# Control of linear systems using pure time-delay 

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#### Abstract

A synthetic technique is derived for control systems governed by linear differential-difference equations. It is shown that such a system is equivalent to an infinite-dimensional difference equation whose matrix elements can be readily calculated by recursive formulas. To accomplish this, we introduce sampling, and replace the differential-difference equation by an infinite-dimensional system of difference equation. The infinite-dimensional system corresponds exactly to the finite dimensional one. The computation of the necessary transition matrices is done by matrix iterations similar to those used to compute the transition matrices of ordinary linear systems. For results, two practical examples are illustrated. From this it takes but a slight extension of present day procedures to calculate a stable system.


Keywords: Linear differential difference equations, infinite-dimensional system, transition matrices and ordinary linear systems

### 1.0 Introduction

In many industrial processes where transportation lags are common, the time behaviour of the system can be adequately derived by linear differential-difference equations. That is, the system is described by $\underline{\ell}(t)=\sum_{i=1}^{n} A_{1} \underline{x}\left(t-T_{i}\right)+\sum_{j=1}^{m} D_{j} \underline{u}\left(t-T_{j}\right)$, where $\underline{x}$ and $\underline{u}$ are the state and input vectors, and $T_{i}$ and $T_{j}$ are some fixed delay times. Techniques now classical in the control field, such as the Laplace transform [1] or the direct method of Lyapunov [2,3], can be used in the analysis of the equation. It is worth mentioning that the utilization of these techniques require extensive computation. This is natural since the state has infinite dimension. It is desirable to do these computations on a digital computer. However, when a computer issued, the design technique should be one which is intended for a computer and unfortunately the classical techniques do not have this desirable property.

Thus, the purpose of this paper is to present a synthesis technique suitable for digital computation of engineering problems. To achieve this we introduce sampling, and replace the differential-difference equation by an infinite-dimensional system of difference equation shown by Conte [4] and Pipes [5].

In this paper we will have occasion to refer to "ordinary" linear difference equations and to a "linear control law" of such equations. By an "ordinary" difference equation, we mean finite dimensional equations such as $\underline{x}(k+1)=\Phi \underline{x}(k)+\Delta \underline{u}(k)$ where $\underline{x}$ and $\underline{u}$ are finite-dimensional state and input vectors, and $\Phi$ and $\Delta$ are constant matrices. By a "linear control law", we mean that $\underline{u}(k)$ is a linear vector functional of the state, that is, $\underline{u}(k)=c \underline{x}(k)$.
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In general, a linear control law is determined from some performance index J . That is, $\underline{u}(k)$ is chosen to minimize some functional $J$ of the state and control vectors along the motion of the system. The theory of such control laws has been expensively studied, and we will assume that the various methods of synthesis are known to the reader [6].

### 2.0 Input-delay problem

Let's consider a simple input-delay problem whose system is governed by

$$
\underline{x}(t)=A \underline{x}(t)+D_{2} \underline{u}(t-T)
$$

If this system is sampled every $\tau$ seconds, with $T / \tau=N$, an integer, and if the input is applied through an appropriate sample-and-hold element, then

$$
\begin{equation*}
\underline{x}\left(t_{k}+\tau\right)=\Phi(\tau) x\left(t_{k}\right)+\Delta(\tau) \underline{U}\left(t_{k}-N \tau\right) \tag{2.1}
\end{equation*}
$$

where $\Phi(\tau)=\exp (A \tau), \Delta(\tau)=\left[\int_{0}^{\tau} \exp (A s) d s\right] D_{2}$.
We wish to present at this stage two methods which can be used to synthesize a control for the above system. In the first approach, we replace (2.1) by $\underline{x}\left(t_{k}+\tau\right)=\Phi(\tau) \underline{x}\left(t_{k}\right)+\Delta(\tau) \underline{V}\left(t_{k}\right)$. Thus, we have defined $\underline{V}\left(t_{k}\right) \equiv \underline{U}\left(t_{k}-N \tau\right)$

The above equation is an ordinary linear difference equation for which a linear control law for $v(t)$ can be found. Assuming that this control law has been found [7], the $\underline{v}\left(t_{k}\right)=c \underline{x}\left(t_{k}\right)$. To arrive at $\underline{u}\left(t_{k}\right)$, we have
$\underline{u}\left(t_{k}\right) \equiv \underline{V}\left(t_{k}+N T\right)=c \underline{x}\left(t_{k}+N \tau\right)=c\left[\Phi^{N}(\tau) \underline{x}\left(t_{k}\right)+\sum_{i=1}^{N} \Phi^{i-1} \Delta(\tau) u\left(t_{k}-\tau\right)\right]$
Thus, we observe that the control law is a function of $\underline{x}\left(t_{k}\right)$ and the N past inputs.
The next alternative approach (which allows for more flexibility is the performance index at the expense of additional complexity), is to enlarge the state space when we define

$$
\underset{-}{z}\left(t_{k}\right)=\left[\begin{array}{c}
x\left(t_{k}\right) \\
u\left(t_{k}-y\right) \\
\mathrm{M} \\
\underline{u}\left(t_{k}-N y\right)
\end{array}\right]
$$

$$
\underset{-}{z}\left(t_{k}+y\right)=\left[\begin{array}{llllll}
\Phi & 0 & 0 & \Lambda & 0 & \Delta \\
0 & 0 & 0 & \Lambda & 0 & 0 \\
0 & 1 & 0 & \Lambda & 0 & 0 \\
\mathrm{M} & \mathrm{M} & \mathrm{M} & \Lambda & \mathrm{M} & \mathrm{M} \\
0 & 0 & 0 & \Lambda & 1 & 0
\end{array}\right] z\left(t_{k}\right)+\left[\begin{array}{c}
0 \\
1 \\
0 \\
M \\
0
\end{array}\right] \underset{-}{u}\left(t_{k}\right)=\Phi^{*} \underset{-}{z}\left(t_{k}\right)+\Delta^{*} \underset{-}{u}\left(t_{k}\right)
$$

where $\Phi^{*}$ and $\Delta^{*}$ are defined by the above equation. Again, we have an ordinary linear-difference equation for which a linear control law can be found by standard techniques. Without difficulty, it is
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possible to apply either of the two procedures outlined here to the case involving multi-delays. Since the procedure is straight forward, we will dispense with further discussion and turn to the state-delay problem

### 3.0 General difference equation

Here we assume that the system is governed by the following differential-difference equation for which we desire a sampled version

$$
\begin{equation*}
\underline{x}(t)=A \underline{x}(t)+B \underline{x}(t-T)+D \underline{u}(t)+D_{2} \underline{u}(t-T) \tag{3.1}
\end{equation*}
$$

where
$\underline{x}(t)=(n \times n)$ vector; referred to hereafter as the state vector
$\underline{u}(t)=(r \times l)$ input vector, assumed constant between samples, i.e., $u(t)=u\left(t_{k}\right)$ for $t_{k} \leq t \leq t_{k+1}$
$A, B=(n \times n)$ constant matrices;
$D_{1} D_{2}=(n \times r)$ constant matrices.
By direct integration, the solution to the homeogeneous part of the above equation is

$$
\underline{x}(t)=\exp (A t)\left[\underline{x}(0)+\int_{0}^{t} \exp (-A s) B \underline{x}(s-T) d s\right]
$$

Now, if it is assumed that $\underline{x}(t), t \leq 0$ is itself a solution, then by integrating the above equation we have

$$
\begin{aligned}
\underline{x}(t)= & \exp (A t)\left[\underline{x}(0)+\left(\int_{0}^{t} \exp (-A s) B \exp (A s) d s\right) \underline{x}(-T)\right] \\
& +\int_{0}^{t} \exp (-A s) B \exp (A s) \int_{0}^{s} \exp (A \lambda) B \underline{x}(\lambda-2 T) d \lambda d s
\end{aligned}
$$

Applying successive iteration, we obtain the infinite-dimensional difference equation

$$
\begin{equation*}
\underline{x}(t)=\sum_{i-o}^{\infty} \Phi_{i}(t) \underline{x}(-i T) \tag{3.2}
\end{equation*}
$$

where $\Phi_{0}(t)=\exp (A t), \Phi_{i+1}=\exp (A t) \int_{0}^{t} \exp (-A s) B \Phi_{i}(s) d s$. From this result, we see that $\Phi_{i}$ can be identified with the terms of the transition matrix A $+B$. In essence,

$$
\left.\lim _{T \rightarrow 0} \sum_{i=0}^{\infty} \Phi_{i}(t) \underline{x}(-i T)=\exp (A+B) t\right) \underline{x}(0)
$$

Because of this identity, we are assured of the existence of the $\Phi_{i}(t)$ for all $i$ and $t$. Also, we realize that the sequence $\left\|\Phi_{i}(\tau)\right\|, i=0,1 \ldots$ for fixed $t-\tau$ is majorized by

$$
e^{\|A\| T}\left[1+(\lambda \tau)+\frac{(\lambda \tau)^{2}}{2!}+\ldots+\frac{(\lambda \tau)^{n}}{n!}+\ldots\right]
$$

where $h$ is a fixed constant greater than zero. Therefore, $\left\|\Phi_{i+1}{ }^{(\tau)}\right\| \leq \frac{(\lambda \tau)}{i+1}\left\|\Phi_{i}(\tau)\right\|$.
Letting $n$ be the first $i$ for which $i+1>\lambda \tau,\left\|\Phi_{i+1}(\tau)\right\| \leq r\left\|\Phi_{i}(\tau)\right\|$ for $i>n ; r<1$.As a result, there exist constants c and $r<1$ such that $\left\|\Phi_{i+1}(\tau)\right\| \leq c r^{i}$. This result will be used later.

Finally, $\Phi_{i}(\tau)$ can be calculated by an infinite series. This follows from the fact that $\Phi_{1}(\tau)$ must be the sum of those terms of $((A+B) \tau)$ involving the $i t h$ powers of $B$. From this and examination of the terms of $((A+B) \tau)$, we have

$$
\Phi_{i}(\tau)=\sum_{k=0}^{\infty} c_{i, k}
$$

where $c_{i, k+1}=\frac{(A \tau) c_{i, k}+(B \tau) c_{i-1, k}}{k+1}, \mathrm{c}_{0,0}=\mathrm{I}, \mathrm{c}_{\mathrm{i}, 0}=0, i>0, \mathrm{c}_{1, \mathrm{k}}=0, k \geq 0$.

Thus, it is possible to compute the $\Phi_{i}(\tau)$ on a digital computer by an infinite series expansion in the same way as $\exp (\mathrm{A} \tau)$. Following the same line of reasoning, when the control is added and considered constant during a sampling interval, we arrive at the complete infinite-dimensional difference equation:

$$
\begin{equation*}
\underline{x}(t+\tau)=\sum_{i=0}^{\infty} \Phi_{i}(\tau) \underline{x}\left(t_{k}-i N \tau\right)+\Delta_{i}(\tau) \underline{u}\left(t_{k}-i N \psi\right) ; N=\tau / \tau \tag{3.3}
\end{equation*}
$$

$\Delta_{i}(\tau)=G_{1, j}+G_{2, i-1 ;} G_{2,-1}=0, G_{j, i}=\sum_{k=1}^{\infty} c_{i, k} \tau / k+1 D_{j}$. The $C_{i, k^{\prime} s}$ are define in the same previous iteration for $\Phi_{i}(\tau)$

### 4.0 Stability

In this section we wish to define stability and derive a theorem necessary for our purposes. For a more thorough analysis in the continuous cases, the reader is referred to a paper by Driver [10] and the works of Ogata [2] and Dorf and Bishop [8].

We shall consider a liner space whose elements, e, are infinite sequences of vectors ( $\left[x_{1} x_{2}\right]$. For notational convenience, we shall agree that if

$$
e_{k}=[x(k), \underline{x}(k-1), \ldots] \text { then } e_{k+1}=\lfloor x(k+1), \underline{x}(k), x(k-1), \ldots]
$$

## Definition 4.1

The null solution, $\mathrm{e}=[\underline{0}, \underline{0}, \underline{0} \ldots]=0$, of the system $e_{k+1}=f\left(e_{k}, k\right) ; f(0, k)=0$ will be said to be stable if given any number $\varepsilon>0$ there corresponds a $\lambda(\varepsilon, \mathrm{k})>0$ and a norm such that if $\left\|e_{k}\right\| \leq \lambda$ then $\left\|e_{k+1}\right\| \leq \varepsilon$ for $i>k$. If $\lambda$ is not a function of $k$, the solution will be said to be uniformly stable.

## Definition 4.2

The null solution of $e_{k+1}=f\left(e_{k, k}\right) ; f(0, k)=0$ will be said to be uniformly asymptotically stable if it is uniformly stable and if given any $\mathrm{M}>0$ there correspond a $\mathrm{T}(\mathrm{M})$ and a norm such that $\left\|e_{k+p}\right\| \leq M$ for all $\mathrm{P}>\mathrm{T}$ whenever $\left\|e_{k}\right\| \leq r$, and $r$ not depending on M or $e_{\mathrm{k}}$.

## Lemma 4.3:

$$
\text { Given the system defined by } \underline{x}(k+1)=\sum_{i=0}^{\infty} \Phi_{i} \underline{x}(k-i) \text {, where } \sum_{i=0}^{\infty}\left\|\Phi_{i}\right\| \leq 1 . \text { Then, the system }
$$ is uniformly stable.

Proof:

Define $\left\|e_{k}\right\|=\operatorname{Max}_{i}\|x(k-i)\|, i=0,1, \ldots . \quad$ Let $\lambda=\varepsilon \rightarrow\left\|e_{k}\right\| \leq \varepsilon$ or $\|x(k-i)\| \leq \varepsilon ; i=0$, $1, \ldots$. From the hypothesis of the lemma, $\|\underline{x}(k+1)\| \leq \sum_{i=0}^{\infty}\left\|\Phi_{i}\right\| \underline{x}(k-i)\left\|\leq \sum_{i=0}^{\infty}\right\| \Phi_{i} \| \mathcal{E} \leq \mathcal{E}$. By induction, the above holds for all $\underline{x}(k+j) ; j>1$, and we obtain $\left\|e_{k+j}\right\| \leq \mathcal{E}$. Hence the lemma is proved.

## Theorem 4.4

Given the system governed by $\underline{x}(k+1)=\sum_{i=0}^{\infty} \Phi_{i} \underline{x}(k)$, where $\sum_{i=0}^{\infty}\left\|\Phi_{i}\right\|<1$ and there exist constants $c$ and $r<1$ such that $\left\|\Phi_{i}\right\| \leq c r^{i}$. Then, the system is uniformly asymptotically stable.

## Proof:

By Lemma 1 the system is uniformly stable. Now we define $\left\|e_{k}\right\|=\max _{i} \frac{\|\underline{x}(k-i)\|}{i^{2}} ; i=0,1$,
$1, \ldots$ It is clear that the theorem is true if we can show that $\|\underline{x}(k)\| ; k=1,2, \ldots$ progressively decreases. In what follows we will establish this. First, we show that if there exists a $\beta>1$ such that $\sum_{i=0}^{\infty}\left\|\Phi_{i}\right\| \beta^{i+1} \leq 1$ then $\|\underline{x}(k-1)\|_{i}$ progressively decreases. Now, we assume at some fixed point $k$ $\|\underline{x}(k-i)\| \leq r ; i=0,1,2 \ldots$, then surely $\|\underline{x}(k-i)\| \leq r \beta^{i} ; i=0,1,2 \ldots$ and $\left\|\underline{x}(k+1) \leq \sum_{i=0}^{\infty}\right\| \Phi_{i}\| \|$ $\underline{x}(k-i)\left\|\leq \sum\right\| \Phi_{i}\left\|r \beta^{i} \leq 1 / \beta \sum_{i=0}^{\infty}\right\| \Phi_{i} \| \beta^{i+1} r \leq r \beta^{-1} . \quad$ By induction, $\|\underline{x}(k+T)\| \leq r \beta^{-T}$ and the result desired is established.

Now, by the conditions of the theorem $\left\|\Phi_{i}\right\| \leq c r^{i} ; r<1$. From the continuous function $f(t) f(t) \leq t c \sum_{i=0}^{\infty}\left\|(r t)^{i}\right\|<\infty$ for $t<1 / 2$ since $f(1) \leq a \leq 1$ and $r<1$, by continuity then exist some $\mathrm{t}=\mathrm{t}^{*}, 1<\mathrm{t}^{*}<\mathrm{r}^{-1}$, such that $f\left(t^{*}\right) \leq 1$. Thus, $\|x(k)\|$ progressively decreases and the theorem is proved.

### 5.0 Synthesis

A basic synthesis development is given in this section. We shall assume that the system is governed by an infinite dimensional difference equation which has been derived from a differentialdifference equation. This assumption is made to insure that the norms of the matrices decrease in a suitable manner.

The general approach will be to terminate the infinite dimensional series after its first term, reducing the equation to an ordinary linear difference equation. Reduction equation we shall develop a linear control law which will facilitate a stability analysis of complete equation by means of the theorem discusses above.

Now, starting with the first terms on the right in (3.3), we obtain
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$$
\begin{equation*}
\underline{x}\left(t_{k}+\tau\right)=\Phi_{0} \underline{x}\left(t_{k}\right)+\Delta_{0} \underline{u}\left(t_{k}\right) \tag{5.1}
\end{equation*}
$$

Assuming a linear control law for this reduced equation has been obtained, then $\underline{u}\left(t_{k}\right)=c \underline{x}\left(t_{k}\right)$. If this is substituted into (3.3), we have $\quad \underline{x}\left(t_{k}+\tau\right)=\sum_{i=0}^{\infty} v_{i} \underline{x}\left(t_{k}-\tau\right)$
where $v_{i}=\Phi_{i}+\Delta_{i} c$. Now, we apply the theorem to the matrices $v_{i}$; that is, find a norm such that $\sum_{i=0}^{\infty}\left\|v_{i}\right\|<1$. Certainly, it would be fortunate if such a norm could be found for any control law. The choice of the control law should be made with this problem in mind. To achieve the desire result, let us for now assume that the performance index that we wish to minimize for the reduced system is

$$
\begin{equation*}
J\left(\underline{x}\left(t_{0}\right)=\sum_{i=0}^{\infty} \beta^{k} \underline{x}\left(t_{k}\right) \underline{x}\left(t_{k}\right) ; \beta>1\right. \tag{5.3}
\end{equation*}
$$

It has been shown in the literature [9] that the linear control law which minimize $J$ can be calculated from the following iteration: $\left\|\Phi_{i}\right\| \leq C r^{r}, r<1$
(Iteration A)
$P_{N+1}=\beta S_{N} P_{N} S_{N}+\mathrm{I}, \quad S_{N}=\Phi_{0}+\Delta_{0} c_{N}, \quad C_{N}=-\left[\Delta_{0} P_{N} \Delta_{0} J^{-1} \Delta_{0} P_{N} \Phi_{0}\right], \quad P_{0}=\mathrm{I}$, $\underline{u}\left(t_{k}\right)=c_{\infty} \underline{x}\left(t_{k}\right)$ is the desired control law.

It is also known that the induced matrix norm to a vector norm defined as $\|\underline{x}\|=(\underline{x} \mathrm{P} \underline{x})^{1 / 2} ; P>0$ is the Euclidian matrix norm of $T A T^{-1}$ where $T T=P$. By examining the equation for $P_{\infty}$ above, we observe that $T^{-1} P_{\infty} T^{-1}=\mathrm{I}=\beta\left(T^{-1} S T\right)\left(T S_{\infty} T^{-1}\right)+T^{-1} T^{-1}$ where $T T=P_{\infty}$. Thus, $T^{-1} T^{-1}=\mathrm{I}-\beta\left(T^{-1} S_{\infty} T\right)=$ positive definite matrix. However, roots $[\mathrm{I}-\beta A]=1-\beta$ roots (A). Therefore, the induced norm $S_{\infty}$ is less than $1 / \sqrt{\beta}$. However, $S_{\infty}$ is $v_{o}$ in (5.1). Thus, by choosing $\beta$ large enough, we can always find a control law such that the norm of $\mathrm{v}_{\mathrm{o}}$ is arbitrarily small. This is illustrated in Example 6.2 below, where we can expand the dimensionality of the state such that the norms of the remaining matrices in (5.1) become small. Thus, it is possible to achieve a stable control system in a straight forward manner.

### 6.0 Examples and results

Example 6.1: Input Delay
Assume that we have the system

$$
\underline{x}(f t)=\left[\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right] \underline{x}(t)+\left[\begin{array}{l}
0 \\
1
\end{array}\right] u(t-T) \text { where } \tau=\pi / 2
$$

The sampled equation with $u$ constant during a sampling interval, is as follows:

$$
x\left(1_{k}+\pi / 2\right)=\left[\begin{array}{ll}
0 & 1 \\
-1 & 0
\end{array}\right] \underline{x}\left(t_{k}\right)+\left[\begin{array}{l}
1 \\
1
\end{array}\right] u\left(t_{k}-\pi / 2\right)
$$

If it is desired to achieve a deadbeat performance, that is, reduce $\underline{x}\left(t_{0}\right) \rightarrow \underline{0}$ in minimum time, we proceed as follows:

$$
v\left(t_{k}\right)=\left[\begin{array}{ll}
-1 & 0
\end{array}\right]\left[\Phi^{-1} \underline{\Delta} \Phi^{-2} \underline{\Delta}\right]^{-1} \underline{x}\left(t_{k}\right)=\left[1 / 2-1 / 2 \underline{x}\left(t_{k}\right)\right.
$$

Thus,

$$
u\left(t_{k}\right)=v\left(t_{k}+\pi / 2\right)=\left[\frac{1}{2} \frac{1}{2}\right] \underline{x}\left(t_{k}\right)\left[\left[\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right] \underline{x}\left(t_{k}\right)+\left[\begin{array}{l}
1 \\
1
\end{array}\right] u\left(t_{k}-\pi / 2\right)\right]=[1 / 2-1 / 2] \underline{x}\left(t_{k}\right)
$$

The system will be returned to the origin in $t=3 \tau$.

## Example 6.2: State-Delay

Consider a system governed by

$$
\underline{x}(t)=\left[\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right] \underline{x}(t)+\left[\begin{array}{cc}
2 & 0 \\
0 & -1
\end{array}\right] \underline{x}(t-T)+\left[\begin{array}{l}
0 \\
1
\end{array}\right] u(t-T)
$$

assuming $T=\tau=\frac{\pi}{4}$. The first six $\Phi_{i} s$ and $\Delta_{i}^{1} s$ calculated by their respective iteration are tabulated
in Table 7.1. (All the others being zero to eight places). We expand the dimensionality of $x$ by defining

$$
\underline{z}\left(t_{k}\right)\left[\begin{array}{l}
\underline{x}\left(t_{k}\right) \\
\underline{x}\left(t_{k}-T\right) \\
u\left(t_{k}-T\right)
\end{array}\right]
$$

Thus,

$$
\begin{gathered}
\underline{z}\left(t_{k}+\tau\right)=\left[\begin{array}{lll}
\Phi_{0} & \Phi_{1} & \Delta_{1} \\
1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right] \underline{z}\left(t_{k}\right)+\left[\begin{array}{l}
\Delta_{0} \\
0 \\
1
\end{array}\right] u\left(t_{k}\right) \\
{\left[\begin{array}{lll}
\Phi_{2} & \Phi_{3} & \Delta_{3} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right] \underline{z}\left(t_{k}-2 \tau\right)+\left[\begin{array}{l}
\Delta_{2} \\
0 \\
0
\end{array}\right] u\left(t_{k}-2 \tau\right)+\ldots} \\
\equiv
\end{gathered}
$$

Using $\beta=2$ in the proposed performance index:
$P_{\infty}$ Matrix
$\left[\begin{array}{ccccc}17.4841 & 13.0768 & 2.3796 & 0.5215 & 4.9132 \\ 13.0768 & 41.4109 & 3.8072 & -0.6420 & 23.4899 \\ 2.3796 & 3.8072 & 1.4952 & .0169 & 1.9932 \\ .5215 & -0.6420 & .0169 & 1.0656 & -.6206 \\ 4.9132 & 23.4899 & 1.9952 & -.6206 & 16.3186\end{array}\right]$
$C_{\infty}$ Vector

$$
\left[\begin{array}{lllll}
.6827 & -1.1927 & -.0003 & .1043 & -1.1060
\end{array}\right]
$$

The norms of the first 5 composite matrices (5.1) are (where the vector norm is $\sqrt{\underline{x}^{1} P \underline{x}}$ ) :

$$
\begin{array}{lll}
.7012 & 244 & <1 / \sqrt{2} \\
0706 & 351 & \\
0000 & 417 & \\
0000 & 000 & \\
0000 & 000 &
\end{array}
$$

Table 7.1

$$
\begin{array}{ll}
\Phi_{0}=\left[\begin{array}{llll}
.7071 & 068 & .7071 & 068 \\
.7071 & 068 & .7071 & 068
\end{array}\right] \quad \Delta_{0}=\left[\begin{array}{ll}
.0000 & 000 \\
.0000 & 000
\end{array}\right] \\
\Phi_{1}=\left[\begin{array}{llll}
.1338 & 340 & .0279 & 680 \\
. .0277 & 680 & -.0782 & 980
\end{array}\right] & \Delta_{1}=\left[\begin{array}{ll}
.2928 & 932 \\
.7071 & 068
\end{array}\right] \\
\Phi_{2}=\left[\begin{array}{llll}
.0109 & 582 & .0022 & 524 \\
-.0022 & 542 & .0026 & 278
\end{array}\right] \quad \Delta_{2}=\left[\begin{array}{ll}
.0075 & 873 \\
-.0308 & 106
\end{array}\right] \\
\Phi_{3}=\left[\begin{array}{llll}
.0003 & 903 & .0000 & 742 \\
-.0000 & 742 & -.0000 & 854
\end{array}\right] \quad \Delta_{3}=\left[\begin{array}{ll}
.0004 & 532 \\
.0007 & 349
\end{array}\right] \\
\Phi_{4}=\left[\begin{array}{cccc}
.0000 & 236 & .0000 & 026 \\
-.0000 & 026 & -.0000 & 013
\end{array}\right] \quad \Delta_{4}=\left[\begin{array}{cc}
.0000 & 119 \\
-.0000 & 163
\end{array}\right] \\
\Phi_{5}=\left[\begin{array}{cccc}
.0000 & 008 & .0000 & 001 \\
-.0000 & 001 & -.0000 & 000
\end{array}\right] \quad \Delta_{5}=\left[\begin{array}{ll}
.0000 & 003 \\
.0000 & 002
\end{array}\right]
\end{array}
$$

### 7.0 Conclusion

A synthesis technique has been derived for control systems governed by linear differentialdifference equations. This technique is particularly suited for a digital computer because the procedure uses matrix iterations exclusively.

For a given control system, stability is usually the most important thing to be considered. If the system is linear and time invariant, criteria are available to include the reguiest stability and Routh's stability. In this study our attention was focus on stability in the sense of Lypunoy. Finally, if the upper limit of integration in the performance index $J$ given in (7) is finite, then it can be shown that the optimal control vector is still a linear function of the state variables, but with time-varying coefficients. Therefore, the determination of the control vector involves that of optimal time-varying matrices.

## References

[1] Bellman, R. and Cooke, K., Differential-Difference Equations, Academic Press, 1963.
[2] Ogata, Katsuhiko; Modern Control Engineering, Prentice-Hall of India, New Delhi, 3rd edition, pp. 896-926, 2000.
[3] Ragamikhin, B. S., "Application of Lyapunous Method to Problems in the stability of systems with a Delay", Automatic and Remote Control, 21, pp. 515-520.
[4] Conte, S. D., Elementary Numerical Analysis-An Algorithmic Approach, McGraw-Hill Book Co. 1965.
[5] Hovanessian, Shahen A. and Pipes, Lovis A., Digital Computer Methods in Engineering, McGraw-Hill Book Со., 1969.
[6] Kalman, R. E. and Koepche, R. W; "Optimal Synthesis of Linear Sampling Control Systems using Generalized Performance Indexes", ASME Paper No. 58 - IRD-6, 1979.
[7] Gantmacher, F. R., The Theory of Matrices, Vol. II, Chelsea Publishing Companying 1974.
[8] Dorf Richard C. and Bishop Robert H; Modern Control Systems, Addison-Wesley Longman, Inc. 1998, 8th edition, pp. 284-325.
[9] Pericles Emmanuel and Edward Leff, Introduction to Feedback Control Systems, McGraw-Hill Kogakusha Ltd., 1976, pp. 62-77.
[10] Driver, R. D. Existence and stability of solutions of Delay. Differential system", Arch. Rational Mech.Anal,Vol. 10,pp 401 - 26.

