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On the precise order of unit groups of Burnside rings of some finite Abelian groups

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Abstract

We determine the precise order of $B^*(G)$, for $G = \oplus_i G_i$, a bounded abelian 2-group, where G_i is a direct sum of r copies of a cyclic group of order 2^n. The cases r = 1 and r = k, for some natural number k, are respectively considered in this paper.

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1.0 Introduction

The mathematical motivation for this paper is as follows: Let *G* be a finite group, the Burnside ring *B*(*G*) of *G*, as introduced by L. Solomon [6] is the Grothendieck group of the category of finite *G*-sets with multiplication given by direct product. Tammo Tom Dieck in [1] constructed congruences between fixed point numbers to determine the order of units of Burnside rings of various finite groups while Matsuda introduced the structure matrix method to determine the order of units of Burnside rings for various finite groups with many normal subgroups. Our principal aim is to prove the following. Let $G := C_{2^n}$, the cyclic group of order 2^n, $n \geq 1$, then we show that the precise order of unit group of its Burnside ring is 2^2, and more generally, when $G := \underbrace{C_{2^n} \oplus C_{2^n} \oplus \ldots \oplus C_{2^n}}_{r-times}$, the abelian 2-group of exponent 2^n and rank r > 1, $n \geq 1$ and it is considered, then we obtain the precise order of unit group of its corresponding Burnside ring to be 2^{2^r}. More precisely, using the congruence method due to Tom Dieck we proved first the following result: (see notations below)

***Theorem* 1.1**

Let $G := C_{2^n}$ *and* $H_i \leq G$ *with* $1 := H_0 < H_1 < \ldots < H_n := G$. *Let* $\gamma(H_i) \in \{\pm 1\}$ *for* $i = 0, 1, \ldots, n-1$, *then*

$$\gamma(H_i) + \gamma(H_{i+1}) + 2\gamma(H_{i+2}) + \ldots + 2^{j-1}\gamma(H_{i+j}) + \ldots + 2^{n-i-2}\gamma(H_{n-1}) + 2^{n-i-1}\gamma(H_n) \cong 0(2^{n-i})$$

for all $i = 0,\ 1, \ldots, n-1$ *if and only if* $\gamma(H_0) = \gamma(H_1) = \ldots = \gamma(H_{n-1}) = \pm\gamma(H_n)$.

***Remark* 1.2**

Theorem 1.1 implies that $|B^*(G)| = 2^2$. Finally, using Matsuda's approach, we proved the following:

***Theorem* 1.3**

Let $G := \underbrace{C_{2^n} \oplus C_{2^n} \oplus \cdots \oplus C_{2^n}}_{k-times}$ *k a natural number greater than 1, then we have* $|B^*(G)| = 2^{2^k}$

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Notes on Notations

In this paper we use the following notations:

1 *the unite element of G*

(H) the conjugacy class of a subgroup H of G

Sub(G) the set of cojugacy classes of all subgroups of G

For a G-sets X and for each $x \in X$ *, the set*

$G_x := \{g \in G \mid gx = x\}$ *is the isotropy subgroup at a point x of a G-set X,*

$X^G = \{x \in X \mid gx = x \ \forall g \in G\}$ *is the set of fixed points of a G-set X*

|X| is the cardinal number of a set X.

[X] is the element of B(G) represented by a finite G-set X,

$1_{B(G)}$ *is the unite element [point] of B(G),*

N(F) is the normalize of a subgroup F of G in G,

R^* *is the unit group of a ring R,*

Z is the ring of rational integers,

Z_2 *is the set {1 - -1},*

Z_2' *is the set {0, -2}.*

2.0 Preliminaries

The following is a summary for the reader's convenience of elementary facts about the Burnside ring of a finite group and its units which will be used in the sequel, most of which are standard materials taken directly from Matsuda [3] and are stated without proof;

***Theorem* 2.1 [5]**

Let G be a finite group and B(G) the Burnside ring of G. Then we have the following:

[1] *B(G) is a commutative ring and a free* Z*–module generated by the set* $\{[G/F] \mid (F) \in Sub(G)\}$.

[2] *Let* $\gamma_F : B(G) \to Z$ *be a map defined by* $\gamma_F([G/H]) = |(G/H)^F|$, *where* $(H),(F) \in Sub(G)$. *Then* γ_F *is a ring homomorphism. Moreover,*

$$\gamma = \prod_{(F) \in Sub(G)} \gamma_F : B(G) \to Z^{|Sub(G)|}$$

is an injective ring homomorphism.

[3] *For each finite G-set X, [X] has the following representation in B(G).*

$[X] = \sum_{(F) \in Sub(G)} \lambda_F [G/F],$ *where* $\lambda_F = |\{x \mid x \in X \text{ and } (G_x) = (F)\}| / |G/F|$

[4] *For an element* $\alpha \in B(G)$, *the following three statements are equivalent:*

(i) $\alpha \in B^*(G)$

(ii) $\alpha^2 = 1_{B(G)}$

(iii) $\gamma\alpha \in Z_2^{|Sub(G)|}$

***Theorem* 2.2 [1]**

The Burnside ring B(G) can be viewed as a subring of Map(Sub(G), Z), where $\gamma \in Map(sub(G), Z)$ *is contained in B(G) if and only if*

$$\sum_{(K)} | N(H)/N(H) \cap N(K) \| K/H^* | \gamma((K)) \cong 0 \bmod | N(H)/H | \ \ for\ all\ (H) \in Sub(G),$$

where the sum is over N(H)-conjugate classes (K) such that H is normal in K and K/H is cyclic, and K/H is the set of generators of K/H.*

***Definition* 2.3:**

A subset *S* of *Sub*(*G*) is called a basic subset if *S* satisfies the following two conditions:

(i) $G \in S, <1> \in S$ and, for $(H) \in S, H$ is a normal subgroup of *G*.

(ii) If $(H),(F) \in S,$ then $(H \cdot F),(H \cap F) \in S$ where $H \cdot F$ is a subgroup of *G* generated by *H* and *F*. Now, for each $H \neq G$ in *S*, put

$$S(H) = \{(F) \in Sub(G) \mid F \supset H, \text{ and } H = H' \text{ if } F \supset H' \supset H \text{ and } H' \in S\}$$

a non-empty set. Next, define a partial order on *Sub*(*G*) by setting $(K) \leqq (P)$ if *K* is conjugate in *G* to a subgroup of *P*. Further, define with respect to this partial order, a bijection

$$t(S(H)) : S(H) \to \{1, \ldots, | S(H) |\}$$

satisfying

$$(K) \lneqq (P)\ if\ \ t(S(H))((K)) < t(S(H))\ ((P)).$$

Finally, we have the following theorem:

***Theorem* 2.4 [5]**

Let S be a basic subset of Sub(G). Then we have

$$| B(G^*) | = 2\left(\prod_{(H) \in S - \{G\}} | M^{-1}_{t(S(H))} \left(Z_2'^{|S(H)|}\right) \cap Z^{|S(H)|} |\right),$$

where $M_{t(S(H))} = (a_{j,i}(t(S(H)))) = (\gamma_P([G/K]))$ *is the* $| S(H) | \text{x} | S(H) |$ *structure matrix of* B(G) *over* S(H) *subordinate to* t(S(*H*)) *and where* $t(S(H))\ ((P)) = j\ \ and\ \ t(S(H))\ ((K)) = i$.

***Theorem* 2.5 [5]**

Let Sub(*G*) *be the set of conjugate classes of all subgroups of G, then we have*

$$\left|B^*(G)\right| = \left|M_t^{-1}\left(Z_2'^{|Sub(G)|}\right) \cap Z_2^{|Sub(G)|}\right|,$$

where M_t *is the* |*Sub(G)*| x |*Sub(G)*| *structure matrix of B(G) over Sub(G) subordinate to a bijection t defined on Sub(G).*

***Theorem* 2.6 [5]**

If G is a finite abelian group, then we have |*B*(G)*|= 2^{m+1}, *where m* = |{*H*|*H is a subgroup of G with* |*G*/*H*| = 2}|.

***Theorem* 2.7 [1]**

If G is a finite group of odd order, then we have |*B*(G)*| = 2.

3.0 Units of Burnside ring of Abelian 2-group of exponent 2^n and rank 1

***Lemma* 3.1**

Let [G:1] = 2^n *then we have for each unique subgroup* H_j *of G*, [$G : H_j$] = 2^{n-j}.

Proof:

This is trivial as *G* is cyclic.

***Lemma* 3.2**:

Let a denote a generator of G and put $a_j := a^{2n-j}$ so that

$$H_0 :=< a_0 >,\ H_j :=< a_j >,\ j \neq 0,\ j = 1, 2, \ldots, n$$

with $$1 :=< a_0 > \leq < a_1 > \leq \ldots \leq < a_n > = < a > = G.$$

Then we have the following list of distinct conjugate classes

$$Sub(G) = \{\{< a_0 >\}, \{< a_1 >\}, \ldots, \{< a_n >\}\}.$$

Proof:

This is trivial because for all $j, N_G(< a_j >) = G.$

***Lemma* 3.3**

Let A_i be set of generators of $H_i, i = 0, 1, 2, \ldots n,$ then we have

$$|A_0| = 1,\ |A_1| = 1, \ldots, |A_{n-1}| = 2^{n-2} \text{ and } |A_n| = 2^{n-1}.$$

Proof:

Let g be an arbitrary element of G, then $g = a^k$ for all k. It also follows from above lemma that $<g> = H_j$ for some j, that $< a^k > = < a^{2n-j} >$. So we can rewrite each member in $Sub(G)$ in terms of its set of generators in the following way:

$$\begin{array}{ll}
A_0 & := \{a^{2n}\} \\
A_1 & := \{a^{2n-1}\} \\
\vdots & \vdots \quad \vdots \\
A_{n-1} & := \{a^2, a^6, \ldots a^{4n-6}, a^{2n-1}\} \\
A_n & := \{a, a^3, \ldots a^{2n-3}, a^{2n-1}\}
\end{array}$$

and hence the result follows.

Now, since $|N(H)/N(H) \cap N(K)| = 1$ in this case, applying Theorem 2.2 we obtain the congruences

$$\begin{array}{r}
\gamma(H_0) + \gamma(H_1) + 2\gamma(H_2) + 4\gamma(H_3) + \ldots 2^{n-2}\gamma(H_{n-1}) + 2^{n-1}\gamma(G) \cong 0(2^n) \\
\gamma(H_1) + \gamma(H_2) + 2\gamma(H_3) + \ldots 2^{n-3}\gamma(H_{n-1}) + \ldots 2^{n-2}\gamma(G) \cong 0(2^{n-1}) \\
\vdots \quad \vdots \quad \ldots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
\gamma(H_{n-1}) + \quad \gamma(G) \quad \cong \quad 0(2)
\end{array}$$

***Theorem* 3.4**

Let $\gamma(H_i) \in \{\pm 1\}$ *for* $i = 0, \ldots, n-1$ *then*

$$\gamma(H_i) + \gamma(H_{i+1}) + 2\gamma(H_{i+2}) + \ldots + 2^{j-1}\gamma(H_{i+j}) + \ldots + 2^{n-i-2}\gamma(H_{n-1}) + 2^{n-i-1}\gamma(H_n) \cong 0(2^{n-i})$$

for all $i = 0, 1, \ldots n-1$ *if and only if* $\gamma(H_0) = \gamma(H_1) = \ldots = \gamma(H_{n-1}) = \pm\gamma(H_n)$

Proof

To see "$\Leftarrow$" is easy, since

$$\gamma(H_i) + \gamma(H_{i+1}) + 2\gamma(H_{i+2}) + \ldots + 2^{n-i-2}\gamma(H_{n-1}) = 2^{n-i-2}\gamma(H_n)$$

and by assumption we must have that

$$\gamma(H_i) + \gamma(H_{i+1}) + 2\gamma(H_{i+2}) + \ldots + 2^{n-i-1}\gamma(H_n) \cong 0(2^{n-i}) \text{ for all } i.$$

To see "$\Rightarrow$" w use induction on $n - i$:

For $n-i=0 \Rightarrow i=n$ it is easy to see that $\gamma(H_0)=\gamma(H_n)$. Similarly for i = n – 1. Now assume that the induction hypothesis is true for $i<n-1$, that is, $n-i>1$, so that we have

$$\gamma_0 := \gamma(H_{i+1}) = \gamma(H_{i+2}) = \Lambda = \gamma(H_{n-1}) = \pm\gamma(H_n).$$

Then we obtain by hypothesis

$$\gamma(H_i) + (2^{n-i-1}-1)\gamma_0 \pm 2^{n-i-1}\gamma(H_n) \cong 0(2^{n-i})$$

This implies, $$\gamma(H_i) + 2^{n-i-1}\left(\gamma_0 \pm \gamma(H_n)\right) - \gamma_0 \cong 0(2^{n-i})$$

But since $\left(\gamma_0 \pm \gamma(H_n)\right)$ is either 0 or ± 2 we get that $2^{n-i-1}(\gamma_0 \pm \gamma(H_n)) \cong 0\left(2^{n-i}\right)$ and $\gamma(H_i) - \gamma_0 \cong 0\left(2^{n-i}\right)$ also since $n-i \ \phi\ 1$, $\gamma(H_i) = \{\pm 1\}$, $\gamma_0 = \{\pm 1\}$ we cannot get that $+1 \not\cong -1(4)$ for instance, so it follows that $\gamma(H_i) = \gamma_0$ and the proof is complete.

***Remark* 3.5**

The above theorem 3.4 implies that $|B^*(G)| = 2^2$

4.0 Units of Burnside ring of Abelian 2-group of exponent 2^n and rank r > 1

***Lemma* 4.1**

Let $G := \underbrace{C_{2^n} \oplus C_{2^n} \oplus K \oplus C_{2^n}}_{r-times}$ $n \geq 1$ and $H \leq G.$ *Then the number of G/H such that |G/H| = 2 is 2r* – 1

Proof:

Let $G := \underbrace{C_{2^n} \oplus C_{2^n} \oplus \Lambda \oplus C_{2^n}}_{r-times}$ $r \ \phi\ 1,\ n \geq 1$ and H a subgroup G of order 2^{nr-1}, then we define a subgroup base for H as $(r-1)$, r-tuples generating H. This can be represented as $(r-1)$-rows of $r \times r$ -matrix whose rows generate G. Now, let $C_2 := \pi\ a\ \phi$, then we can choose the following number of r subgroup bases, for each H of G and through each subgroup base, the number of cyclic quotients satisfying $\left|G/H = 2\right|$, is determined. Thus, the precise number of r distinct subgroup bases, for each H of G is determined from the following set.

$$\left\{\begin{pmatrix} a^2 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & a & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a & \varepsilon_k & 1 & \cdots & 1 & 1 & 1 \\ 1 & a^2 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & a & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a & 1 & \varepsilon_k & \cdots & 1 & 1 & 1 \\ 1 & a & \varepsilon_k & \cdots & 1 & 1 & 1 \\ 1 & 1 & a^2 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \ldots,\right.$$

$$\left.\begin{pmatrix} a & 1 & 1 & \cdots & \in_\kappa & 1 & 1 \\ 1 & a & 1 & \cdots & \in_\kappa & 1 & 1 \\ 1 & 1 & a & \cdots & \in_\kappa & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a^2 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a & 1 & 1 & \cdots & 1 & \in_\kappa & 1 \\ 1 & a & 1 & \cdots & 1 & \in_\kappa & 1 \\ 1 & 1 & a & \cdots & 1 & \in_\kappa & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & \in_\kappa & 1 \\ 1 & 1 & 1 & \cdots & 1 & a^2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a & 1 & 1 & \cdots & 1 & 1 & \in_\kappa \\ 1 & a & 1 & \cdots & 1 & 1 & \in_\kappa \\ 1 & 1 & a & \cdots & 1 & 1 & \in_\kappa \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a & 1 & \in_\kappa \\ 1 & 1 & 1 & \cdots & 1 & a & \in_\kappa \\ 1 & 1 & 1 & \cdots & 1 & 1 & a^2 \end{pmatrix}\right\},$$

where $\in_\kappa \in \{a^l\},\ 0 \le l \le 1.$ We obtain a total sum of number of cyclic quotients form the above r distinct subgroup bases, for each H of G as:

$$1 + 2 + 2^2 + \cdots + 2^{r-3} + 2^{r-2} + 2^{r-1},$$

which yields the formula:

$$2^r - 1, \text{ and any integer } r > 1$$

Finally, the main result of this paper counts the number of factor groups of order 2 in abelian group G in order to write down the order of $B^*(G)$ by Matsuda's Theorem is seen in the following

***Theorem* 4.2.**

Let $G := \underbrace{C_{2^n} \oplus C_{2^n} \oplus \cdots \oplus C_{2^n}}_{r-times}$ $r > 1,\ n \ge 1$ *then we have* $|B^*(G)| = 2^{2^r}$

Proof:

This follows from Lemma 4.1, and from Matsuda's Theorem.

5.0 Conclusion

It is desirable to generalize the computations of $B^*(G)$, G a cyclic group of order 2^n to more general cyclic groups, or more generally to finite nilpotent and solvable groups of even order.

6.0 Acknowledgement

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