam are obtained for all other examples of classical boundary conditions.

8.0 Numerical calculation and discussion of results

For the purpose of Numerical analysis of our dynamical system, the uniform beam of length 12.192m is considered. Also $\frac{EI}{\mu} = 2200\ m^4/s^2$, speed of the mass is 8.128 *m/s* and the ratio of the mass of the load to the beam is 0.2. The transverse deflections of the beam are calculated and plotted against time for various values of axial force *N* and subgrade *K*. Values of *N* between 0 and 20,000,000 were used while the values of K were varied between 0 N/m^3 and 400,000 N/m^3. The results are as shown on the various graphs below.

Figure 8.1 displays transverse displacement response of a simply supported uniform beam under the action of distributed forces moving at variable velocities for various values of axial force N for fixed values of foundation moduli *K* = 40,000. The figure shows that as *N* increases the deflection of the uniform beam decreases. In a similar way, for various time *t*, the deflection profile of the beam for various values of foundation moduli *K* and for fixed axial force *N* are shows in figure 8.2. It is observed that higher values of foundation moduli reduce the deflection profile of the beam. In figure 8.3 and 8.4, the corresponding curves due to moving masses of the uniform beam clamped at both ends are presented. Figure 8.5 displays transverse displacement response of a clamped-clamped uniform beam under the action of distributed forces moving at constant velocity for various values of axial force *N* for fixed value of foundation moduli K=40,000. The figure shows that as N increases the deflection of the uniform beam decreases. The deflection profile of the beam for various values of foundation moduli and for fixed axial force N are shown in figure 8.6. It is observed as that as foundation modulus increases the the deflection of the beam decreases.

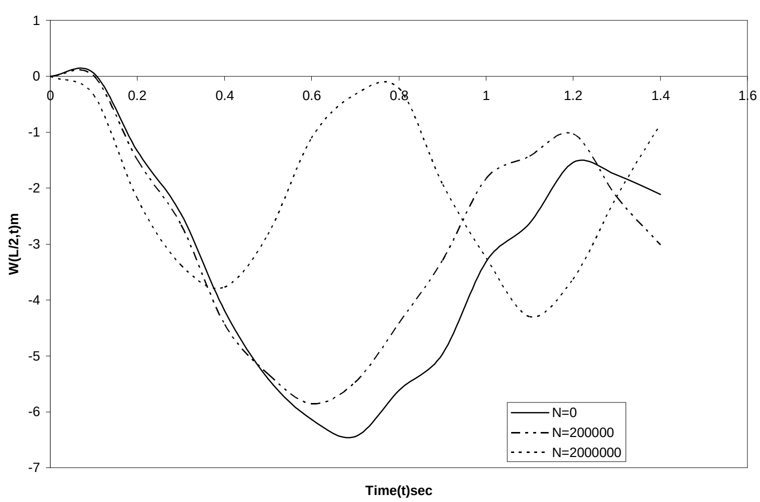

Figure 8.1: Transverse displacement of the simply supported beam under the action of forces moving at constant velocity for various values of axial force N for fixed value of foundation moduli K (40000).

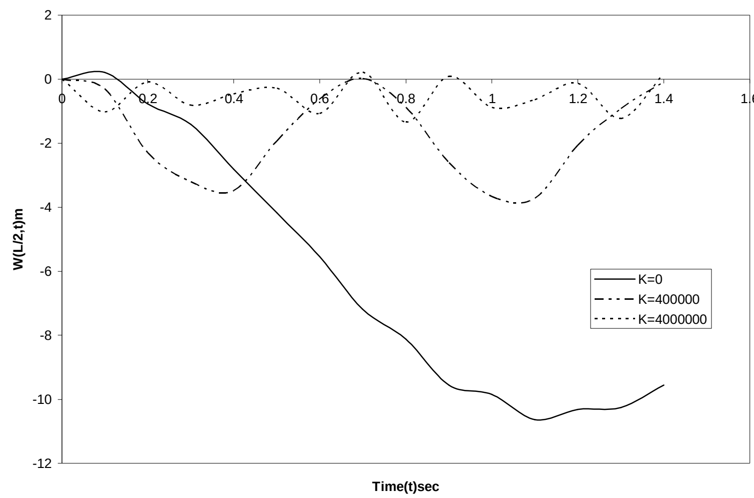

Figure 8.2: Deflection profile of the simply supported beam under the action of force moving at constant velocity for various values of foundation moduli K for fixed value of axial force N (20000)

Finally, figure 8.5, 8.6 and 8.7 show the comparison of the transverse displacement of moving force and moving mass cases for N=200,000 and K=40,000 for simply supported end condition, Clamped-Clamped end condition and Clamped-free (Cantilever beam) end condition respectively. The figure shows that the response amplitude of moving mass is higher than that of the moving force.

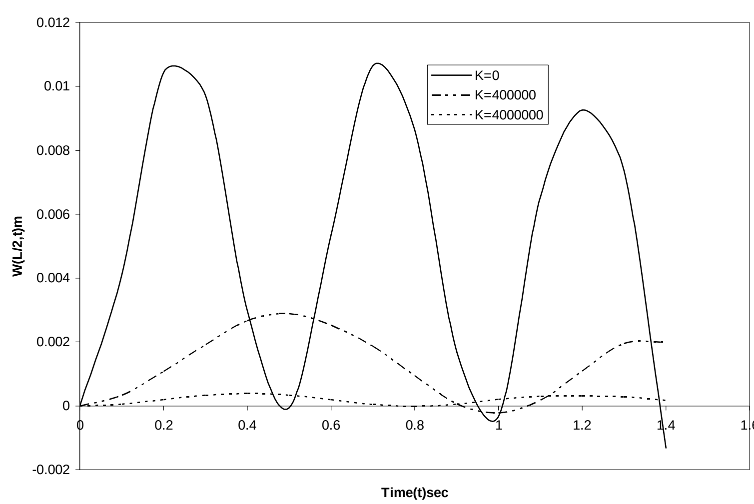

Figure 8.3: Deflection profile of the clamped-clamped uniform beam under the action of distributed forces moving at constant velocity for various values of foundation moduli K and for fixed value of axial force N (200000).

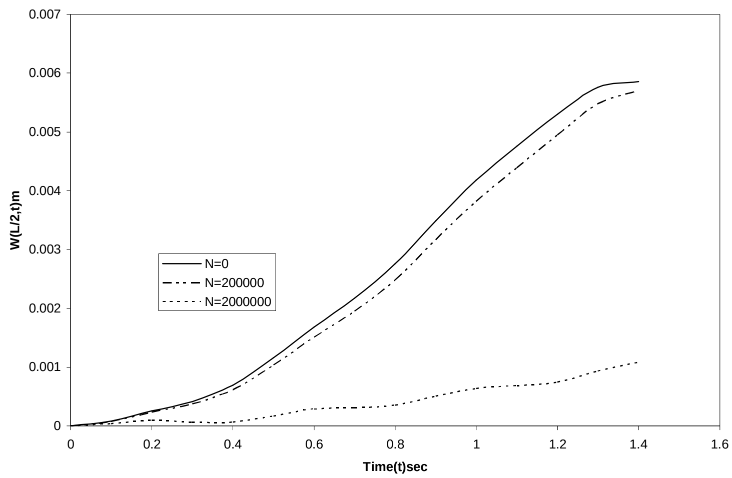

Figure 8.4: Transverse displacement of the clamped-clamped uniform beam under the action distributed masses moving at constant velocity for various values of axial force N and for fixed value of foundation moduli K (40000)

9.0 Conclusion

The problem of dynamical analysis of finite prestressed Bernoulli-Euler beam with general boundary conditions when it is under the action of travelling loads is considered in this paper. The governing equation is a non-homogeneous fourth order partial differential equation with variable and singular coefficients. At the right hand side is the so-called Heaviside function which describes the arrival of a continuous load distributed along the beam. The main objective is to obtain a closed form solution valid for all variants of classical boundary conditions to the cumbersome partial differential equations. The solution technique is based on generalized integral transforms, the use of the properties of the Heaviside function $H(x-ct)$ as the generalized derivative of the Dirac Delta function $\delta(x-ct)$

in the distributed sense and a modification of the asymptotic method of struble. The analytical and numerical analyses show that (*i*) for the same natural frequency, the critical speed of the moving mass problem is smaller than that of the moving force problem. Hence, resonance is reached earlier in moving mass problem of a uniform beam under the action of a distributed moving load. (*ii*) as the axial force *N* increases, the response amplitudes of uniform Bernoulli-Euler beams under the action of moving distributed loads moving with constant velocities decrease. (iii) when the axial force N is fixed, the displacement of a uniform Bernoulli-Euler beam resting on elastic foundation and traversed by distributed masses travelling with constant speed decreases as the foundation moduli increase for all variants of the boundary conditions.

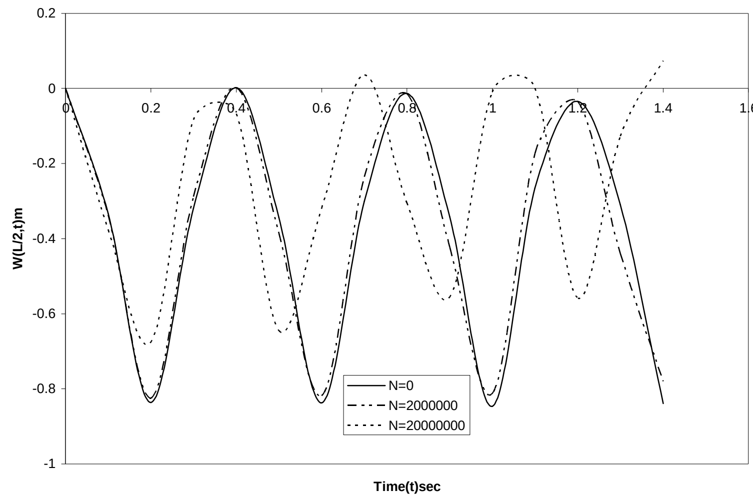

Figure 8.5: Transverse displacement of the clamped-free uniform beam under the action of distributed forces moving at constant velocity for various values of axial force N for fixed value of foundation moduli K (40000).]

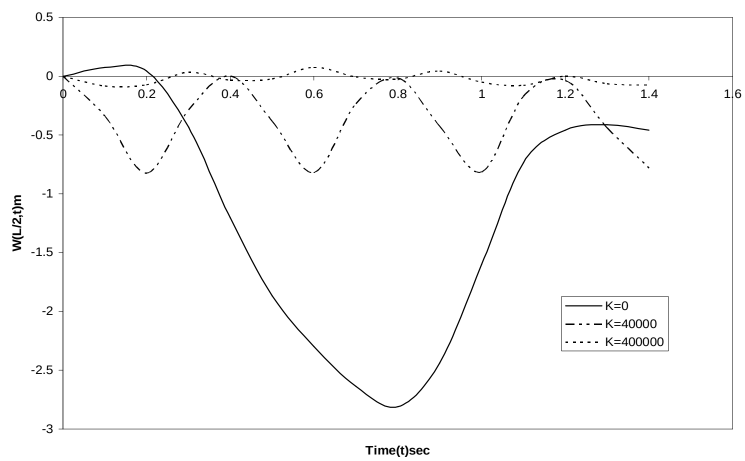

Figure 8.6: Deflection profile of the clamped-free uniform beam under the action of distributed forces moving at constant velocity for various values foundation moduli *K* and for fixed value of axial force *N* (200000).

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