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Iterative methods involving composed operators of the accretive type

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Abstract

In this paper, the equation

$$Lu = f,$$

where L=A + B, A a K-positive definite operator, B an L-positive definite operator, is solved in a Banach space. An iterative scheme which converges to the unique solution of this equations is also constructed. Finally, a composed equation involving other operators of the accretive type is also solved in a Banach space.

Keywords: K-positive definite, Iteration, weakly contractive, accretive.

1.0 Introduction

Let H_o be a dense subspace of a Hilbert space, H. AN operator T with domain $D(T) \supseteq H_o$ is said to be continuously H_o-invertible if the range of T, $R(T)$ with T considered as an operator restricted to H_o is dense in H and T has a bounded inverse on $R(T)$. Let H be a complex and separable Hilbert space and A be a linear unbounded operator defined on a dense domain $D(A)$ in H with the property that there exist a continuously $D(A)$-invertible closed linear operator K with $D(A) \subseteq D(K)$, and a constant $\alpha > 0$ such that

$$(Au, Ku) \geq \alpha \|Ku\|^2, \; u \in D(A) \tag{1.1}$$

then A is called K-positive definite (Kpd) (see e.g. [7]). If $K = 1$ (the identity operator on H) inequality 1 reduces to $< Au, u > \geq \alpha \|Ku\|^2$ and in this case A is called positive definite. Positive definite operators have been studied by various authors (see e.g. [2, 3, 4, 7]). It is clear that the class of K-pd operators contains among others, the class of positive definite operators and also contains the class of invertible operators (when K = A) as its subclass.

The class of K-positive definite operators was first studied by W.V. Petryshyn, who proved interalia, the following theorem, (see [7]).

***Theorem* P**

If A is K-pd operator and D(A) = D(K), then there exists a constant $\alpha > 0$ *such that for all* $u \in D(K)$

$$\|Au\| \leq \alpha \|Ku\|.$$

Furthermore, the operator A is closed, $R(A) = H$ and the equation $Au = f, f \in H$, has a unique solution. The author and C. E Chidume extended this result to different Banach spaces and obtained convergence results in different directions (see [3, 4]). We proved, among others, the following theorem.

***Theorem* CA (see [4])**

Suppose X is a real uniformly smooth Banach space. Suppose A is an asymptotically K-positive

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definite operator defined in a neighborhood $U(x_o)$ *of a real uniformly smooth Banach space, X. Define the sequence* $\{x_n\}$ *by* $x_0 \in U(x_0),\ x_{n+1} = x_n + r_n,\ n \geq 0,\ r_n = K^{-1}y - K^{-1}Ax_n, y \in R(A)$. *Then* x_n *converges strongly to the unique solution of* $Ax = y \in U(x_0)$.

The present paper looks at a composed operators of the K-pd and also of the K-pd and the weakly contractive map. It is shown that each of the composed equations has a unique solution. Furthermore, an iterative scheme that converges to the unique solution of the equation $Lu = f$, where $L = A + B$ is constructed.

2.0 Preliminaries

For a Banach space *X*we shall denote by J the duality mapping from *X* to 2^{x^*} given by

$$Jx = \{ f \in X^* :< x, f >= \|x\|^2 = \|f\|^2 \},$$

where X^* denotes the dual space of X and <,> denotes the generalized duality pairing. It is well known that if X* is strictly convex then *J* is single valued and if *X** is uniformly smooth (equivalently if *X** is uniformly convex) then *J* is uniformly continuous on bounded subsets of *X** (see e.g. [6]). We shall denote the single valued duality mapping by *j*. Thus, by a single-valued normalized duality mapping we shall mean a mapping $j: X \to X^*$ such that for each $x \in X$, $j(x)$ is an element of X^* which satisfies the following two conditions:

$$< x, j(x) >= \|j(x)\|\|x\|,\ \|j(x)\| = \|x\|.$$

***Lemma* 2.1** (see e.g. [6]

Let X be a real Banach space and let J be the normalized duality map on X. Then for any given $x, y \in X$ *, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2 < y, j(x+y) > \quad \forall j(x+y) \in J(x+y).$$

***Definition* 2.2**

Let *X*be a Banach space and let *A* be a linear unbounded operator defined on a dense domain. $D(A) \in X$. An operator *A* will be called K-positive definite (kpd) if there exist a continuously D(A)-invertible closed linear operator *K* with $D(A) \subseteq D(K)$, and a constant $c \geq 0$ such that for $j \in J(Ku)$,

$$< Au, j(Ku) > \geq c\|Ku\|^2,\ u \in D(A)..$$

***Definition* 2.3**

A mapping *T* with domain *D*(*T*) and range *R*(*T*) in *X* is called weakly contractive if there exists a continuous and nondecreasing function $\phi : [0, \infty] := R^+ \to R^+$ such that ϕ is positive on $R^+ - 0, \phi(0) = 0, \lim_{t \to 0} \phi(t) = \infty$ and for $x, y \in D(t)$ there exists $j(x-y) \in J(x-y)$ such that

$$\|Tx - Ty\| \leq \|x - y\| - \phi(\|x-y\|).$$

It is called d-weakly contractive if

$$|< Tx - Ty, j(x-y) >| \leq \|x-y\|^2 - \phi(\|x-y\|).$$

The weakly contractive and d-weakly contractive operators were first studied by Alber and Guerre-Delabriere [1].

3.0 Main results

We establish the following results: 1. Let Abe K-pd, i.e. there exists a continuously D(K) – invertible closed operator K and a constant α:

$$< Au, Ku >\geq \alpha\|Ku\|^2, u\in D(A)$$

Let B be L-pd, then there exist L and β such that $Bu, Lu > \geq \beta\|Lu\|^2$;

$$\begin{aligned}<(A+B)u,(K+L)u> &= < Au, Ku > + < Au, Lu > + < Bu, Ku > + < Bu, Lu > \\ &\geq \alpha\|Ku\|^2 + < Au, Lu > + < Bu, Ku > + \beta\|Lu\|^2 \\ &\geq \alpha\|Ku\|^2 + \beta\|Lu\|^2 \quad provide \quad A\perp L, B\perp K \\ &\geq \min(\alpha,\beta)\|(K+L)\|^2, \ if \quad K\perp L.\end{aligned}$$

Thus, $A+B$ is $(K + L)$- positive definite provided the following orthogonality conditions hold:

$$A\perp L, \ B\perp K \ and \ K\perp L.$$

***Theorem* 3.1**

Let X be a real separable strictly convex Banach space and let A be a K-pd operator, B and L-pd operator with domain $D(A)\cup D(B) = D(K)\cup D(L)$. *Then there is a constant* $\theta\geq 0$ *such that*

$$\|(A+B)u\| \leq \theta\|(K+L)u\|.$$

Furthermore, the equation

$$(A+B)u = f$$

has a unique solution.

Proof:

Let $P = A+B, Q = K+L.$ We take $D(P) = D(A)\Upsilon D(B) D(Q) = D(K)\Upsilon D(L)$

Then $$D(Q)\supseteq D(K)\supseteq D(L).$$

We introduce in $D(Q)$ a new inner product and norm defined respectively by:

$$[u,v]_0 =< Qu, Qv >=<(K+L)u,(K+L)v>; \ |u|_0 = \|Qu\|_0.$$

Clearly P is closed since addition is continuous.

Also P is invertible (same reason). $R(Q)\supseteq R(K)\Rightarrow R(Q)$ is dense in H. K is continuously D(A) invertible; L is continuously $D(B)$ invertible. This implies K+L is continuously invertible in $R(K)\cup R(L)$. The rest of the proof follows as in theorem P, to establish that $Lu = f$ has a unique solution. In the next result, we construct an iteration process which converges strongly to the unique solution of the equation $Lu = f$ in a Banach space. This result generalizes all others (see e.g. [3, 4, 7]).

***Theorem* 3.2**

Suppose X is a real Banach space and $A\,|\,D(A)\subseteq X\rightarrow X$ *is a K-pd operator;* $B\,|\,D(B)\subseteq X\rightarrow X$ *is an L-pd operator with*

$$D(A)\cup D(B) = D(K)\cup D(L) = R(K)\cup R(L).$$

Define the sequence u_n *by* $u_o\in D(A), u_{n+1} = u_n + r_n, n\geq 0.$ $r_n = Q^{-1}f - Q^{-1}Px_n$, *where* $P = A+B, Q = K+L; f\in R(K+L)$. *Then* u_n *converges strongly to the unique solution of* $pu = f, P = A+B.$

Proof:

Clearly $P = A+B$ and $Q = K+L$ are invertible.

$$Qr_n = f - Px_n$$

$$Qr_{n+1} = f - P\left(x_{n+1}\right) = Qr_n - \Pr_n$$

$$\left\|Qr_{n+1}\right\|^2 = \left\|Qr_n - \Pr_n\right\|^2 = \left\|Qr_n\right\|^2 - 2\ \pi\ \Pr_n\ j\left(Qr_n - \Pr_n\right)\phi \ \text{(by Lemma 2.1)}$$

$$\leq \left\|Qr_n\right\|^2 - 2\ \pi\ Qr_n,\ j\left(Qr_{n+1}\right)\phi$$

$$\leq \left\|Qr_{n+1}\right\|^2 - 2\gamma\left\|Qr_{n+1}\right\|^2.$$

Hence

$$(1+2\gamma)\left\|Qr_{n+1}\right\|^2 \leq \left\|Qr_n\right\|^2$$

$$\left\|Qr_{n+1}\right\|^2 \leq (1+2\gamma)^{-1}\left\|Qr_n\right\|^2.$$

It follows that the sequence $\{Qr_n\}$ is monotonically decreasing, and hence it converges to a real number δ > 0. Since Q has a bounded inverse, then $r_n \to 0$ and thus $Qx_n \to f$, i.e., $(A+B)x_n \to f$ or $x_n \to Q^{-1}f$, the unique solution of the equation $Qu = (A+B)u = f$.

In the next section we study some composed operators involving the K-pad and some accretive type operators such as the weakly contractive, (see [3, 4, 7] for detailed study of these class of operators and consequent convergent results proved. In particular, we show that a composed equation involving the weakly contractive and the K-pd has a unique solution.

***Theorem* 3.3**

Let *A* be a K-pd operator equation and *B* a weakly contractive operator. Then the equation

$$BAu = f,\ f \in X$$

has a unique solution.

Proof A satisfies: $$Au,\ j(Ku) >\geq \alpha\|Ku\|^2,$$

and B satisfies the following inequalities:

$$\|Bx - By\| \leq \|x-y\| - \phi(\|x-y\|),$$

where ϕ is as defined above.

$$\|BAx - BAy\| \leq \|Ax - Ay\| - \phi(\|Ax - Ay\|)$$

$$= \|Au\| - \phi(\|Au\|)$$

$$\leq \alpha\|Ku\| - \phi(\|Au\|)$$

$$\leq \alpha\|Ku\|$$

It follows that the convergence of $\{ku\}$ implies that of $\{BAu\}$. But *K* is continuously invertible. Hence as in [7] *BAu* = *f* has a unique solution.

References

[1] On the projection methods for fixed point problems, Analysis 21 (2001, 17-39).

[2] F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl. 20 (1967), 197-288.

[3] C. E. Chidume and S. J. Aneke, Existence, uniqueness and approximation of a solution for a K-positive definite operator equations, Appl. Anal. 50 (1993), 285-294.

[4] C. E. Chidume and S. J. Aneke, A local approximation method for the solution K-positive definite operator equations, Bull. Korean. Math. Soc. 40 (2003), 603-611.

[5] C. E. Chidume and S. J. Aneke and H. Zegeye, Approximations of fixed points of weakly contractive Non-self maps in Banach spaces, J. Math. Anal. 270 (2002), 189 – 199.

[6] J. Lindenstrauss. L. Tzafriri, Classical Banach spaces II, Springer Verlag, Belin-Heidelber, New York, 1979.

[7] W. V. Petryshyn, Direct and iterative methods for the solution of linear operator equations in Hilbert spaces, Trans. Amer. Math. Soc. 105 (1962), 136-175.