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On the existence of global optimization solution on abstract spaces

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Abstract

Herein is characterized the solution to global optimization problems using differential conditions and convexity assumptions. The uniqueness of the existence theorems of solution are proposed and proved. An application of the results obtained is shown to best approximation problem. The results here compliment similar results in our work in* [15] *where existence of solution to statistical estimation problems were established using classical projection theorem.

***Keywords*: Global Optimization, Differential Conditions, Convexity.**

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1.0 Introduction

The fields of application of Global Optimization Problems (GOP) include concave minimization, reverse convex programming, difference of two convex functions programming, Lipschitzian optimization, global integer programming etc.

It is important to note that all standard techniques in nonlinear optimization can at most locate a local minima. Moreover, there is no criterion for deciding whether a local solution is global. Due to the inherent difficulties mentioned above, the methods devised for solving multiextremality global optimization problems are quite diverse and significantly different from standard optimization tools.

Convexity plays a vital role in this aspects and its effect is been realized in two ways. First, any candidate for an extremum of a convex functional is automatically a minimum. Second, any local minimum of a convex functiona is automatically a global minimum.

Virtually all the main classes of GOP stated above are concave minimization problems including the Lipschitzian optimization. According to Rockfellar[19] relates the fact that a convex functional is Lipschitzian on a compact subset of the relative interior domain which seems apparently to be a general problem. Thus, most problems stated earlier are actually Lipschitzian.

The investigation of the existence of global solution and their applications have been an important preoccupation in the analysis of the global optimization problems defined on finite and infinite dimensional Euclidean spaces. Research in this subject has attracted considerable attention in the literature. Some well known results and their applications in the finite and infinite dimensional Euclidean settings can be found in [1,2,4,7,9,10,11,12,13,15,16,19,21,22].As in the [1,2,7,9,15,16],such problems have been found useful in approximation theory, statistical estimation problems, signal and image reconstruction as well as in other engineering applications. The authors in the above references used different methods in proving the existence of solution to the various types of optimization problems considered which include duality principle, Fixed point theorem, Newton-type method, classical projection theorem etc.

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We established in this work the existence of local solution using differential conditions and the global solution based on the convexity arguments.

2.0 Notation and preliminary results.

***Definition* 2.1**

Let $f : D \subseteq X \to \Re$ be a functional on the real locally convex space X, and u_o is a given interior point of $D(f)$.Then, the functional f is Gateaux differentiable at u_o if and only if there exists a continuous linear functional $a \in X^*$,that we denote by $f'(u_o)$,such that

$$\lim_{t \to o} \frac{f(u + th) - f(u_o)}{h} = \langle f'(u_o), h \rangle \text{ for all } h \in X \tag{2.1}$$

$f^{'}(u_o)$ is called the G-derivative of f at u_o.

The G-derivative $f^{'}(u_o)$ exists if and only if $\delta f(u_{o;}h)$ exists for all $h \in X$ and $h \to \delta f(u_{o;}h)$ is a continuous linear functional on X .So that

$$\delta f(u_{o;}h) = \langle f'(u_o), h \rangle \text{ for all } h \in X \tag{2.2}$$

Note that this definition requires no norm on X hence ,properties of G-derivatives are not related to continuity ,when X is a normed space, a more satisfactory tool is given below.

***Definition* 2.2**

Let $f : D \subseteq X \to \Re$ be a functional on the real normed space X ,and u_o is given fixed interior point of $D(f)$.The functional f is frechet differentiable at u_o if and only if there exists a continuous linear functional $a \in X^*$, denoted by $f^{'}(u_o)$,such that an expansion of the for

$$f(u_o + h) = f(u_o) = \langle f'(u_{o,} h) \rangle + 0\|h\|, as \quad h \to 0 \tag{2.3}$$

holds for all h in a neighbourhood of zero. $f^1(u_o)$ is F-derivatives of f at u_o and is denoted by $df(u_o; h) = \langle f'(u_o), h \rangle u$.

***Proposition* 2.1** [15].

If the Frechet differential of f exists at u_o ,then the Gateaux differential exists at u_o and they are equal.

Proof

Denote the Frechet differential by $\delta f(u_{o;}h)$.By definition, we have for all *h*.

$$\frac{\| f(u_o + th) - f(u_o) - \delta f(u_o; h)\|}{t} \to 0, \text{as} \, t \to 0 \tag{2.4}$$

Thus, by the linearity of $\delta f(u_{o;}h)$ with respect to t,

$$\lim_{t \to 0} \frac{f(u_o + th) - f(u_o)}{t} = \delta f(u_{o,} h) \tag{2.5}$$

which completes the proof.

The following theorem will be the main tools for deriving necessary and sufficient conditions for the existence of extremal points.

***Theorem* 2.1**

Let X be a real locally convex space , $f : D \subseteq X \to \Re$ be given and $u_o \in \operatorname{int} D(f)$

Then the following assertions hold:

(a) **Necessary conditions**.

If f has a free local minimum at u_o , then

$$\delta f(u_{o;}h) = 0 \tag{2.6}$$

$$\delta^2 f(u_o;h) \geq 0 \tag{2.7}$$

for all $h \in X$,where these variations exist. For (2.6) to hold, it suffices that $\delta f(u_{o;}h)$ exists for all $h \in X$.

(b) **Sufficient conditions**.

Let n be an even number , $n \geq 2$,and let X be a Banach space. Then f has a free strict local minima at u_o provided the following hold:

(i) For all $h \in X$ and fixed $c \geq 0$,

$$\delta^{(k)} f(u_o;h) = 0, k = 1,2,\ldots,n-1 \tag{2.8}$$

$$\delta^{(n)} f(u_o;h) \geq c\|h\|^n \tag{2.9}$$

(ii) $u \to \delta^{(n)} f(u;h)$ is continuous at u_o and indeed uniformly continuous with respect to *h*, i.e to be precise ,for each $\varepsilon > 0$ there exists an $\eta(\varepsilon) > 0$ such that

$$\left|\delta^{(n)} f(u;h) - \delta^{(n)} f(u_o;h) \leq \varepsilon\|h\|^n\right| \tag{2.10}$$

for all $h \in X$ and all $u \in X$ such that $\|u - u_o\| < \eta(\varepsilon)$.Here ,it is assumed that all variations that appear exists.

Proof

(a) For every $h \in X$, the function

$$\varphi_h(t) = f(u_o + th) \tag{2.11}$$

of the real variable *t*, achieves an extreme at *t* = 0.Thus,

$$\frac{d}{dt} f(u_o + th)_{t=0} = \varphi_h^{1}(t) = 0 \tag{2.12}$$

i.e. $\delta f(u_{o;}h)$=0 for all *h* and the result follows for (2.7)

(b) Let $\varphi_h(t) = f(u_o + th)$,therefore

$$\varphi_h^{(k)}(t) = \delta^{(k)} f(u_o + th;h) \text{ for all } h \tag{2.13}$$

in a neighbourhood of zero ,using Taylor theorem about φ_h in a neighbourhood of t=0,

$$\varphi_h(t) = \varphi_h(0) + \sum_{k=1}^{n} \frac{t^k \varphi_h^{(k)}(0)}{k} + R_n \tag{2.14}$$

yields for $t = 0$,

$$f(u_o+h)-f(u_o)=\varphi_h(1)-\varphi_h(0)=\frac{\varphi_h{}^n(\theta)}{h}, 0<\theta<1 \quad (2.15)$$

By (2.8) and (2.9) with $\varepsilon=\frac{c}{2}$ for h with $\|h\|<\tau\left(\frac{c}{2}\right)$ we thus have

$$f(u_o+h)-f(u_o)=\frac{\delta^{(n)}f(u_o+th;h)}{n}\geq\frac{c}{2n}\|h\|^n \quad (2.16)$$

3.0 Existence and uniqueness of global solution

Consider the standard global optimization problem

(A) $$\min f(u), u\in D \quad (3.1)$$

where $f: X\to\Re$ is a convex functional on a convex set $D\subset X$ called the set of constraints. To problem (A) we can associate a problem

(B) $$\min \bar{f}(u), u\in X, where. \bar{f}=f+\iota_D \quad (3.2)$$

we call value of problem (A) the extended real

$$v(f,D)=\inf\{f(u): u\in D\}\in\Re$$

we denote optimal solution of problem (A) by an element $\bar{u}\in D$ with the property that $f(\bar{u})=v(f,D)$. We call $S(f,D)$ the set of optimal solutions of problem (A).Therefore

$$S(A)=\{\bar{u}\in D:\forall u\in X; f(\bar{u})\leq f(u)\}=\{\bar{u}\in X:\forall u\in X; \bar{f}(u)\leq f(u)\}=S(B) \quad (3.3)$$

if $D\cap domf\neq\phi$.The set $S(f,D)$ is denoted by $\arg\min f$.

3.1 Main results

An important issue is that of the existence of global solution for (A) and (B) respectively. The major results that assures the existence of solution to (A) is Weirstrass's theorem. But we may use for the same purpose some coercivity conditions because the underlying space is not compact.

***Definition* 3.1**

Let $f: X\to R$,we say f is coercive if

$$\lim_{\|x\|\to\infty} f(x)=\infty \quad (3.4)$$

***Lemma* 3.1**

Let $(X,\|.\|)$ be a normed space and $f: X\to\overline{R}$. f is coercive if and only if the level set is bounded.

Proof

Since *f* is coercive i.e $\lim_{\|x\|\to\infty} f(x)=\infty$ if and only if $\exists$ a scalar $r>0$ such that

$$\|x\|>r,\ f(x)>\lambda\forall\lambda\in\Re$$

$$\Leftrightarrow\forall\lambda\in\Re,\exists r>0:[f\leq\lambda]\subset B(0,r)$$

which completes the proof.

***Theorem* 3.1**

Let $f\in\Gamma(X)$,where $\Gamma(X)$ is the class of lower semicontinuous proper functions $f: X\to R$

(i) If there exists $\lambda>v(f,X)$ such that $[f\leq\lambda]$ is w-compact ,then $\arg\min f\neq\phi$.

(ii) If X is a reflexive Banach spaces and f is coercive, then $\arg\min f\neq\phi$.

Proof

(i) Let $v(f,X) := \inf\{f(u) : u \in X\} = v(f,[f \le \lambda])$.since f is lower semicontinuous and convex, f is w-lower semicontinuous. The conclusion follows using Weirstrass theorem applied to the function $f_{[f \le \lambda]}$.

(ii) Because f is coercive , $[f \le \lambda]$ is bounded .Using Lemma (3.1) for $\lambda \in \Re$ since $[f \le \lambda]$ is *w*- closed and X is reflexive ,we have that $[f \le \lambda]$ is w-compact for every $\lambda \in \Re$. Thus $\arg\min f \ne \phi$.

***Proposition* 3.1**

Let $f \in \Lambda(X)$,where $\Lambda(X)$ is the class of convex functions $f : X \to R$.Then $S(f,X)$ is a convex set. Furthermore ,if f is strictly convex then $S(f,X)$ has at most one element.

Proof

Let $\bar{u} \in S(f,X)$,then $S(f,X) = [f \le f(\bar{u})]$,whence $S(f,X)$ contains at least two distinct elements u_1, u_2; since f is a proper function , $S(f,X) \subset domf$, then we obtain a contradiction,

$$v(f,X) \le f\left(\frac{1}{2}u_1 + \frac{1}{2}u_2\right) < \frac{1}{2}\{f(u_1) + f(u_2)\} = v(f,X)$$

Therefore $S(f,X)$ has at most one element.

4.0 Application to best approximation problem

Let $D \subset (X, \|.\|)$be a non-empty set and consider the distance function

$$d_D : X \to \Re, \quad d_D(u) = d_D(u,c) = inf\{\|u - c\| : c \in D\} \tag{4.1}$$

Having $u \in X$, an important problem consists of determining the set

$$P_D(u) = \{c \in D \mid \|\bar{u} - u\| = d_D(u)\} \tag{4.2}$$

$\bar{u} \in P_D(x)$ is called a minimum solution of *x* by elements of *D*.

***Proposition* 4.1**

Let $d_D : D \to \Re$ be a distance function and $D \subset (X, \|.\|)$ be a non-empty set. Then d_D is Lipschitzian.

Proof

Let $$d_D(u) = inf\{\|u - c\| : c \in D\}, d_D(y) = inf\{\|y - c\| : c \in D\} \tag{4.3}$$

$$d_D(u) - d_D(y) = \|u - c\| - \|y - c\| \le \|u - y\| \tag{4.4}$$

Using triangle inequality

$$|d_D(u) - d_D(y)| = \big|\|u - c\| - \|y - c\|\big| \le \|u - y\| \tag{4.5}$$

$$\Rightarrow |d_D(u) - d_D(y)| \le \|u - y\| \tag{4.6}$$

For all $u, y \in X \Rightarrow d_D$ Is Lipschitzian with Lipschitz constant 1.

4.1 Existence and uniqueness of best approximation

***Theorem* 4.1**

Let $D \subset X$ be a non-empty closed convex set and $u_0 \in X$

(i) If X is a reflexive Banach space then $P_D(u_0) \neq \phi$

(ii) If X is a strictly convex normed space then $P_D(u_0)$ has at most one element.

Proof

(i) Let us consider the function $f := \|- u_0\| + l_D$. Since f is Lipschitzian, then it is necessarily convex and lower semicontinuous. By Theorem 3.1, there exists $\bar{u} \in X$ such that $f(\bar{u}) \leq f(u)$ for every $u \in X$ i.e. $\bar{u} \in P_D(u_0)$

(ii) We have already seen that $P_D(u)$ = {u} for $u \in D$. Let $u \notin D$ and suppose that there are two distinct elements x_1, x_2 in $P_D(u)$. Then $\frac{1}{2}(x_1 + x_2) \in D$, and

$\|x_1 - u\| = \|x_2 - u\| = d_D(u) > 0$. Since X is strictly convex, we obtain

$$\left\|\frac{1}{2}(x_1 + x_2) - u\right\| = \left\|\frac{1}{2}(x_1 - u) + \frac{1}{2}(x_2 - u)\right\| < \frac{1}{2}\|x_1 - u\| + \frac{1}{2}\|x_2 - u\| = d_D(u)$$

This contradiction proves that $P_D(u)$ has at most one element. □

5.0 Conclusion

In this paper, an attempt has been made to characterized global optimization solution of more general objective functional using convexity assumptions and differential conditions. The existence and uniqueness theorem of the solution was proposed and proved while the results obtained were later applied to best approximation problem.

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References

[1] Bamigbola, O. M and Osinuga, I. A. (2003). Resolution of Extremization Problems Using Duality Principle. Proceedings African Mathematical Union-International Conference of Mathematical Scientists, University of Agriculture, Abeokuta, Nigeria, 207-213.

[2] Bamigbola, O. M and Osinuga, I. A (2007):On the Minimum Norm Solution to Weber's Problem. Proceedings of the SAMSA (Southern Africa Math. Sci. Ass.) Submitted.

[3] Borwein, J. M and Lewis, A.S (2000):Convex Analysis and Non-linear Optimization(Theory and Examples),Springer-Verlag, New York

[4] Boyd, S. and Vandersberghe, L. (2004): Convex Optimization, Cambridge University Press, United Kingdom.

[5] Brechtken-Manderscheid, U (1991):Introduction to the Calculus of Variations. Chapman and Hall, London.

[6] Clarke, F. H. (1983):Optimization and Nonsmooth Analysis, Wiley, New York

[7] Ferreira, Paulo Jorge S. G. (1996). The existence and uniqueness of the minimum norm solution to certain linear and non linear problems. Signal Processing, 55, 137-139.

[8] Horst, R et al (1990):Global Optimization (deterministic approaches),Springer-Verlag, New-York.

[9] Kanzow, C., Qi, H. and Qi, L. (2000). On the minimum Norm Solution of linear Programs. Applied mathematics Report, AMROO/ 14, 1-11

[10] Klans, k. and Hannu, S. (2005). Minimum Norm Solution for Cooperative Games. Research Report, Department of Economics, University of Turku, Finland, 115, 1-23.

[11] Kreyzig, E. (1978). Introductory Functional Analysis with Applications. Wiley, New York.

[12] Lebedev, V. I. (1997). An Introduction to Functional Analysis in Computational Mathematics , Birkhauser Boston, Cambridge.

[13] Luenberger, D. G. (1969). Optimization by vector space methods. Wiley, New York.

[14] Maddox, I. J. (1988). Elements of functional Analysis Cambridge University Press, Cambridge.

[15] Oniyide, O. R. and Osinuga, I. A. (2006). On the Existence of Best Sample in Simple Random Sampling, ABACUS(J. Math. Ass. Nig.), 33(2B), 290-294.
[16] Osinuga, I. A. and Bamigbola, O. M.(2004): On Global Optimization for Convex Problems on Abstract Spaces, ABACUS (J. Math .Ass. Nig.),31(2A),94-98.
[17] Osinuga, I. A (2007):.Further Results on the Classes of Convex Functions, J. Nig. Ass. Math. Phys. 11, 515-518.
[18] Osinuga, I. A. (2000):On Lp Bound Lipschitzian Optimization, M.Sc.Thesis. University of Ibadan. Ibadan
[19] Rockafellar, R. T. (1970): Convex Analysis, Princeton University Press .New Jersey.
[20] Wan, F. Y. M (1995):Introduction to the Calculus of Variations and its Application, Chapman and Hall, New York
[21] Zeidler, E (1985):Nonlinear Functional Analysis and its Application(Variational Methods and Optimization),Springer-Verlag, New York.
[22] Zalinescu, C. (2002). Convex Analysis in Generl Vector Spaces, World Scientific, Singapore.