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Combinatorial results for Subgroup of orientation preserving mappings

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Abstract

Let OPD_n be the subgroup of all orientation preserving bijective mappings of n-element set. It is shown that for n-odd there are n-even derangements, and for n-even there are $\frac{n}{2}$ even and $\frac{n}{2}$ odd derangements, respectively.

Keywords: *Permutation, derangement, symmetries, full transformation, partial transformation, partial one-one transformation*.

1.0 Introduction and preliminaries

Let X_n denote the set $\{1, 2, \Lambda\ , n\}$ considered with standard ordering and let T_n, P_n and O_n be the full transformation semigroup, the partial transformation semigroup and the submonoid of T_n consisting of all order preserving mappings of X_n, respectively. Another closely related algebraic structure to O_n and P_n are S_n and D_n the symmetric and dihedral groups on the set X_n, respectively.

Catarino and Higgins [12] introduced a new subsemigroup of X_n containing O_n which is denoted by OP_n and its elements are called orientation preserving mappings. Also, they introduced a semigroup $P_n = OP_n \cup OR_n$ where OR_n denotes the collection of all orientation reversing mappings. Fernandes [14] studied the monoid of orientation preserving partial transformations of a finite chain, concentrating in particular on partial transformations which are injective. Here, we consider the subgroup of orientation preserving bijective mappings. In particular, we pay attention to a subgroup of the Dihedral group D_n of the order $2n$ defined as $D_n = \left\{x, y \mid x^n = 1,\ y^2 = 1 \quad xy = x^{-1}y\right\}$.

Combinatorial properties of T_n and S_n and some of their semigroups and subgroups respectively, have been studied over a long period and many interesting and delightful results have emerged (see for example [11], [8], [9]). Recently, inspired by the works of Laradji and Umar [8] and Bashir and Umar [6], Bashir and Umar [7], obtained a geometric proof for the number of even and odd permutations having exactly k fixed points in the Dihedral group D_n. However, the algebraic proof of this result along the lines of Catarino and Higgins [12] seem not to have been studied.

At the end of this introductory section we gather some known results that we shall need in later sections. In Section 2 we prove some results which we will need in the proof of the main result which is

Theorem 3.4. Finally, in Section 3, we obtain the number of even and odd permutations and the number of fixed points in the subgroup.

The semigroup of all order-preserving self maps of X_n consist of all maps $\alpha : X_n \to X_n$ with the property that $x \le y \Rightarrow x\alpha \le y\alpha$. A map α is order decreasing if $x\alpha \le x$ for all x in X_n. Let $A = (a_1, a_2, \ldots, a_s)$ be a finite sequence from the chain X_n. We say that A is cyclic or has clockwise orientation if there exist not more than one subscript i such that $a_i > a_{i+1}$ where a_{s+1} denotes a_1. We say that $A = (a_1, a_2, \ldots, a_s)$ is anti-cyclic or has anticlockwise orientation if there exists no more than one subscript i such that $a_i < a_{i+1}$. Note that a sequence A is cyclic if and only if A is empty or there exist $i \in \{0, 1, \ldots, s-1\}$ such that $a_{i+1} \le a_{i+2} \le \cdots \le a_s \le a_1 \le \cdots \le a_i$. i is unique unless the sequence is a constant.

Let $n \in \mathbb{N}$. Let us consider the following permutation of $\{1, \ldots, n\}$:

$$\alpha_n = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix}.$$

Notice that $\alpha_n \in OPD_n$ and $\alpha_n^k = \left(\begin{array}{cccc|ccc} 1 & 2 & \cdots & n-k & n-k+1 & \cdots & n \\ k+1 & k+2 & \cdots & n & 1 & \cdots & k \end{array}\right)$

for $0 \le k \le n-1$.

Recall from [1] that an even permutation is a permutation which can be expressed as a product of an even number of cycles of even length and/or a product of any number of cycles of odd length. A permutation that is not even is called odd. The set of even permutations of X_n, called the alternating group is usually denoted by A_n.

Recall also that, a derangement σ is a permutation such that $\sigma(x) \ne x$, that is, a permutation without fixed points.

***Result* 1.1**

Let A be any cyclic (anti-cyclic) sequence. Then A is anti-cyclic (cyclic) if and only if A has no more than two distinct values.

If $A = (a_1, a_2 \ldots, a_t)$ is any sequence then we denote by A^τ sequence $(a_t, a_{t-1}, \ldots, a_1)$, called the reversed sequence of A.

***Result* 1.2**

Let $A = (a_1, a_2 \ldots, a_t)$ be any sequence from X_n. Then A is cyclic (anti-cyclic) if and only if A^τ is anti-cyclic (cyclic).

***Result* 1.3**

If $(a_1, a_2 \ldots, a_t)$ is cyclic (anti-cyclic) then so is

(a) the sequence. $(a_{i_1}, a_{i_2}, \ldots, a_{i_r})$ $(i_1 < i_2 < \cdots < i_r)$

(b) and the sequence. $(a_j, a_{j+1}, \ldots a_t, a_1, \ldots, a_{j-1})$, for all $1 \le j \le t$.

***Result* 1.4**

For non-constant $\alpha \in OP_n$, α is an order-preserving mapping if and only if $1\alpha < n\alpha$.

2.0 Subgroup of orientation preserving mapping

We will use the following results, adapted to the subgroup of Orientation preserving bijective mapping case, which is easily proved.

We list some known results which may be found in [12], [14] that we shall need later.

***Proposition* 2.1**

Any restriction of a member of OPD_n is also a member of OPD_n.

***Proposition* 2.2**

Let $\alpha \in OPD_n$ and let $(a_1 \mathrm{K} \ a_m)$, $m \geq 1$ be any cyclic sequence of members of X_n, then the sequence $(a_1\alpha \mathrm{K} \ a_m\alpha)$ is also cyclic. Similarly $((a_1\alpha)\alpha \mathrm{K} \ (\alpha_m\alpha)\alpha)$ is cyclic.

***Proposition* 2.3 [11, *Lemma* 4.8]**

Let $\alpha \in OPD_n$ .Then the digraph of α cannot have a non-trivial cycle and a fixed point.

***Proposition* 2.4 [11, *Lemma* 4.9]**

Let $\alpha \in OPD_n$. Then the digraph of α cannot have two cycles of different length.

3.0 Even (odd) permutations

We turn our attention to the number of even (odd) permutation of X_n, n $-$even (odd).

***Lemma* 3.1**

The set of all $\alpha \in OPD_n$ is a cyclic subgroup of D_n.

Proof

Every subgroup of a cyclic group of order less than or equal to the order of the group. $\alpha = (a_1 \ \ a_2 \Lambda \ \ a_n)$ is a cyclic subgroup of X_n, and since $\alpha \in OPD_n$ then by Proposition 2.2, the sequence $(a_1\alpha \quad a_2\alpha \quad \Lambda \quad a_m\alpha)$ is cyclic and if $\tau \in OPD_n$ then $(a_1\alpha\tau \ \ a_2\alpha\tau \Lambda \ \ a_m\alpha\tau)$ is cyclic.

***Lemma* 3.2**

Every $\alpha \in \mathrm{OPD}_n$ is either a derangement or an identity, $\quad f(e) = n,\ \ f(\alpha) = 0$

Proof

It is clear from Proposition 2.4 that the digraph of α cannot have two cycles of different length and Proposition 2.3 implies the result.

***Lemma* 3.3**

If n is odd, the set of all $\alpha \in \mathrm{OPD}_n$ forms a cyclic subgroup of A_n of order n .

Proof

$A_n = \{\alpha \in S_n |\ \ \alpha \ \ is \ \ even\}$, since every, $\alpha(\neq e) \in OPD_n$ is a derangement, and α is of odd length. Then every permutation of odd length is even and a product of even or odd number of even permutations is even. Hence OPD_n is a set of even permutations and Lemma 3.1 implies the result.

***Theorem* 3.4**

If n is even, there are exactly $\frac{n}{2}$ even permutations and exactly $\frac{n}{2}$ odd permutations.

Proof

Every $\alpha_n^m \in OPD_n$ $1 \leq m \leq n-1$ is defined as,

$$\alpha_n^m = \prod_{i-1}^{\frac{n}{2}} (i,\ n-i+1) = (1\ 1+m\ \ \ 1+2m\ \Lambda\ \ 1+n-m)$$

Let $|T_k|$ be the length of one of the cycles of α_n^k and $|a^k|$ be the number of disjoint cycles in α_n^k. If n is even we first consider even values of m, $m = n = 2k$ .and then carry out the induction process of the proof.

First consider $m = n = 2k$, we have,

Case I

$$m = n = 2k$$

$$\alpha^n = (1\ 1+n\ \ 2+n\ \ 3+n\,\Lambda\ \ 1+n-n) = (1),$$

implies that α^n has n fixed points.

Case II

$$\alpha^{n-2} = (1\ \ n-1\ \ n-3\ \ n-5\,\Lambda\ \ n-(n-3)=3)(2\ \ n\ \ n-2\ \ n-4\,\Lambda\ \ n-(n-4)=4)$$
$$\Lambda\ \ (k\ \ k+n\ \ (k+2n)+n\ \ (k+n)+2n\ \Lambda\ \ n-(n-k)-2=k+2)$$

To determine the nature of the permutation α^{n-2}, we only need to determine the length of one of the cycles in the product of disjoint cycles of α^{n-2}

Now, let

$$T_1 = (1\ \ n-1\ \ n-3\ \ n-4\,\Lambda\ \ (n-(n-3)))$$

be one of the cycles of α^{n-2}.

$$|T_1| = \frac{n}{n-(n-3)-1} = \frac{n}{2}$$

Since by Proposition 2.4, any $\alpha^m \in OPD_n$ cannot have two cycles of different length, we obtain that $|T_1| = \frac{n}{2}$. By the same Proposition , we can only have $n \left| \frac{n}{2} \right.$ cycles each of length $\frac{n}{2}$, which is a product of even number of odd (even) length cycles. Hence α^{n-2} is a product of even number of even (odd) length cycles.

Case III

$$m = n-4$$

$\alpha^{n-4} = (1\ \ n-3\ \ n-7\ \ n-11\,\Lambda\ \ \ n-(n-5))\,\Lambda\ \ (4\ \ n\ \ n-4\,\Lambda\ \ \ n-(n-8))$. Then the length of one of the cycles, says $T_1 = (1\ \ n-3\,\Lambda\ \ \ n-(n-5))$ of the permutation α^{n-4} is $\frac{n}{4}$. By a similar argument as in case 1, we have $|\alpha^{n-4}| = 4,$ Thus for any value of n the permutation α^{n-4} is a product of four even numbers of even (odd) length cycles. Hence α^{n-4} is an even permutation for $\frac{n}{4}$ even (odd).

Case IV

We now consider a general case for $m - n - m_k$

$$m_k = \{2, 4, \Kappa , n-2\}$$

$$\alpha^{n-m_k} = \prod_{i=1}^{m_k} (i \; n-m_k+i \; n-2m_k+i \; n-3m_k+i \; \Lambda \; m_k+i)$$

Denote one of the cycles of α^{n-m_k} by T_K,

$$T_K = (1 \; n-m_k+1 \; n-2m_k+1 \; n-3m_k+1 \Lambda \; m_k+1)$$

such that the length of T_k is $|T_k| = \frac{n}{m_k}$. By similar argument as in cases I-III, for $\frac{n}{m_k}$ even (odd), the permutation α^{n-m_k} is a product of m_k (an even number) of even (odd) length cycles.

It is clear that for $n = 4k$ there is $\frac{n}{2}$ even numbers and $\frac{n}{2}$ odd numbers. We conclude from Cases I - IV, that if $n = 4k$ and $m = 2k$, then there are $\frac{n}{2}$ even permutations.

We now pay attention to the remaining $\frac{n}{2}$ permutations. By a similar argument as in the case of $m=2k$ we consider, $m = n - m_\tau$, m_τ is an odd number, $m_\tau = \{1, 3, \Kappa , n-1\}$ such that for any cycle, say, T_τ of α^{n-m_τ} we have $|T_k| = \frac{n}{m_\tau}$. Since n is even and m_τ is odd we consider two cases:

Case I

m_τ does not divide, n.

Then α^{n-m_τ} is a cyclic permutation of length n, $n -$ even.

Case II

m_τ divides n

Let $n = m_\tau d$ .Since n is even and m_τ is odd, and then it is clear that d is an even number. α^{n-m_τ} is a product of m τ cycles each of length d, is a product of odd number of even length cycle.

Finally we conclude that if n is even, then for any value of m satisfying Cases I and II α^m is an odd permutation, and there are $\frac{n}{2}$ m_τ`s in n.

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