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Relative controllability of nonlinear perturbed neutral systems with distributed and multiple delays in control

R. A. Umana
Department of Mathematics and Computer Science
Federal University of Technology, Owerri, Imo State, Nigeria

Abstract

We establish a set of sufficient conditions for the relative controllability of nonlinear perturbed neutral systems with distributed and time varying multiple lumped delays in control. The results are established by using the Schauder fixed point theorem.

***Keywords*:** Controllability, nonlinear perturbed neutral systems, distributed delays, multiple lumped delays, Schauder's fixed point theorem.

1.0 Introduction

A neutral functional differential equation is one in which the derivatives of the past history or derivatives of functionals of the past history are involved as well as the present state of the system. Neutral differential equations arise in many areas of applied mathematics and such equations have received much attention in recent years.

Systems with delayed control are natural models for the study of some economic, biological and physiological systems as well as electromagnetic systems composed of subsystems interconnected by hydraulic and various other linkages [4]. More specifically, models for systems with distributed delays in the control occur in the study of agricultural economics and population dynamics [8].

The theory of controllability of functional differential equations has been extensively studied in the literature. Fu [1] studied the controllability and the local controllability of abstract neutral functional differential systems with unbounded delay by using the fractional power of operators and the Sadovskii fixed point theorem. Mahmudov and Zorlu [6] studied the approximate controllability of semilinear neutral systems in Hilbert spaces by using the Schauder fixed point theorem. Balachandran and Anandhi [5] established sufficient conditions for the controllability of neutral functional integrodifferential infinite delay systems in Banach spaces by using the analytic semigroup theory and the Nussbaum fixed point theorem. Balachandran et al [2] obtained some existence results for nonlinear abstract neutral differential equations with time varying delays by using the Schaefer fixed point theorem and as an application the controllability problem is discussed. Balachandran and Anandhi [7] investigated the controllability problem for neutral functional integrodifferential control systems in Banach spaces.

The main purpose of this paper is to extend the results of [3] by considering a more general class of nonlinear perturbed neutral time varying systems with distributed and multiple delays in control. Using Schauder's fixed point theorem sufficient conditions for the relative controllability in a given time interval are formulated and proven.

2.0 Preliminaries

Let E be the real line $(-\infty,\infty)$ and E^n the Euclidean n-dimensional vector space. Let

e-mail: reubenandy@yahoo.com

$h>0$ be a given real number. For a function $x:[t_0-h,t_1]\to E^n$ and $t\in[t_0,t_1]\subset E$, we use the symbol x_t to denote the function on $[-h,0]$ defined by $x_t(s)=x(t+s)$ for $s\in[-h,0]$. Similarly, for a function $u:[t_0-h,t_1]\to E^m$ and $t\in[t_0,t_1]\subset E$, we use the symbol u_t to denote the function on $[-h,0]$ defined by $u_t(s)=u(t+s)$ for $s\in[-h,0]$. The symbol $C=C([-h,0],E^n)$ is the space of continuous functions from $[-h,0]\to E^n$, with sup norm.

Let us consider the nonlinear neutral system with distributed and multiple time varying delays in control, represented by the following differential equation:

$$\frac{d}{dt}D(t,x_t)=f(t,x_t,u(t))+\int_{-h}^{0}d_\theta B(t,\theta,x(t),u(t))u(t+\theta)$$
$$+F(t,x(t),u(w_0(t)),u(w_1(t)),...,u(w_i(t)),...,u(w_N(t)))\ \ x(t)=\phi(t),\qquad t\in[-h,0]\tag{2.1}$$

satisfied everywhere on the interval $[t_0,t_1]$, $(t_0<t_1)$, and where $x\in E^n$, u is an m-dimensional control function with $u\in C_m[t_0-h,t_1]$, $B(t,\theta,x,u)$ is an $n\times m$-dimensional matrix, continuous in (t,x,u) for fixed θ, and of bounded variation in θ on $[-h,0]$ for each $(t,x,u)\in[t_0,t_1]\times E^{n\times m}$, $f,F:E\times C\times E^m\to E^n$ are nonlinear real n-vector functions which are continuous and Lipschitzian in both x and u. The integral is in the Lebesgue – Stieltjes sense which is denoted by the symbol d_θ. The continuous strictly increasing functions $w_i(t):[t_0,t_1]\to E$, $i=0,1,2,...,N$, represent deviating arguments in the control, that is, $w_i(t)=t-h_i(t)$,where $h_i(t)$ are lumped time varying delays for $i=0,1,2,...,N$. The operator $D, D:E\times C\to E^n$ is atomic at 0 and uniformly atomic at 0 in the sense of Hale [10]. Instead of the atomicity assumption on D, we may assume that D is of the form $D(t,\phi)=\phi(0)-g(t,\phi)$, where $g:E\times C\to E^n$ is continuous and uniformly nonatomic at zero on $E\times C$ in the following sense.

***Definition* 2.1**

For any $(t,\phi)\in E\times C$, and $\mu\ge 0$, $s\ge 0$, let $Q(t,\phi,\mu,s)=\{\psi\in C:(t,\psi)\in E\times C,\|\psi$ - $\phi\|\le\mu,\psi(\theta)=\phi(\theta),\theta\ \pi\ -s,\theta\in[-h,0]\}$.

We say that a continuous function $g:E\times C\to E^n$ is uniformly nonatomic at zero on $E\times C$ if, for any $(t,\phi)\in E\times C$, there exist $s_0>0$, $\mu_0>0$ independent of (t,ϕ), and a scalar function $\rho(t,\phi,\mu,s)$, defined and continuous for (t,ϕ), for $s\in[0,s_0]$, $\mu\in[0,\mu_0]$, nondecreasing in μ,s such that

$$\rho_0=\rho(E\times C,\mu_0,s_0)=\sup_{E\times C}\rho(t,\phi,\mu_0,s_0)<1$$

and
$$|g(t,\psi)-g(t,\phi)|\le\rho_0\,\|\psi-\phi\|$$

for $t \in E$, $\psi \in Q(t,\phi,\mu,s)$ and all $s \in [0,s_0]$, $\mu \in [0,\mu_0]$.

***Definition* 2.2**

The set $z(t) = \{x(t), x_t, u_t\}$ is said to be the complete state of the system (2.1) at time t.

***Definition* 2.3**

The system (2.1) is said to be relatively controllable on $[t_0, t_1]$ if, for every initial complete state $z(t_0)$ and every vector $x_1 \in E^n$, there exists a control $u \in C_m[t_0, t_1]$ such that the solution of the system (2.1) satisfies $x(t_1) = x_1$.

It is known ([9] and the references therein) that under the prevailing assumptions on $D, f, g, B, F,$ and u for each $\phi \in C$ there is a unique solution of (2.1) with initial value ϕ at t_0. The solution is continuous with respect to initial data and parameter u.

A function x is a solution of (2.1) through (t_0, ϕ) if and only if there exists a $t_1 > 0$ such that x satisfies the equation

$$D(t,x_t) = D(t_0,\phi) + \int_{t_0}^{t} f(s,x_s,u(s))ds + \int_{t_0}^{t} X(t,s,x,u)\left(\int_{-h}^{0} d_\theta B(s,\theta,x,u)u(s+\theta)\right)ds$$

$$+\int_{t_0}^{t} X(t,s,x,u)F(s,x(s),u(w_0(s)),u(w_1(s)),...,u(w_i(s)),...mu(w_N(s)))ds, \quad t \in [t_0,t_1],$$

$$x(t_0) = \phi, \tag{2.2}$$

where $X(t,s,x,u)$ is an $n \times n$ matrix function defined for $0 \le s \le t+h$, continuous in s from the right, of bounded variation in s, $X(t,s,x,u) = 0$, $t < s \le t+h$.

Since

$$D(t,x_t) = x(t) - g(t,x_t),$$

we deduce that the solution $x(t)$ of (2.1) is given by $x(t+t_0) = \phi(t)$, $t \in [-h,0]$,

$$x(t) = D(t_0,\phi) + g(t,x_t) + \int_{t_0}^{t} f(s,x_s,u(s))ds + \int_{t_0}^{t} X(t,s,x,u)\left(\int_{-h}^{0} d_\theta B(s,\theta,x,u)u(s+\theta)\right)ds$$

$$+\int_{t_0}^{t} X(t,s,x,u)F(s,x(s),u(w_0(s)),u(w_1(s)),...,u(w_i(s)),...,u(w_N(s)))ds, \quad t \ge t_0 \tag{2.3}$$

Observe that the second to the last term of (2.3) contains the values of the control for $t < t_0$, as well as for $t > t_0$. The values of the control $u(t)$ for $t \in [t_0 - h, t_0]$ enter into the definition of the initial complete state $z(t_0)$. Thus, the second to the last term of (2.3) must be transformed to take care of this by interchanging the order of the integration. Using the unsymmetric Fubini theorem, as in [11], we have the following:

$$x(t+t_0) = \phi(t), t \in [-h,0], x(t) = D(t_0,\phi) + g(t,x_t) + \int_{t_0}^{t} f(s,x_s,u(s))ds$$

$$+\int_{-h}^{0} d_{B_\theta} \int_{t_0+\theta}^{t_0} X(t,s-\theta,x,u)B(s-\theta,\theta,x,u)u_{t_0}(s)ds$$

$$+\int_{t_0}^{t}\left(\int_{-h}^{0} X(t,s-\theta,x,u)d_\theta, B_t(s-\theta,\theta,x,u)\right)u(s)ds$$

$$+\int_{t_0}^{t} X(t,s,x,u)F(s,x(s),u(w_0(s)),u(w_1(s)),...,u(w_i(s)),...,u(w_N(s)))ds \tag{2.4}$$

where the symbol d_{B_θ} denotes that the integration is in the Lebesgue – Stieltjes sense with respect to the variable θ in B and

$$B_t(s,\theta,x,u) = \begin{cases} B(s,\theta,x,u), & s \le t \\ 0, & s > t. \end{cases}$$

Define $p(t,x,u) = D(t_0,\phi) + g(t,x_t) + \int_{t_0}^{t} f(s,x_s,u(s))ds,$

$$q(t,x,u_{t_0}) = \int_{-h}^{0} d_{B_\theta} \int_{t_0+\theta}^{t_0} X(t,s-\theta,x,u)B(s-\theta,\theta,x,u)u_{t_0}(s)ds$$

$$Z(t,s,x,u) = \int_{-h}^{0} X(t,s-\theta,x,u)d_\theta B_t(s-\theta,\theta,x,u)$$

and the $n\times n$-dimensional controllability matrix

$$W(t_0,t_1,x,u) = \int_{t_0}^{t_1} Z(t_1,s,x,u)Z^T(t_1,s,x,u)ds$$

where T denotes the matrix transpose. Note that $W(t_0,t_1,x,u)$ is symmetric and nonnegative-definite. The solution of (2.1) with u as a control is given by (2.3). Let $x \in C([t_0-h,\tau],E^n)$, $0 \le \tau < \infty$. Define

$$\bar{x}_{t_0}(t) = x(t), \quad t \in [t_0-h,t_0],$$
$$\bar{x}_{t_0}(t) = x(t_0), \quad t \in [t_0,t_0+\tau],$$

and define $\bar{\phi} : [-h,\infty) \to E^n$ by

$$\bar{\phi}(t) = \phi(t), \quad t \in [t_0-h,t_0],$$
$$\bar{\phi}(t) = \phi(0), \quad t \in [0,\infty).$$

It follows from these definitions and from (2.3) that x is a solution of (2.1) on $[t_0,t_0+\tau]$, if and only if

$$x(t_0+t) = \bar{\phi}(t) + k(t), \quad -h \le t \le \tau,$$

where $k(t)$ satisfies

$$k(t) = g(t+t_0,\bar{\phi}_t + k_t) - g(t_0,\phi) + \int_0^t f(s+t_0,\phi_s+k_s,u(s+t_0))ds$$
$$+\int_0^t X(t+t_0,s+t_0,\phi+k,u)\left(\int_{-h}^{0} d_\theta B(s+t_0,\theta,\phi+k,u)u(s+t_0+\theta)\right)ds$$
$$+\int_0^t X(t+t_0,s+t_0,\phi+k,u)F(s+t_0,\bar{\phi}(s)+k(s),u(w_0(s+t_0)),u(w_1(s+t_0)),...,$$
$$u(w_i(s+t_0)),...,u(w_N(s+t_0)))ds, \quad k_0 = 0.$$

3.0 Main result

We are now ready to obtain our main results on the relative controllability of the nonlinear perturbed neutral system (2.1). For this, we will take

$$p = (x,u_0,u_1,...,u_i,...,u_N) \in E^n \times E^{(N+1)m}$$

and let $$|p| = |x| + |u_0| + |u_1| + ... + |u_i| + ... + |u_N|,$$

where $|\cdot|$ denotes the standard norm in the finite dimensional Euclidean space.

***Theorem* 3.1**

Let the continuous function F satisfy the so called growth condition

$$\lim_{|p|\to\infty}\frac{|F(t,p)|}{|p|}=0 \tag{3.1}$$

uniformly in $t\in[t_0,t_1]$, and suppose that the function g is continuous and uniformly nonatomic at zero and f is continuous and uniformly Lipschitzian in the last two arguments. Then the system (2.1) is relatively controllable on $[t_0,t_1]$ if there exists a positive constant l such that for each pair of functions $(x,u)\in C_n[t_0,t_1]\times C_m[t_0,t_1]$

$$\det W(t_0,t_1,x,u)\ge l\,.$$

***Proof*:**

Let $Q=C_n[t_0,t_1]\times C_m[t_0,t_1]$ and define the nonlinear continuous operator

$$T:Q\to Q \text{ by } T(x,u)=(y,v)\,,$$

where

$$v(t)=Z^T(t_1,t,x,u)W^{-1}(t_0,t_1,x,u)[x_1-p(t_1,x,u)-q(t_1,x,u_{t_0})$$
$$-\int_{t_0}^{t_1}X(t_1,s,x,u)F(s,x(s),u(w_0(s)),u(w_1(s)),...,u(w_i(s)),...,u(w_N(s)))ds]$$

for $t\in[t_0,t_1]$, and $v(t)=0$ for $t\in[-h,0]$;

$$y(t)=p(t,x,u)+q(t,x,u_{t_0})+\int_{t_0}^{t}Z(t,s,x,u)v(s)ds$$
$$+\int_{t_0}^{t}X(t,s,x,u)F(s,x(s),u(w_0(s)),u(w_1(s)),...,u(w_i(s)),...,u(w_N(s)))ds$$

for $t\in[t_0,t_1]$, and $y(t)=\phi(t)$ for $t\in[-h,0]$.

Let

$$a_1=\sup\{|Z(t_1,t,x,u)|:(x,u)\in X,t\in[t_0,t_1]\},$$
$$a_2=\sup\{|W^{-1}(t_0,t_1,x,u)|:(x,u)\in X,t\in[t_0,t_1]\}$$
$$a_3=\sup\{|p(t_1,x,u)|+|q(t_1,x,u_{t_0})|+|x_1|:(x,u)\in X,t\in[t_0,t_1]\},$$
$$a_4=\sup\{|X(t,s,x,u)|:u\in C_m[t_0,t_1]\},$$
$$a_5=\sup\{|F(s,x(s),u(w_0(s)),u(w_1(s)),...,u(w_i(s)),...,u(w_N(s)))|:(x,u)\in X,s\in[t_0,t_1]\},$$

$b=\max\{(t_1-t_0)a_1,1\}$, $\quad c_1=(2N+4)ba_1a_2a_4(t_1-t_0)$, $\quad c_2=(2N+4)a_4(t_1-t_0)$,

$d_1=(2N+4)a_1a_2a_3b$, $\quad d_2=(2N+4)a_3$, $c=\max\{c_1,c_2\}$, $\quad d=\max\{d_1,d_2\}$.

Then,

$$|v(t)|\le a_1a_2[a_3+a_4a_5(t_1-t_0)]\ =d_1[(2N+4)b]^{-1}+c_1a_5[(2N+4)b]^{-1}$$

$$\leq (d + ca_5)[(2N+4)b]^{-1}$$

and

$$|y(t)| \leq a_3 + (t_1 - t_0)a_1\|v\| + (t_1 - t_0)a_4a_5 = b\|v\| + d_2(2N+4)^{-1} + c_2a_5(2N+4)^{-1}$$
$$\leq b\|v\| + (d + ca_5)(2N+4)^{-1}.$$

It follows from the growth condition (3.1) that for each pair of positive constants c and d, there exists a positive constant r such that, if $|p| \leq r$, then

$$d + c|F(t,p)| \leq r, \quad \text{for all } t \in [t_0, t_1]. \tag{3.2}$$

Now, take c and d as given above, and let r be chosen so that (3.2) is satisfied.
Therefore, if we take

$$\|x\| \leq r(N+2)^{-1}, \quad \|u\| \leq r(N+2)^{-1},$$

and moreover,

$$\|u_{t_0}\| \leq r(N+2)^{-1}, \text{ then } |p| = |x(s)| + \sum_{i=0}^{N} |u(w_i(s))| \leq r \quad \text{for all } s \in [t_0, t_1].$$

It follows that $d + ca_5 \leq r$. Therefore, $|v(t)| \leq r[(2N+4)b]^{-1}$ for all $t \in [t_0, t_1]$ and hence $\|v\| \leq r[(2N+4)b]^{-1}$. It follows that $|y(t)| \leq r(2N+4)^{-1} + r(2N+4)^{-1} = r(N+2)^{-1}$ for all $t \in [t_0, t_1]$ and hence $\|y\| \leq r(N+2)^{-1}$. Thus we have proved that if

$$H = \{(x,u) \in Q : \|x\| \leq r(N+2)^{-1} \text{ and } \|u\| \leq r(N+2)^{-1}\},$$

then T maps H into itself. Since all the functions involved in the definition of the operator T are continuous, it follows that T is continuous and hence it is completely continuous by the application of Arzela – Ascoli theorem. Since the set H is closed, bounded, and convex, then by Schauder's fixed point theorem there exists at least one fixed point $(x,u) \in H$ such that $T(x,u) = (x,u)$. It follows that, for $(x,u) \equiv (y,v)$, we have

$$x(t) = p(t,x,u) + q(t,x,u_{t_0}) + \int_{t_0}^{t} Z(t,s,x,u)u(s)ds$$
$$+\int_{t_0}^{t} X(t,s,x,u)F(s,x(s),u(w_0(s)),u(w_1(s)),\ldots,u(w_i(s)),\ldots,u(w_N(s)))ds.$$

Thus $x(t)$ is a solution of the system (2.1) and

$$x(t_1) = p(t,x,u) + q(t,x,u_{t_0}) + \int_{t_0}^{t_1} Z(t_1,s,x,u)Z^T(t_1,z,x,u)W^{-1}(t_0,t_1,x,u)[x_1 - p(t_1,x,u)$$
$$-q(t_1,x,u_{t_0}) - \int_{t_0}^{t_1} X(t_1,s,x,u)F(s,x(s),u(w_0(s)),u(w_1(s)),\ldots,u(w_i(s)),\ldots,u(w_N(s)))ds]ds$$
$$+\int_{t_0}^{t_1} X(t_1,s,x,u)F(s,x(s),u(w_0(s)),u(w_1(s)),\ldots,u(w_i(s)),\ldots,u(w_N(s)))ds = x_1.$$

Hence, (2.1) is relatively controllable on $[t_0, t_1]$.

4.0 Conclusion

Using Schauder's fixed point theorem, sufficient conditions for the relative controllability on the time interval $[t_0, t_1]$, of a certain special class of nonlinear perturbed neutral systems with both distributed and lumped multiple time varying delays in the control have been derived. A similar method may be applied to derive sufficient conditions for the so called absolute or functional controllability of the systems considered.

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