

On an improved convergence rate estimate obtained by Liwei Liu

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Abstract

It is shown that any fixed point of a Lipschitzian strictly pseudo-contractive mapping T on a closed, convex subset K of a Banach space X may be norm approximated by an iterative procedure. Our argument improved the convergence rate estimate obtained by Liu [2, Theorem 2].

Keywords: Banach space, strictly pseudo-contractive mapping, Mann iterative process with errors.

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1.0 Preliminaries

Let X be a Banach space , K a non-empty ,convex subset of X, T a selfmap of K, let $x_0 \in K$. The Mann iteration, (see[3]), is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n = 0, 1, 2 \dots \quad (1.1)$$

The sequence $\{\alpha_n\} \subset (0, 1)$ satisfy

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = +\infty \quad (1.2)$$

Definition 1.0

Let X be a Banach space . A mapping T is said to be Lipshitzian if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L \|x - y\| \text{ for all } x, y \in K \quad (1.3)$$

A mapping T is said to be strictly pseudo-contractive [3] if there exists a number $t > 1$ such that the inequality

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\| \quad (1.4)$$

holds for all $x, y \in K$ and $r > 0$. We also have that T is strictly pseudo-contractive following Liu[2] this may be stated as there exists $k \in (0, 1)$ for which

$$\|x - y\| \leq \|x - y + r[(I - T - kI)x - (I - T - kI)y]\| \quad (1.5)$$

for all $r > 0$ and $x, y \in K$.

The following result generalizes Liu [2, Theorem 1 and 2] and provides a convergence rate estimate when compared with the result of Liu [2, Theorem 2].Our result extends and improves the convergence rate estimate of Liu [2].

2.0 Main result

Theorem 2.1

Let X be a Banach space, K a non-empty, convex subset of X . Let $T : K \rightarrow K$ be a Lipschitzian strictly pseudo-contractive mapping such that $T\rho = \rho$ for some $\rho \in X$. Suppose that $(\alpha_n)_{n \in N}$ is a sequence in $(0,1]$ such that for some $\eta \in (0, k)$, for all $n \in N$

$$\alpha_n < \frac{k - \eta}{(1 + L)L}; \text{ while } \sum_{n=1}^{\infty} \alpha_n = \infty. \text{ Fix } x_1 \in K. \text{ Define for all } n \in N$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n$$

$$\text{Then, there is } \frac{\alpha_n [k - (1 + L)L\alpha_n]}{1 - (1 - k)\alpha_n} \geq \frac{\eta\alpha_n}{1 + k^2}, \quad \forall n \geq 0$$

$$\text{Such that } \|x_{n+1} - \rho\| \leq \prod_{j=1}^n \left(1 - \frac{\eta}{1 + k^2}\alpha_j\right) \|x_1 - \rho\| \text{ Converges strongly to } \rho.$$

Proof

Since T is strictly pseudo-contractive we have from (1.5) that

$$\|x - y\| \leq \|x - y + r[(I - T - kI)x - (I - T - kI)y]\|$$

for any $x, y \in K$ and $r > 0$. Using (1.1), we obtain that,

$$\begin{aligned} (1 - \alpha_n)x_n &= x_{n+1} - \alpha_n Tx_n, & n = 0, 1, 2 \dots \\ &= [1 - (1 - k)\alpha_n]x_{n+1} + \alpha_n(I - T - kI)x_{n+1} + \alpha_n Tx_{n+1} - \alpha_n Tx_n \end{aligned} \quad (2.1)$$

We note that

$$(1 - \alpha_n)\rho = [1 - (1 - k)\alpha_n]\rho + \alpha_n(I - T - kI)\rho \quad (2.2)$$

By using the inequality (1.5) with $x = x_{n+1}$ and $y = \rho$

$$\begin{aligned} \|x_{n+1} - \rho\| &\leq \|x_{n+1} - \rho + x_n[(I - T - kI)x_{n+1} - (I - T - kI)\rho]\| \\ &= \|x_{n+1} - \rho + \alpha_n(I - T - kI)(x_{n+1} - \rho)\| \end{aligned}$$

$$\text{Thus, } 0 \leq \alpha_n(I - T - kI)\|(x_{n+1} - \rho)\| \quad (2.3)$$

In view of (2.3), it follows from (2.1) and (2.2) that

$$\begin{aligned} (1 - \alpha_n)\|(x_n - \rho)\| &\geq [1 - (1 - k)\alpha_n]\|(x_n - \rho)\| + \frac{\alpha_n}{1 - (1 - k)\alpha_n}[(I - T - kI)x_{n+1} \\ &\quad - (I - T - kI)\rho] - \alpha_n\|Tx_{n+1} - Tx_n\| \\ &\geq [1 - (1 - k)\alpha_n]\|(x_{n+1} - \rho)\| - \alpha_n\|Tx_{n+1} - Tx_n\| \end{aligned} \quad (2.4)$$

Since T is Lipschitzian, we have

$$\begin{aligned} \|(Tx_n - x_n)\| &= \|Tx_n - \rho + \rho - x_n\| = \|(Tx_n - \rho) + (\rho - x_n)\| \\ &\leq \|Tx_n - \rho\| + \|x_n - \rho\| \\ &\leq (1 + L)\|x_n - \rho\| \end{aligned}$$

$$\begin{aligned}
\|Tx_{n+1} - Tx_n\| &\leq L \|x_{n+1} - x_n\| \\
&= L \|(1 - \alpha_n)x_n + \alpha_n Tx_n - x_n\| \\
&= L \|\alpha_n Tx_n - \alpha_n x_n\| \\
&= \alpha_n L \|Tx_n - x_n\| \\
&\leq \alpha_n L (1 + L) \|x_n - \rho\|
\end{aligned} \tag{2.5}$$

Substituting (2.5) into (2.4) We get

$$\begin{aligned}
[1 - (1 - k)\alpha_n] \|(x_{n+1} - \rho)\| &\leq (1 - \alpha_n) \|x_n - \rho\| + \alpha_n^2 L (1 + L) \|x_n - \rho\| \\
&= (1 - \alpha_n) + \alpha_n^2 L (1 + L) \|x_n - \rho\|
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
\|x_{n+1} - \rho\| &\leq \frac{[(1 - \alpha_n) + \alpha_n^2 L (1 + L)]}{1 - (1 - k)\alpha_n} \|x_n - \rho\| \\
&= \left[1 - \frac{\alpha_n (k - (1 + L)) L \alpha_n}{1 - (1 - k)\alpha_n} \right] \|x_n - \rho\| \\
&\geq \frac{\alpha_n [(k - (1 + L)) L \alpha_n]}{1 - (1 - k)} \geq \frac{\eta \alpha_n}{1 + k^2} \text{ for some } \eta \in (0, k)
\end{aligned} \tag{2.7}$$

Since

$$\|x_{n+1} - \rho\| \leq \left[1 - \frac{\eta \alpha_n}{1 + k^2} \right] \|x_n - \rho\| \tag{2.8}$$

$$\leq \prod_{j=1}^n \left(1 - \frac{\eta \alpha_j}{1 + k^2} \right) \|x_1 - \rho\| \tag{2.9}$$

$$\text{Clearly, } \sum_{n=1}^{\infty} \frac{\eta}{1 + k^2} \alpha_n = \frac{\eta}{1 + k^2} \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and so } \prod_{j=1}^n \left(1 - \frac{\eta \alpha_j}{1 + k^2} \right) = 0$$

Hence, $\lim_{n \rightarrow \infty} \|x_n - \rho\| = 0$ which implies that the sequence $(x_n)_{n \in N}$ strongly converges to ρ .

3.0 Numerical example.

Let $X, K, T, L, k, \rho, \eta$ and $(x_n)_{n \in N}$ be as the hypothesis of theorem 2.1, where $(\alpha_n)_{n \in N}$ is the sequence in $(0, 1]$ given for every $n \in N$ by

$$\alpha_n = \frac{k - \eta}{(1 + L)L}$$

Then, we have the following geometric convergence rate estimate for $(x_n)_{n \in N}$ for all $n \in N$

$$\|x_{n+1} - \rho\| \leq \lambda^n \|x_1 - \rho\|$$

where

$$\lambda = 1 - \frac{\eta \alpha_1}{1 + \alpha_1^2}$$

We remark that the choice $\eta = \frac{k}{2}$ where η varies over $(0, k)$, our λ yields

$$\lambda = 1 - \frac{k^2}{4L(1+L) + \frac{k^2}{L(1+L)}}$$

and the λ value of Liu [2,Theorem 2] was given as

$$\lambda = 1 - \frac{k^2}{4(3 + 3L + L^2)}$$

Comparing the convergence rate estimate of λ 's values of our result and that of Liu [2, Theorem 2], using C++ programming language, thus:

```
#include <iostream.h>
int main()
{
    int n, l;
    float k, Liu, MogbadeAkin;
    l = 1; n = 1; k = 1;
    cout<<"Liu\t" << "MogbadeAkin" << endl;
    while (n<=6)
    {
        k = k/n;
        Liu = 1-((k*k)/(4*(3+(3*l)+(l*l))));
        MogbadeAkin = 1-(k*k)/(((4*l)*(l+1))+((k*k)/(l*(l+1)))); 
        cout<<Liu<<"\t" << MogbadeAkin << endl;
        n++;
        l++;
    }
    return 0;
}
```

We have (see Table 3.1).

Table 3.1

ITERATION	λ LIU	λ MOGBADEMU, AKINFENWA
Step 1	0.964286	0.882353
Step 2	0.995192	0.989601
Step 3	0.999669	0.999421
Step 4	0.999986	0.999978
Step 5	1.000000	0.999999
Step 6	1.000000	1.000000

We observed in the above Table 3.1 that our minimal value as η varies over $(0, k)$ is less than that of Liu thus improving the convergence rate estimate obtained by Liu [2].

4.0 Conclusion

We have shown that Mann iteration procedure can be used to approximate the fixed point of a Lipschitzian strictly pseudo-contractive mapping defined on a closed convex subset of a Banach space. Our argument therefore, improves the convergence rate estimate obtained by Liu [2].

References

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