Sylvester's series and Egyptian fractions

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Abstract

In number theory, Sylvester's sequence is a sequence of integers in which each member of the sequence is the product of the previous members, plus one. The first few terms of the sequence are 2, 3, 7, 43, 1807 ..., its values grow doubly exponentially, and the sum of its reciprocals (Sylvester's Series) forms a series of unit fractions that converges to 1 i.e. $\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + ...$ to 1. We examine the connection between Sylvester's Series and Egyptian fractions [1], as both of them can be used in number theory to represent any $x \in (0,1)$, in the form $x = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + ...$ where a < b < c... Besides the difficulty in generating a, b, c... using Sylvester's Series is overcome by developing Algorithms for Egyptian fractions [1].

1.0 Introduction

Sylvester's Sequence is named after James Joseph Sylvester, who first investigated it in 1880. Definition 1: Formally, Sylvester's Sequence can be defined by the formula

$$S_n = 1 + \prod_{i=0}^{n-1} S_i \tag{1.1}$$

The product of an empty set is 1,, so $S_0 = 2$. Also S_1 and S_2 will be

$$S_{1} = 1 + \prod_{i=0}^{n-1} S_{i} = 1 + \prod_{i=0}^{0} S_{0} = 1 + 2 = 3$$
$$S_{2} = 1 + \prod_{i=0}^{2-1} S_{i} = 1 + \prod_{i=0}^{1} S_{i} = 1 + S_{0} \times S_{1} = 1 + 2 \times 3 = 7$$

Alternatively, one may define the sequence by the recurrence formula

$$S_i = S_{i-1}(S_{i-1} - 1) + 1$$
 with $S_0 = 2$ (1.2)

where S_1 and S_2 will be

$$S_1 = S_0(S_0 - 1) + 1 = 2(2 - 1) + 1 = 3$$

$$S_2 = S_1(S_1 - 1) + 1 = 3(3 - 1) + 1 = 7$$

It is straight forward to show by mathematical induction that (1.1) and (1.2) are equivalent, as seeing in the examples above.

Journal of the Nigerian Association of Mathematical Physics Volume 12 (May, 2008), 33 - 36 Sylvester's series and Egyptian fractions Orobosa Izevbizua J of NAMP Beside taking the sum of the reciprocals of Sylvester's sequence Sylvester also developed a method for converting any fraction $\frac{P}{q}q \neq 0$ into its Egyptian form.

2.0 Sylvester's method

Sylvester's method provides a simple way to convert any proper fraction $\frac{P}{q}$ to a series of unit

fractions. *Example 2.1*

$$\frac{3}{8} = \frac{3}{9} + a \text{ bit}$$
$$\frac{3}{8} = \frac{1}{3} + \frac{3 \times 3 - 8 \times 1}{3 \times 8}$$
$$\frac{3}{8} = \frac{1}{3} + \frac{1}{24}$$

In the example above, the second and final unit fraction occurred on the first application of Sylvester's method. More often the method is applied recursively until a one appears in the final numerator

e.g.
$$\frac{7}{11} = \frac{7}{14} + abit$$
$$\frac{7}{11} = \frac{1}{2} + \frac{7 \times 2 - 11 \times 1}{11 \times 2}$$
$$\frac{7}{11} = \frac{1}{2} + \frac{3}{22}$$
Now
$$\frac{3}{22} = \frac{3}{24} + abit$$
$$\frac{3}{22} = \frac{1}{8} + \frac{3 \times 8 - 22 \times 1}{22 \times 8}$$
$$\frac{3}{22} = \frac{1}{8} + \frac{1}{88}$$
Therefore
$$\frac{7}{11} = \frac{1}{2} + \frac{1}{8} + \frac{1}{88}$$
we express Sylvester's method mathematically as
$$\frac{P}{q} = \frac{P}{q + x} + abit$$
, where

(i) x is define such that q + x is the first integer divisible by P, greater than q.

(ii) The bit is obtained by first expressing $\frac{P}{q+x}$ as a unit fraction and then cross multiplying to get

$$\frac{P \times (q+x) - q \times 1}{P(q+x)}$$

3.0 Connection with Egyptian fractions

Journal of the Nigerian Association of Mathematical Physics Volume 12 (May, 2008), 33 - 36 Sylvester's series and Egyptian fractions Orobosa Izevbizua J of NAMP The unit fractions formed by the reciprocals of the values in Sylvester's sequence generate an infinite series.

$$\sum_{i=0}^{\infty} \frac{1}{S_i} = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \dots$$
(3.1)

The partial sums of this series have a simple form,

$$\sum_{i=0}^{j-1} \frac{1}{S_i} = \frac{S_j - 2}{S_j - 1}$$
(3.2)

as may be proved by induction. Clearly this identity is true for j = 0, as both sides are zero. For larger j, expanding the left side of the identity in (3.2), using the induction hypothesis produces.

$$\sum_{i=0}^{j-1} \frac{1}{S_i} = \frac{1}{S_{j-1}} + \sum_{i=0}^{j-2} \frac{1}{S_i} = \frac{1}{S_{j-1}} + \frac{S_{j-1} - 2}{S_{j-1}} = \frac{S_{j-1}(S_{j-1} - 1) - 1}{S_{j-1}(S_{j-1} - 1)} = \frac{S_j - 2}{S_j - 1}$$

as in equation (3.2)

Since this sequence of partial sums $\frac{S_j - 2}{S_j - 1}$ converges to one, the Overall series form an infinite

Egyptian fraction representation of the number one:

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \dots \bigg\}$$
(3.3)

from (3.3 one can find finite Egyptian Fraction representations of one, of any length by truncating this series and subtracting one from the last denominator for example

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$$

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42}$$

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42} + \frac{1}{1807}$$

We observe that the sum of the first k terms of the infinite series in (3.3) provides the closest possible underestimate of 1 by any *k*-term Egyptian Fraction. For example, the first four terms of (3.3) add to $\frac{1805}{1806}$, and therefore any Egyptian Fraction for a number $\frac{P}{q}$ in the open interval $\left(\frac{1805}{1806}, 1\right)$ requires

of least five terms.

It is possible to interpret the Sylvester's Sequence as the result of a greedy algorithm [2] for Egyptian Fractions that at each step chooses the smallest possible denominator that makes the partial form of the series be less than one. Alternatively, the terms of the sequence after the first can be viewed as the

denominators of the sequence the denominators of the odd greedy Egyptian expansion of $\frac{1}{2}$ [3].

4.0 Uniqueness of quickly growing series with rational sums

As observed, Sylvester's Sequence seems to be unique in having such quickly growing values, of reciprocals that converged to a rational number. This is seeing clearly, if a sequence of integers grows quickly enough that

$$a \ge a_{n-1}^2 - a_{n-1} + 1 \tag{4.1}$$

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and if the series $A = \sum \frac{1}{a_i}$ converges to a rational number A, then, for all n after some point, this

sequence must be define by the same recurrence $a_{n-1}^2 - a_{n-1} + 1$ that can be used to define Sylvester's sequence

5.0 Conclusion

Not only is Sylvester's series a form of Egyptian fraction, it's indeed an improved form of it, converging faster than any other form of unit fractions of 1 in Egyptian form. The series also gives the best underestimate of 1 in Egyptian form.

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