Journal of the Nigerian Association of Mathematical Physics
Volume **12 (May, 2008), 69 – 78**

Relative null controllability of nonlinear systems with multiple delays in state and control

R. A. Umana
Department of Mathematics and Computer Science
Federal University of Technology, Owerri, Imo state, Nigeria.

Abstract

Sufficient conditions for the relative null controllability of nonlinear systems with time varying multiple delays in the state and control are developed. Conditions are placed on the perturbation function which guarantee that if the linear control base system is proper and if the uncontrolled linear system is uniformly asymptotically stable, then the nonlinear delay system is relatively null controllable. As application, an example is given to illustrate the obtained results.

***Keywords*:** Controllability, uniform asymptotic stability, nonlinear systems, multiple delays.

1.0 Introduction

The concept of controllability plays a major role in finite-dimensional control theory. Controllability is the property of being able to steer between two arbitrary points in the state space. On the other hand, null controllability is the property of being able to steer all points exactly to the origin. This has important connections with the concept of stabilizability. Investigation into the controllability of functional differential systems to the origin has attracted great attention in recent years with the growing interest in disease control models in which the number of infected individuals is desired to be controlled to zero. Several authors have studied the null controllability of various kinds of dynamical systems. Balachandran et al [2] studied the local null controllability of nonlinear functional differential systems in Banach spaces whereas in [5] Balachandran and Leelamani investigated the null controllability of neutral evolution integrodifferential systems with infinite delay. Iyai [9] discussed the Euclidean null controllability of linear systems with delays in state and control. Iheagwam and Onwuatu [4] derived a set of sufficient conditions for the relative controllability and null controllability of linear systems with distributed delays in the state and control whereas in [7] Onwuatu studied the null controllability of nonlinear infinite neutral system. Eke [3] established a set of conditions for the null controllability of linear control systems. Umana and Nse [6] studied the null controllability of nonlinear integrodifferential systems with delays whereas in [1] Umana discussed the relative null controllability of linear systems with multiple delays in state and control.
In this research we develop sufficient computable criteria for the relative null controllability on a bounded interval $[0, t_1]$ of nonlinear systems with time varying multiple delays in the state and control variables. Our results extend those of [9, 4, 1] to nonlinear systems with multiple delays in state and control.

2.0 Preliminaries

Let n and m be positive integers, E the real line $(-\infty, \infty)$. We denote by E^n the space of real n-tuples with the Euclidean norm denoted by $|\cdot|$. If J is any interval of E, the usual Lebesgue

e-mail: reubenandy@yahoo.com

space of square integrable (equivalent classes of) functions from J to E^n will be denoted by $L_2(J, E^n)$. $L_1([0,t_1], E^n)$ denotes the space of integrable functions from $[0,t_1]$ to E^n. N_{nm} will be used for the collection of all real $n \times m$ matrices with a suitable norm.

Let $h > 0$ be given. For functions $x:[-h,t_1] \to E$, $t \in [0,t_1]$ we use the symbol x_t to denote the function on $[-h,0]$ defined by $x_t(s) = x(t+s)$ for $s \in [-h,0]$. $C = C([-h,0], E^n)$ is the space of continuous functions mapping the interval $[-h,0]$ into E^n. Similarly, for functions $u:[-h,t_1] \to E^m$, $t \in [0,t_1]$, we use the symbol u_t to denote the function on $[-h,0]$ defined by $u_t(s) = u(t+s)$ for $s \in [-h,0]$.

Consider the nonlinear system with time varying multiple delays of the form:

$$\dot{x}(t) = \sum_{i=0}^{P} A_i(t)x(t-h_i) + \sum_{i=0}^{P} B_i(t)u(t-h_i) + f(t, x(t), x(t-h), u(t), u(t-h)) \tag{2.1}$$

$$x(t) = \phi(t) \quad t \in [-h,0]$$

where $x \in E^n$, $u \in E^m$, $t \in [0,t_1]$ for $t_1 > 0$.

$A_i(t)$ are $n \times n$ continuous matrices, $B_i(t)$ are $n \times m$ continuous matrices and $\phi(t)$ is a continuous vector function on the interval $[-h,0]$. Throughout the sequel, the control sets of interest are $IB = L_2\left([0,t_1], E^m\right)$, $IU \subseteq L_2\left([0,t_1], E^m\right)$ a closed and bounded subset of IB with zero in the interior relative to IB.

We shall show that if the free system

$$\dot{x}(t) = \sum_{i=0}^{P} A_i(t)x(t-h_i) \tag{2.2}$$

is uniformly asymptotically stable, and the linear control system

$$\dot{x}(t) = \sum_{i=0}^{P} A_i(t)x(t-h_i) + \sum_{i=0}^{P} B_i(t)u(t-h_i) \tag{2.3}$$

is relatively controllable, then system (2.1) is relatively null controllable provided the continuous function f satisfies some smoothness and growth conditions.

The above conditions on A_i and B_i ensures that for each initial data $(0,\phi)$ a unique solution $x(t)$ of system (2.1) exists through $(0,\phi)$ which is continuous in $(0,\phi)$. This solution is given by

$$x(t) = X(t,0)\phi(0) + \sum_{i=0}^{P} \int_{-h_i}^{0} X(t, s+h_i)A_i(s+h_i)\phi(s)ds + \int_0^t X(t,s)\sum_{i=0}^{P} B_i(s)u(s-h_i)ds$$
$$+ \int_0^t X(t,s) f(s, x(s), x(s-h), u(s), u(s-h))ds \tag{2.4}$$

This formula can be rewritten as

$$x(t)=X(t,0)\phi(0)+\sum_{i=0}^{P}\int_{-h_i}^{0}X(t,s+h_i)A_i(s+h_i)\phi(s)ds+\sum_{i=0}^{P}\int_{-h_i}^{0}X(t,s+h_i)B_i(s+h_i)u_0(s)ds$$

$$+\sum_{i=0}^{P}\int_{0}^{t-h_i}X(t,s+h_i)B_i(s+h_i)u(s)ds+\int_0^t X(t,s)f(s,x(s),x(s-h),u(s),u(s-h))ds \qquad (2.5)$$

where $X(t,s)$ is the fundamental solution of system (2.2) which satisfies the equations:

$$\frac{\partial}{\partial t}X(t,s)=\sum_{i=0}^{P}A_i(t)X(t-h_i,s), t>s \qquad (2.6)$$

$$X(t,s)=\begin{cases} I, & t=s \\ 0, & t<s \end{cases} \qquad (2.7)$$

or
$$\frac{\partial}{\partial t}X(t,s)=-\sum_{i=0}^{P}X(t,s+h_i)A_i(s+h_i), t>s \qquad (2.8)$$

Define
$$Z(t,s)=\sum_{i=0}^{P}X(t,s+h_i)B_i(s+h_i) \qquad (2.9)$$

and the controllability matrix
$$W(0,t)=\int_0^t Z(t,s)Z^T(t,s)ds \qquad (2.10)$$

where *m* denotes the matrix transpose.

***Definition* 2.1**

The system (2.1) is said to be relatively controllable on $[0,t_1]$ if for any function $\phi\in C$ and any vector $x\in E^n$, there exists a control $u\in IB$ such that the solution $x(t)=x(t,0,\phi,u,f)$ of (2.1) satisfies $x(\cdot,0,\phi,u,f)=\phi$, $x(t_1,0,\phi,u,f)=x_1$. It is relatively null controllable on $[0,t_1]$ if $x_1=0$ in the above definition.

***Definition* 2.2**

The reachable set $R(t,0)$ of (2.3) at time t is a subset of E^n given by

$$R(t,0)=\left\{\int_0^t Z(t,s)u(s)ds : u\in IB\right\}.$$

***Definition* 2.3**

The system (2.3) is said to be proper in E^n on $[0,t_1]$ if $\eta^T Z(t_1,s)=0$ almost everywhere $s\in[0,t_1]$, $\eta\in E^n$ implies $\eta=0$, where η^T is the transpose of η.

***Definition* 2.4**

The domain D of relative null controllability of system (2.1) is the set of all initial functions $\phi\in C$ which can be steered to the origin $0\in E^n$ in finite time, using $u\in IU$.

From the above developments, we now proceed to establish some crucial facts leading to the main results of this paper.

Firstly, we have the following:

***Proposition* 2.1**

The following statements are equivalent for system (2.3) *(i)* $W(0,t_1)$ *is nonsingular for each* $t_1 > 0$

(ii) *system (2.3) is proper in* E^n *for each interval* $[0,t_1]$

(iii) *system (2.3) is relatively controllable on each interval* $[0,t_1]$.

Proof:

First we show that (i) => (ii)

Let $W(0,t_1) = \int_0^{t_1} Z(t_1,s)Z^T(t_1,s)ds$. Define the operator

$$K : L_2([0,t_1],E^m) \to E^n \text{ by } K(u) = \int_0^{t_1} Z(t_1,s)u(s)ds$$

where K is a continuous linear operator from a Hilbert space to another. Thus $R(K) \subset E^n$ is a linear subspace and its orthogonal complement satisfies the relation $\{R(K)\}^\perp = N(K^*)$ where K^* is the adjoint of K, $K^* : E^n \to IU \subset L_2$. By the nonsingularity of $W(0,t_1)$, the symmetric operator $KK^T = W(0,t_1)$ is positive definite and hence

$$\{R(K)\}^\perp = \{0\} \text{ i.e } N(K^*) = \{0\}.$$

For any $\eta \in E^n$, $u \in L_2$ the inner product

$$<\eta, Ku> = <K^*\eta, u>$$

$$<\eta, Ku> = <\eta, \int_0^t Z(t,s)u(s)ds> = \int_0^t \eta^T[Z(t,s)]u(s)ds.$$

Thus K^* is given by $\eta \to \eta^T[Z(t_1,s)]$; $s \in [0,t_1]$.

$N(K^*)$ is therefore the set of all such $\eta \in E^n$ such that

$$\eta^T[Z(t_1,s)] = 0,$$

almost everywhere in $[0,t_1]$. Since $N(K^*) = \{0\}$, all such η are equal to zero, i.e $\eta = 0$.
This establishes the properness of system (2.3).

Next we show that (ii) => (iii).

We now show that if system (2.3) is proper then it is relatively controllable on each interval $[0,t_1]$. Let $\eta \in E^n$, if system (2.3) is proper then $\eta^T[Z(t,s)] = 0$
almost everywhere $s \in [0,t_1]$ for each t_1 implies $\eta = 0$.

Thus $$\int_0^t \eta^T[Z(t,s)]u(s)ds = 0 \text{ for } u \in L_2.$$

It follows that the only vector orthogonal to the set

$$R(t_1,0) = \left\{ \int_0^t Z(t,s)u(s)ds : u \in L_2 \right\}$$

is the zero vector. Hence $\{R(t_1,0)\}^\perp = \{0\}$, i.e $R(t_1,0) = E^n$.

This establishes relative controllability on $[0, t_1]$ of system (2.3).
Finally, we show that (*iii*) => (*i*).

We now show that if the system (2.3) is relatively controllable then $W(0, t_1)$ is nonsingular. Let us assumed for a contradiction that $W = W(0, t_1)$ is singular. Then there exists an n vector $v \neq 0$ such that

$$vWv^T = 0.$$

Then

$$\int_0^t \| v[Z(t,s)] \|^2 \, ds = 0.$$

This implies that

$$\| v[Z(t,s)] \|^2 = 0,$$

Hence

$$v[Z(t,s)] = 0$$

almost everywhere for $t \in [0, t_1]$.

This implies that $v \neq 0$, which contradicts the assumption of properness of the system. This completes the proof.

We also have the following:

***Theorem* 2.1**

System (2.3) *is relatively controllable if and only if* $0 \in IntR(t_1, 0)$ *for each* $t_1 > 0$.

***Proof*:**

$R(t_1, 0)$ is a closed and convex subset of E^n. Therefore a point y_1 on the boundary of $R(t_1, 0)$ implies there is a support plane π of $R(t_1, 0)$ through y_1. This means that $\eta^T (y - y_1) \leq 0$ for each $y \in R(t_1, 0)$ where $\eta \neq 0$ is an outward normal to π. If u_1 is the control corresponding to y_1 we have

$$\eta^T \int_0^t [Z(t,s)] u(s) ds \leq \eta^T \int_0^t [Z(t,s)] u_1(s) ds$$

for each $u \in IU$. Since IU is closed and bounded, it is assumed to be a unit sphere and the last inequality holds for $u \in IU$ if and only if

$$\eta^T \int_0^t [Z(t,s)] u(s) ds \leq \int_0^t \eta^T [Z(t,s)] u_1(s) ds = \int_0^t \left| \eta^T Z(t,s) \right| ds$$

and

$$u_1(t) = \operatorname{sgn} \eta^T Z(t,s)$$

as y_1 is on the boundary. Since we always have $0 \in R(t_1, 0)$, if 0 were not in the interior of $R(t_1, 0)$, then it is on the boundary. Hence from the preceding argument, this implies that

$$0 = \int_0^t \left| \eta^T Z(t,s) \right| ds$$

so that

$$\eta^T Z(t,s) = 0 \;\; a.e \;\; s \in [0, t_1].$$

This by definition of properness of system (2.3) implies that the system is not proper since $\eta \neq 0$, hence if $0 \in IntR(t_1, 0)$

$$\eta^T Z(t,s) = 0 \;\; a.e \;\; t \in [0, t_1]$$

would imply $\eta = 0$ proving properness and by Proposition2.1, it is concluded that system (2.3) is relatively controllable for each interval $[0,t_1]$.
In the next section we harness the results put together above to establish the main result of this paper.

3.0 Main result

***Theorem* 3.1**

In (2.1), assume that

(i) *the zero solution of system* (2.2) is *uniformly asymptotically stable so that every solution of* (2.2) *satisfies* $\|x(t)\| \le M\|\phi\|e^{-\alpha t}$ *where* $\alpha > 0,\ M > 0$ *are constants;*

(ii) *system* (2.3) *is relatively controllable on* $[0,t_1]$ *for each* $t_1 > 0$;

(iii) *the continuous function f satisfies all smoothness conditions for the existence and uniqueness of solutions;*

(iv) $f(t,0,0,0,0) = 0$ *;*

Then system (2.1) *is relatively null controllable with constraints.*

Proof:

Suppose that the solution of system (2.1) with $x_0(\phi,u,f) = \phi$ satisfies $x(t,\phi,u,f) = 0$ for some $u \in IU$, then from equation (2.5)

$$0 = x_L(t,\phi) + \int_0^{t-h_i} Z(t,s)u(s)ds + \int_0^t X(t,s)f(s,x(s),x(s-h),u(s),u(s-h))ds$$

where

$$x_L(t,\phi) = X(t,0)\phi(0) + \sum_{i=0}^{P}\int_{-h_i}^{0} X(t,s+h_i)A_i(s+h_i)\phi(s)ds + \sum_{i=0}^{P}\int_{-h_i}^{0} X(t,s+h_i)B_i(s+h_i)u_0(s)ds$$

so that

$$x_L(t,\phi) = -\int_0^{t-h_i} Z(t,s)u(s)ds - \int_0^t X(t,s)f(s,x(s),x(s-h),u(s),u(s-h))ds\,.$$

Recall the definition of $R(t_1,0)$ and now define

$$Y(t_1,0) = \left\{-\int_0^t X(t,s)f(s,x(s),x(s-h),u(s),u(s-h))ds; u \in IU\right\}.$$

If we now set

$$V(t_1,0) = \left\{-\int_0^{t-h_i} Z(t,s)u(s0ds - \int_0^t X(t,s)f(s,x(s),x(s-h),u(s),u(s-h))ds : u \in IU\right\}$$

then
$$V(t_1,0) \subseteq R(t_1,0) + Y(t_1,0).$$

By definition, the domain D of relative null controllability of system (2.1) is the collection of all initial functions $\phi \in C$ such that there exists t_1 and $u \in IU$ such that the solution of system (2.1) with $x_0(0,\phi,u,f) = \phi$ satisfies $x(t,0,\phi,u,f) = 0$. By (ii) and Theorem 3.1, $0 \in IntR(t_1,0)$ and so there is an open ball S such that $0 \in s \subseteq R(t_1,0)$. Hence $S + Y(t_1,0)$ is a ball around $Y(t_1,0)$. Therefore, $0 \in Y(t_1,s) \subseteq IntV(t_1,0)$, for $t_1 > 0$, so that $0 \in IntD$.

Also by (iv), $0 \in D$. Suppose $0 \notin IntD$, then there exists a countable sequence $\{\phi_i\}_1^\infty \subseteq C$ such that $\phi_i \to 0$ as $i \to \infty$ and no ϕ is in D for any i so that $\phi_i \neq 0$. Let $x(t,\phi_i,0) = x_i$, then since $\phi_i \in D$ for any i, $x(t,\phi_i,u) \neq 0$ for any i so, by the variation of constant formula, we have a sequence $\{x_i\}_1^\infty \subseteq E^n$ such that $x_i \to 0$ as $i \to \infty$ and no x_i is in $V(t_1,0)$ for any t_1, therefore $0 \notin IntV(t_1,0)$ - a contradiction. This contradiction shows that $0 \in IntD$. Therefore there exists a ball B_1 around the origin contained in D such that $0 \subseteq B_1 \subseteq IntD$. By conditions (i) and (ii), every solution of the system

$$\dot{x}(t) = \sum_{i=0}^{P} A_i(t)x(t-h_i) + f(t,x(t),x(t-h),0,0)$$

(which is a solution of system (2.1) with $u = 0$) satisfies $x(t,0) \to 0$ as $t \to \infty$. Hence at some $t_2 < \infty$, we have $x_{t_2}(\cdot,0) \in B_1$. Therefore for some $u \in IU$, and some $t_3 > t_2$, the solution $x(t_2, x_{t_2}(\cdot,0), u, f)$ of system (2.1) satisfies $x(t_3, x_{t_2}, u, f) = 0$. Hence system (2.1) is relatively null controllable.

4.0 Application

Assuming piecewise continuity of $Z(t,s)$ with respect to t, and making use of arguments as in Proposition 2.1, we get the following:

***Lemma* 4.1**

The system (2.3) is relatively controllable on $[0,t_1]$ if and only if for $\eta \in E^n$, $\eta^T Z(t,s) = 0$ on $[0,t_1]$ implies $\eta = 0$.

Proof:

Immediate from Proposition 2.1.

***Lemma* 4.2**

The system (2.3) is relatively controllable on $[0,t_1]$ if rank $\left[\int_0^t Z(t,s)Z^T(t,s)ds\right] = n$.

Proof:

Relative controllability of system (2.3) implies that the controllability matrix $W(0,t_1)$ be nonsingular. By the non-singularity of $W(0,t_1)$, the symmetric operator $\int_0^t Z(t,s)Z^T(t,s)ds$ is positive definite. But this holds if and only if rank $\left[\int_0^t Z(t,s)Z^T(t,s)ds\right] = n$.

These criteria are difficult to use, since the matrix valued function $Z(t,s)$ can be analytically obtained only in exceptional cases. In order to handle easily with many lags h_i, we introduce as in [8] the following determining equation.

Let $J=(j_0, j_1,..., j_p)$ be a multi-index defined as a vector, where $j_i, i=0,..., p$ are integers (not necessarily positive) and define $|J|=\sum_{i=0}^{P} j_i$. Let E_i be a multi-index with $j_k=\delta_{ik}, k=0,1,..., p.$ Clearly $|E_i|=1.$ Let $H=(h_0, h_1,..., h_p)$ be a vector of delays with

$$(J,H)=\sum_{i=0}^{P} j_i h_i. \tag{4.1}$$

Assume that $A_i(t)$ and $B_i(t)$ are respectively $(q-2)$ and $(q-1)$ continuously differentiable on $[0,t_1]$. For systems with delays in state and control the determining equation is

$$Q_k(J,t)=\sum_{i=0}^{P} A_i(t) Q_{k-1}(J-E_i, t-(E_i,H))-\frac{d}{dt}Q_{k-1}(J-E_0,t) \tag{4.2}$$

for $k=1,2,...q-1,\quad t\in[0,t_1]$ with initial conditions

$$Q_0(J,t)=\begin{cases} B_i(t),\ t\in[0,t_1], & \mathit{for}\ J=E_i \\ 0, & \mathit{for\ other}\ J\end{cases} \tag{4.3}$$

where $Q_k(J,t)$ is some matrix. We deduce from (4.2) and (4.3) that

(i) $Q_k(J,t)=0$ for $|J|\neq k+1$ or for $J<0$;

(ii) if some $A_i(t)$ or $B_i(t)$ are undefined for $t<0$, then $Q_k(J,t)$ with $J\geq 0, |J|=k+1$, $k=1,2,...,q-1$ is undefined for $t<0$;

(iii) from (4.2) and (ii) it follows that $Q_k(J,t), |J|=k+1$, is undefined also for $t-(E_i,H)<0$ and by induction for $t-(J,H)<0$.

***Theorem* 4.3**

Assume $A_i(t), B_i(t)$ *are respectively* $(n-2)$ *and* $(n-1)$ *continuously differentiable on* $[0,t_1]$. If

$$rank\,\hat{Q}_n(t_1)=n \tag{4.4}$$

where

$$\hat{Q}_n(t_1)=\left\{\hat{Q}_k(J,t_1), k=0,1,...,n-1, J: t_1-(J,H)\geq 0\right\} \tag{4.5}$$

and $\hat{Q}_k(J,t_1)=\sum_{k=0}^{q-1} Q_k(R,t_1)$ for $|R|=|J|=k+1$ and $(R,H)=(J,H)$, *then system* (2.3) *is relatively controllable on* $[0,t_1]$.

Proof:

For the proof see [8].

***Example* 4.4**

Consider the system

$$\dot{x}(t)=A_0x(t)+A_1x(t-1)+A_2x(t-2)+B_0u(t)+B_1u(t-1)+B_2u(t-2)$$
$$+f(t,x(t),x(t-h),u(t),u(t-h)) \tag{4.6}$$

where

$$A_0=\begin{pmatrix}2 & 2\\ -2 & -5\end{pmatrix},\ A_1\begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix},\ A_2\begin{pmatrix}1 & 0\\ -1 & 0\end{pmatrix},\ B_0\begin{pmatrix}1 & 0\\ -1 & 0\end{pmatrix},\ B_1\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\ B_2\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$f=(e^{-at}\sin(x(t)+x(t-h))\cos(u(t)+u(t-h))).$$

Here $n = k = 2$.

Take $H = (h_0, h_1, h_2) = (0,1,2), J = (j_0, j_1, j_2) = (2,0,1), R = (r_0, r_1, r_2) = (1,2,0)$. From the definition, $E_i = \partial_{ik}, i = 0,1,2$. Thus $E_0 = 0, E_1 = 0, E_2 = 1$.

$$(J,H) = \sum_{i=0}^{2} j_i h_i = 0+0+2 = 2$$

$$(R,H) = \sum_{i=0}^{2} r_i h_i = 0+2+0 = 2$$

$$|J| = \sum_{i=0}^{2} j_i = 2+0+1 = 3$$

$$|R| = \sum_{i=0}^{2} r_i = 1+2+0 = 3$$

$$\hat{Q}_n = \left\{ \hat{Q}_k(J), k = 0,1,...,n-1, J : t_1 - (J,H) \geq 0 \right\}$$

$$\hat{Q}_2 = \left\{ \hat{Q}_k(J), k = 0,1, J : t_1 - (J,H) \geq 0 \right\} = \left\{ \hat{Q}_0(J), \hat{Q}_1(J), J : t_1 - 2 \geq 0 \right\}$$

By definition $\hat{Q}_k(J) = \sum_{k=0}^{1} Q(R)$ for $|R| = |J| = k+1$ and $(R,H) = (J,H)$.

Thus $\hat{Q}_0(J) = Q_0(R)$ and $\hat{Q}_1(J) = Q_1(R)$.

But $Q_k(J) = \sum_{i=0}^{2} A_i Q_{k-1}(J - E_i)$, for $k = 1,2,..,q-1$,

$$Q_0(J) = \begin{cases} B_i, & \text{for } J = E_i \\ 0, & \text{for other } J \end{cases}.$$

Hence $Q_0(R) = B_0 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$

$$Q_1(R) = A_0 B_0 + A_1 B_0 + A_2 B_0$$

$$= \begin{pmatrix} 2 & 2 \\ -2 & -5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$$

rank $\hat{Q}_2 =$ rank $\left\{ \hat{Q}_0(J), \hat{Q}_1(J) : t_1 - 2 \geq 0 \right\}$

rank $\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 3 & 0 \end{pmatrix}$ = rank $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix} = 2 = n$.

By Theorem 4.1, the linear base system of (4.6) is relatively controllable for $t_1 \geq 2$. We now show that the free part of system (4.6) is uniformly asymptotically stable. The free part of system (4.6) is

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + A_2 x(t-2) \tag{4.7}$$

The solution of system (4.7) is uniformly asymptotically stable if $x(t) \to 0$ as $t \to \infty$. This can happen if the roots of the characteristic equation have negative real parts.

$$\det\left[\lambda I - (A_0 + A_1 e^{-\lambda} + A_2 e^{-2\lambda})\right]$$

$$\det\left[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \left\{\begin{pmatrix} 2 & 2 \\ -2 & -5 \end{pmatrix} + \begin{pmatrix} 0 & e^{-\lambda} \\ 0 & -e^{-\lambda} \end{pmatrix} + \begin{pmatrix} e^{-2\lambda} & 0 \\ -e^{-2\lambda} & 0 \end{pmatrix}\right\}\right] = 0$$

$$\det\begin{pmatrix} \lambda - 2 - e^{-2\lambda} & -(2 + e^{-\lambda}) \\ 2 + e^{-2\lambda} & \lambda + 5 + e^{-\lambda} \end{pmatrix} = 0$$

$$\lambda^2 + 3\lambda + \lambda e^{-\lambda} - (\lambda + 3)e^{-2\lambda} - 6 = 0.$$

Comparing this result with the equation $\lambda^2 + b\lambda + q\lambda e^{-\lambda h} + k = 0$ whose roots will have negative real part if $b > q, b > 0, q > 0$ (see Driver [10], pp.327), we conclude that the roots of the characteristic equation of system (4.7) have negative real part. Hence the zero solution of system (4.7) is uniformly asymptotically stable. Moreover

$$\left| f(t, x(t), x(t-h), u(t), u(t-h)) \right| = \left| e^{-at} \sin(x(t) + x(t-h)) \cos(u(t) + u(t-h)) \right| \le e^{-at} \cdot 1.$$

Hence by Theorem 3.1, system (4.6) is relatively null controllable with constraint.

5.0 Conclusion

The paper contains sufficient conditions for the relative null controllability in a given finite time interval for nonlinear systems with time varying multiple delays in state and control. These conditions are given with respect to the uniform asymptotic stability of the free linear base system and the controllability of the linear controlled base system, with the assumption that the perturbation function f satisfies some smoothness and growth conditions. A similar method may be applied to derive sufficient conditions for the so called absolute or functional controllability of the systems considered.

References

[1] R. A. Umana, Relative null controllability of linear systems with multiple delays in state and control, *J. Nigerian Ass. Math. Physics*, 10 (2006) 517 – 522.

[2] K. Balachandran, J. P. Dauer and P. Balasubramanian, Local null controllability of nonlinear functional differential systems in Banach spaces, *J. Optim. Theory Appl.*, 88 (1995) 61 – 75.

[3] A. N. Eke, Null controllability for linear control systems, *J. Inst. Math.& Comp. Sci. (Math. Ser.)*, 3 (1990) 149 – 154.

[4] V. A. Iheagwam and J. U. Onwuatu, Relative controllability of linear delay systems with distributed delays in the state and control, *J. Nigerian Ass. Math. Physics*, 9 (2005) 221 – 238.

[5] K. Balachandran and A. Leelamani, Null controllability of neutral evolution integrodifferential systems with infinite delay, *Math. Problems in Engineering*, 2006 (2006) 1 – 18.

[6] R. A. Umana and C. A. Nse, Null controllability of nonlinear integrodifferential systems with delays, *J. Advances in Modelling (Automatic Control)*, 61 (2006) 73 – 84.

[7] J. U. Onwuatu, Null controllability of nonlinear infinite neutral system, *Kybernetika*, 29 (1993) 325 – 336.

[8] A. Manitius and A. W. Olbrot, Controllability conditions for linear systems with delayed state and control, *Archiwum Automatyki Telemechaniki*, 17 (1972) 119 – 130.

[9] D. Iyai, Euclidean null controllability of linear systems with delays in state and control, *J. Nigerian Ass. Math. Physics*, 10 (2006) 553 – 558.

[10] R. D. Driver, *Ordinary and delay differential equations*, Applied Mathematical Sciences 20, Springer-Verlag, New York, (1977).