

## On non-commutative $L_p$ spaces over a quasilocal Von Neumann algebra

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### *Abstract*

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*This paper considers operators of the  $\rho_{\wedge}^{\frac{1-t}{2}} \cdot \rho_{\wedge}^{\frac{1-t}{2}}$  in the context of non-commutative Integration, and construct interpolating family of  $L_p$  spaces over a quasilocal von Neumann algebra generated by such operators. We defined a norm and showed that the norm satisfy the Holder's and Minkowski inequality.*

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**Keywords:** Quasilocal Von Neumann, non commutative  $L_p$  spaces, density matrix.

### 1.0 Introduction

The construction of a non-commutative  $L_p$  space was carried out in [3, 4, 10, 12, 13]. The study of classical Markov Semi groups, their construction and ergodicity uses the analysis of the Interpolation family of an  $L_p$  space associated to a probability measure. In this paper we consider the interpolating family of  $L_p$  spaces in the non-commutative context associated to a quantum Gibbs state on a quasilocal von Neumann algebra. By using the thermodynamic limit of the density matrix and normalized trace, we define a norm and show that the Holder and Minkowski inequality hold with respect to our defined norm.

### 2.0 Preliminaries

Let  $H$  be a Hilbert Space,  $B(H)$  the algebra of all bounded linear operators on  $H$ . A von Neumann algebra is a \*-subalgebra  $M$  of  $B(H)$  which is self-adjoint, and it contains the identity operator  $1$  and is closed in the weak operator topology. Let  $M_+$  denote the positive elements of  $M$ . A linear map  $\varphi$  on  $M_+$  defined by  $\varphi: M_+ \rightarrow [0, \infty]$  satisfying

- (i)  $\varphi(x+y) = \varphi(x) + \varphi(y)$  for  $x, y \in M_+$
- (ii)  $\varphi(\lambda x) = \lambda \varphi(x)$  for  $x \in M_+, \lambda \geq 0$ , is called a weight.
- (iii) If  $\varphi(x) = 0 \Rightarrow x = 0$ , then the weight  $\varphi$  is said to be faithful
- (iv) If  $\varphi(x) = \text{Sup}(x_i)$  whenever  $x$  is the ( $\sigma$ -strong \*) limit of a monotone increasing net  $\{x_i\}$  in  $M_+$  then the weight  $\varphi$  is said to be normal.
- (v) If  $\varphi(x^* x) = \varphi(xx^*)$  then the weight  $\varphi$  is called a trace
- (vi)  $\forall x \in M_+ \exists x_i \in M$  with  $x_i \uparrow x$  ( $\sigma$ -strong \*) and  $\varphi(x_i) < \infty$ . Then the weight  $\varphi$  is semi-finite.

A state  $\varphi$  on  $M$  is a weight such that  $\|\varphi\| = 1$ . The Von Neumann algebra  $M$  is said to be a Quasi Local Algebra if it has a net  $\{M_\wedge\}$  of  $W^*$ -Subalgebras having the following properties,

- (i) If  $\wedge_1 \subset \wedge_2$  then  $M_{\wedge_1} \subset M_{\wedge_2}$

- (ii)  $M = \overline{\cup M_\wedge}$ , where the bar denote the uniform closure.
- (iii) The algebras  $M_\wedge$  have a common identity.

More detailed exposition is given in [2], [5], [8]. The construction is as follows: Let  $Z^d$  be a d-dimensional integer lattice and let  $\wedge \subset Z^d$  be a finite subset and consider the set

$$M_\wedge = \left\{ \rho_\wedge^{\frac{1-t}{2}} x \rho_\wedge^{\frac{1-t}{2}} : x \in M_+, t \in (0,1), p_t = \frac{1}{1-t} \right\}.$$

Then the density matrix  $\rho$  given by  $\rho \equiv \frac{\partial \phi}{\partial \tau}$  is a positive self-adjoint operator such that  $\text{Tr}\rho = 1$  and  $M_\wedge = \rho_\wedge^{\frac{1-t}{2}} M \rho_\wedge^{\frac{1-t}{2}}$ ,  $\forall x \in M_+$ . If  $\wedge_1 \leq \wedge_2$  then  $M_{\wedge_1} \subset M_{\wedge_2}$ , that is  $M_\wedge$  is increasing. We have the subsets  $M_\wedge$  as a  $\sigma$ -weakly closed von Neumann subalgebras of  $M$ . Also  $M = \overline{\cup M_\wedge}$ <sup>11-11</sup>. We therefore call  $M$  a Quasi local Von Neumann Algebra. Here our state is a locally normal state given by  $\phi_\wedge(x) = \text{Tr}_x(\rho_\wedge x)$ ;  $x \in M_\wedge$ , where  $\text{Tr}_x$  is a partial normalized traced. In this work we assumed that  $\rho_\wedge^{-1}$  are locally measurable operators and  $\phi$  is regular and locally finite in Trunov sense [11]. Hence the operators of the form  $\rho_\wedge^{\frac{1-t}{2}} \bullet \rho_\wedge^{\frac{1-t}{2}}$  are closed operators.

### 3.0 The construction

We begin by defining the spaces

$$L_\rho(\phi) \text{ and } L_\rho(\tau)$$

where  $\tau$  faithful normal semi-finite trace and  $\phi$  is a faithful normal state. Let  $M$  be a semi-finite von Neumann algebra. Then  $L_\rho(\tau)$  is the Banach space of measurable affiliated operators which are integrable in degree  $p$  with respect to  $\tau$   $L_\rho(\tau) = \{x \in M : |x|^\rho \in L_1\}$  with norm  $\|x\|_p^r = (\tau|x|^p)^{\frac{1}{p}}$ ,  $x \in L_p(\tau)$   $1 \leq p \leq \infty$  if  $\phi$  is a faithful normal state, then there exist a self-adjoint operator  $\rho \geq 0$  associated with  $M$  called a density operator such that  $\phi = \tau(p)$ . Then  $L_p(\phi)$  is the Banack space of measurable operators which are integrable in degree  $p$  with respect to  $\phi$ .  $L_\rho(\phi) = \{x \in M : \rho^{\frac{1}{2p}} x \rho^{\frac{1}{2p}} \in L_p(\tau)\}$  with norm  $\|x\|_\rho = \left\| \rho^{\frac{1}{2p}} x \rho^{\frac{1}{2p}} \right\|_p^{\frac{1}{p}}$ ,  $x \in L_\rho(\phi)$ . The Banach space  $L_\rho(\phi)$  is isometrically isomorphic to the Banach space  $L_\rho(\tau)$ . Define an isometric isomorphism  $J_\rho : L_\rho(\phi) \rightarrow L\rho(r)$  by  $J_\rho(x) = \rho^{\frac{1}{2p}} \rho x \rho^{\frac{1}{2p}}$  where  $\rho_\wedge$  is a non singular operator called the density matrix or the Radon-Nikodym derivative, for a finite volume  $\wedge \subset Z^d$ , satisfying the condition  $\text{Tr}\rho_\wedge = 1$  where  $\rho_\wedge = \rho_\wedge^{\frac{1-t}{2}} \rho \rho_\wedge^{\frac{1-t}{2}}$  is the density matrix for the local algebra  $M_\wedge$ . We begin the definition of the norm on  $L_\rho(\phi, \tau)$  as follows:

Let  $x_\wedge \in M_\wedge$  with  $x_\wedge \equiv \rho_\wedge^{\frac{1-t}{2}} x \rho_\wedge^{\frac{1-t}{2}}$

$$\|x_{\wedge}\|_{\rho}^{\varphi} = \|J_{\rho}^{-1} f_{xz}(z)\|_{\rho}^{\varphi} = \left\| \rho_{\wedge}^{\frac{xt}{2}} U_{x,t} \left| \rho_{\wedge}^{\frac{(1-t)}{2}} x \rho_{\wedge}^{\frac{(1-t)}{2}} \right|^{\frac{1-z}{1-t}} \rho_{\wedge}^{\frac{z}{2p}} \right\|_p^{\varphi} \quad (3.1)$$

We take  $z = 1/p + i s$  and  $p_t = \frac{1}{1-t} 0 < t < 1$ .

Then we will have:  $\|x_{\wedge}\|_{\rho}^{\varphi} = \left\| \rho_{\wedge}^{\frac{(1+p_{is})}{2p}} U_{x,t} \left| \rho_{\wedge}^{\frac{(1-t)}{2}} x \rho_{\wedge}^{\frac{(1-t)}{2}} \right| p(t-is) \rho_{\wedge}^{\frac{(1+p_{is})}{2p}} \right\|_p^{\varphi} \quad (3.2)$

$$\|x_{\wedge}\|_{\rho}^{\varphi} = \left( \text{Tr} \left| \rho_{\wedge}^{\frac{(1+p_{is})}{2p}} U_{x,t} \left| \rho_{\wedge}^{\frac{(1-t)}{2}} x \rho_{\wedge}^{\frac{(1-t)}{2}} \right| p(t-is) \rho_{\wedge}^{\frac{(1+p_{is})}{2p}} \right|^p \right)^{\frac{1}{p}} \quad (3.3)$$

We take  $Z = \frac{1}{p}$ ; then we will have for simplicity the norm

$$\|x_{\wedge}\|_{\rho}^{\varphi} = \left( \text{Tr} \left| \rho_{\wedge}^{\frac{1}{2p}} U_{x,t} \left| \rho_{\wedge}^{\frac{(1-t)}{2}} x \rho_{\wedge}^{\frac{(1-t)}{2}} \right|^{\frac{p}{q}} \rho_{\wedge}^{\frac{1}{2p}} \right|^p \right)^{\frac{1}{p}} \quad (3.4)$$

Now we claim that (3.4) is a norm, the closure of which gives our  $L_p(\varphi)$  spaces and the polar decomposition in place makes the element  $\rho_{\wedge}^{\frac{(1-t)}{2}} x \rho_{\wedge}^{\frac{(1-t)}{2}} \in L_p(\varphi)$  unique and well defined. Hence we have

$$L_p(\varphi, t) = \left\{ \rho_{\wedge}^{\frac{1-t}{2}} x \rho_{\wedge}^{\frac{1-t}{2}} \in M_{\wedge} : x \in M_+, \left| \rho_{\wedge}^{\frac{1-t}{2}} x \rho_{\wedge}^{\frac{1-t}{2}} \right|^{pt} \in L_1 \right\} \text{ and } 0 < t < 1 \quad (3.5)$$

And for  $p = 2$  we have

$$L_2(\varphi, t) = \left\{ \rho_{\wedge}^{\frac{1-t}{2}} x \rho_{\wedge}^{\frac{1-t}{2}} \in M_{\wedge} : x \in M_1, \text{Tr} \left| \rho_{\wedge}^{\frac{1}{2}} x \rho_{\wedge}^{\frac{1}{2}} \right|^{pt} \leq \infty \right\} \text{ - A Hilbert Space} \quad (3.6)$$

We are going to show the Holder inequality and the Minkowski inequality for this norm.

#### 4.0 Holder's inequality

For  $p, q \in [1, \infty]$  satisfying  $1/p + 1/q = 1$   $x_{\wedge} \in L_q$ ,  $y_{\wedge} \in L_p$  and  $x_{\wedge} y_{\wedge} \in L_1$  we have

$\|x_{\wedge} y_{\wedge}\|_{L_1}^{\varphi} \leq \|x_{\wedge}\|_{L_q}^{\varphi} \|y_{\wedge}\|_{L_p}^{\varphi}$ . Using the trace property and noting that  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\frac{1}{p} = 1-t$ ,  $\frac{1}{q} = t$ , we

have  $\|x_{\wedge} y_{\wedge}\|_{L_1} = \left| \text{Tr} \left( \rho_{\wedge}^{\frac{1}{2q} + \frac{1}{2p}} U_{x,t} \left| \rho_{\wedge}^{\frac{1}{2}} x^* \rho_{\wedge}^{\frac{1}{2}} y \right| \rho_{\wedge}^{\frac{1}{2p} + \frac{1}{2q}} \right) \right| \quad (4.1)$

$$= \left| \text{Tr} \left( \rho_{\wedge}^{\frac{1}{2q} + \frac{1}{2p}} U_{x,t} \left| \rho_{\wedge}^{\frac{(1-t)}{2} + \frac{t}{2}} x^* \rho_{\wedge}^{\frac{(1-t)}{2} + \frac{t}{2}} y \right| \rho_{\wedge}^{\frac{1}{2p} + \frac{1}{2q}} \right) \right| \quad (4.2)$$

$$= \left| \text{Tr} \left( \rho_{\wedge}^{\frac{1}{2q} + \frac{1}{2p}} U_{x,t} \left| \rho_{\wedge}^{\frac{(1-t)}{2}} \rho_{\wedge}^{\frac{t}{2}} x^* \rho_{\wedge}^{\frac{(1-t)}{2}} \rho_{\wedge}^{\frac{t}{2}} y \right| \rho_{\wedge}^{\frac{1}{2p} + \frac{1}{2q}} \right) \right| \quad (4.3)$$

$$= \left| Tr \left( \rho_{\wedge}^{\frac{1}{2q} + \frac{1}{2p}} U_{x,t} \left| \rho_{\wedge}^{\frac{(1-t)}{2}} \rho_{\wedge}^{\frac{t}{2}} x^* \rho_{\wedge}^{\frac{(1-t)}{2}} \rho_{\wedge}^{\frac{t}{2}} y \right| \rho_{\wedge}^{\frac{1}{2p} + \frac{1}{2q}} \right) \right| \quad (4.4)$$

$$= \left| Tr \left( \rho^{\frac{1}{2q} + \frac{1}{2p}} U_{x,t} \left| \rho^{\frac{t}{2}} x^* \rho^{\frac{t}{2}} \right| U_{x,t} \left| \rho^{\frac{(1-t)}{2}} y \rho^{\frac{(1-t)}{2}} \right| \right) \rho^{\frac{1}{2p} + \frac{1}{2q}} \right| \quad (4.5)$$

$$= \left| Tr \left( \rho^{\frac{1}{2q}} \left( U_{x,t} \left| \rho^{\frac{t}{2}} x^* \rho^{\frac{t}{2}} \right| \right) \rho^{\frac{1}{2q} + \frac{1}{2p}} \left| U_{x,t} \left| \rho^{\frac{(1-t)}{2}} y \rho^{\frac{(1-t)}{2}} \right| \right) \rho^{\frac{1}{2p}} \right| \quad (4.6)$$

$$= \left| Tr \left( \rho^{\frac{t}{2q}} U_{x,t} \left| \rho^{\frac{t}{2}} x^* \rho^{\frac{t}{2}} \right| \rho^{2q} \right) \right| \left| Tr \left( \rho^{\frac{1}{2p}} U_{x,t} \left| \rho^{\frac{(1-t)}{2}} y \rho^{\frac{(1-t)}{2}} \right| \rho^{\frac{1}{2p}} \right) \right| \quad (4.7)$$

$$\leq \left| Tr \left( \rho^{\frac{t}{2q}} U_{x,t} \left| \rho^{\frac{t}{2}} x^* \rho^{\frac{t}{2}} \right| \rho^{\frac{1}{2q}} \right) \right| \left| Tr \left( \rho^{\frac{1}{2p}} U_{x,t} \left| \rho^{\frac{(1-t)}{2}} y \rho^{\frac{(1-t)}{2}} \right| \rho^{\frac{1}{2p}} \right) \right| \quad (4.8)$$

$$\leq \left| Tr \left( \rho^{\frac{t}{2q}} U_{x,t} \left| \rho^{\frac{t}{2}} x \rho^{\frac{t}{2}} \right| \rho^{\frac{1}{2q}} \right) \right| \left| Tr \left( \rho^{\frac{1}{2p}} U_{x,t} \left| \rho^{\frac{(1-t)}{2}} y \rho^{\frac{(1-t)}{2}} \right| \rho^{\frac{1}{2p}} \right) \right| \quad (4.9)$$

$$\|x_{\wedge} y_{\wedge}\|_{L_r}^{\phi} \leq \|x_{\wedge}\|_{L_q} \|y_{\wedge}\|_{L_p} \quad (4.10)$$

## 5.0 Minkowski's inequality

To prove the inequality  $\|x_{\wedge} + y_{\wedge}\|_p^{\phi} \leq \|x_{\wedge}\|_p^{\phi} + \|y_{\wedge}\|_p^{\phi}$ . We follow Majewski and Zegarlinski Paper [4]

$$\|x_{\wedge} + y_{\wedge}\|_{\phi}^p = Tr \left( \left| \rho_{\wedge}^{\frac{1}{2p}} U_{x,t} \left| \rho_{\wedge}^{\frac{(1-t)}{2}} x \rho_{\wedge}^{\frac{(1-t)}{2}} \right|^{\frac{p}{q}} \rho_{\wedge}^{\frac{1}{2p}} \right| + \left| \rho_{\wedge}^{\frac{1}{2p}} U_{x,t} \left| \rho_{\wedge}^{\frac{(1-t)}{2}} y \rho_{\wedge}^{\frac{(1-t)}{2}} \right|^{\frac{p}{q}} \rho_{\wedge}^{\frac{1}{2p}} \right|^p \right)^p \quad (5.1)$$

For simplicity we let:  $A_x = U_{x,t} \left| \rho_{\wedge}^{\frac{(1-t)}{2}} x \rho_{\wedge}^{\frac{(1-t)}{2}} \right|^{\frac{p}{q}}$  (5.2)

$$B_y = U_{x,t} \left| \rho_{\wedge}^{\frac{(1-t)}{2}} y \rho_{\wedge}^{\frac{(1-t)}{2}} \right|^{\frac{p}{q}} \quad (5.3)$$

and so we have  $\|x_{\wedge} + y_{\wedge}\|_{L_p(\phi)}^{\phi} = Tr \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^p$  (5.4)

Note that we have  $A_x^* = U_{x,t} \left| \rho_{\wedge}^{\frac{(t-t)}{2}} x^* \rho_{\wedge}^{\frac{(t-t)}{2}} \right|^{\frac{p}{q}}$  (5.5)

$$B_y^* = U_{x,t} \left| \rho_{\wedge}^{\frac{(t-t)}{2}} y^* \rho_{\wedge}^{\frac{(t-t)}{2}} \right|^{\frac{p}{q}} \quad (5.6)$$

The  $\|x_{\wedge} + y_{\wedge}\|_{\phi}^p =$

$$= Tr \left| \left( \rho_{\wedge}^{\frac{1}{2p}} A_x^* \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y^* \rho_{\wedge}^{\frac{1}{2p}} \right) \left( \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right) \right| \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2} \quad (5.7)$$

$$= Tr \left| \left( \rho_{\wedge}^{\frac{1}{2p}} A_x^* \rho_{\wedge}^{\frac{1}{2p}} \right) \left( \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} \right) \right| \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2} \quad (5.8)$$

$$+ Tr \left| \left( \rho_{\wedge}^{\frac{1}{2p}} A_x^* \rho_{\wedge}^{\frac{1}{2p}} \right) \left( \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right) \right| \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2}$$

$$+ Tr \left| \left( \rho_{\wedge}^{\frac{1}{2p}} B_y^* \rho_{\wedge}^{\frac{1}{2p}} \right) \left( \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} \right) \right| \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2}$$

$$+ Tr \left| \left( \rho_{\wedge}^{\frac{1}{2p}} B_y^* \rho_{\wedge}^{\frac{1}{2p}} \right) \left( \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right) \right| \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2}$$

For  $p > 2$  we use the Holder's inequality on each term on the right of equation (5.8) – (5.11) for the first term.

$$\left( \rho_{\wedge}^{\frac{1}{2p}} A_x^* \rho_{\wedge}^{\frac{1}{2p}} \right) \left( \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} \right) \text{ and } \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2}$$

$$0 \leq Tr \left| \rho_{\wedge}^{\frac{1}{2p}} A_x^* \rho_{\wedge}^{\frac{1}{2p}} \right| \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} \right| \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2} \quad (5.9)$$

$$\leq \left( Tr \left| \left( \rho_{\wedge}^{\frac{1}{2p}} A_x^* \rho_{\wedge}^{\frac{1}{2p}} \right) \left( \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} \right) \right| \right) \left( Tr \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2} \right) \quad (5.10)$$

$$\leq \left( Tr \left| \left( \rho_{\wedge}^{\frac{1}{2p}+\frac{1}{2p}} A_x^* \right) \left( A_x \rho_{\wedge}^{\frac{1}{2p}+\frac{1}{2p}} \right) \right| \right) \left( Tr \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2} \right) \quad (5.11)$$

$$\leq Tr \left| \left( \rho_{\wedge}^{\frac{1}{p}} A_x^* A_x \rho_{\wedge}^{\frac{1}{p}} \right) \right| Tr \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2} \quad (5.12)$$

$$\leq Tr \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} \right|^{\frac{2}{p}} Tr \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2} \quad (5.13)$$

$$\leq Tr \left\| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} \right\|_{L_p}^2 \left\| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right\|_{L_p}^{p-2} \quad (5.14)$$

To show the second estimate on equation (5.8) we have

$$\leq \left| Tr \left| \left( \rho_{\wedge}^{\frac{1}{2p}} A_x^* \rho_{\wedge}^{\frac{1}{2p}} \right) \left( \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right) \right| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2} \right| \quad (5.15)$$

$$\leq \left| Tr \left[ \left| \left( \rho_{\wedge}^{\frac{1}{2p}} A_x^* \rho_{\wedge}^{\frac{1}{2p}} \right) \left( \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} \right) \right| + \left| \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right| \right]^{p-2} \right| \times \left| \left( \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right) \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2} \right| \quad (5.16)$$

$$\leq \left( Tr \left( \rho_{\wedge}^{\frac{1}{p}} A_x^* A_x \rho_{\wedge}^{\frac{1}{p}} \right) \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2} \right)^{\frac{1}{2}} \times \left( Tr \left( \rho_{\wedge}^{\frac{1}{p}} B_y^* B_y \rho_{\wedge}^{\frac{1}{p}} \right) \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2} \right)^{\frac{1}{2}} \quad (5.17)$$

Now, we use Holder's inequality for each term in (5.17). For the first term in we have

$$\leq \left( Tr \left( \rho_{\wedge}^{\frac{1}{p}} A_x^* A_x \rho_{\wedge}^{\frac{1}{p}} \right) \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2} \right)^{\frac{1}{2}} \quad (5.18)$$

$$\leq \left( Tr \left( \rho_{\wedge}^{\frac{1}{p}} A_x^* A_x \rho_{\wedge}^{\frac{1}{p}} \right)^{\frac{1}{2}} Tr \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2} \right)^{\frac{1}{2}} \quad (5.19)$$

$$\leq \left( Tr \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} \right|^{\frac{2}{p}} Tr \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{\frac{2}{p-2}} \right)^{\frac{1}{p}} \quad (5.20)$$

$$\leq \left\| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} \right\|_{Lp(\varphi)}^2 \left\| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right\|_{Lo(\varphi)}^{p-2} \quad (5.21)$$

Likewise the second term is given by the same argument and we have

$$\leq \left( Tr \left( \rho_{\wedge}^{\frac{1}{p}} B_y^* B_y \rho_{\wedge}^{\frac{1}{p}} \right) \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2} \right)^{\frac{1}{2}} \quad (5.22)$$

$$\leq \left\| \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right\|_{Lp(\varphi)}^2 \left\| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right\|_{Lo(\varphi)}^{p-2} \quad (5.23)$$

Therefore we have

$$\leq \left( Tr \left( \rho_{\wedge}^{\frac{1}{2p}} A_x^* \rho_{\wedge}^{\frac{1}{2p}} \right) \left| \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right| \left| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right|^{p-2} \right)^{\frac{1}{2}} \quad (5.24)$$

$$\leq \left\| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} \right\|_{Lp(\varphi)}^2 \left\| \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right\|_{Lp(\varphi)}^2 \left\| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right\|_{Lo(\varphi)}^{p-2} \quad (5.25)$$

Hence we repeat the same argument for the two remaining terms in combining all, gives us:

$$\begin{aligned} \|x_{\wedge} + y_{\wedge}\| &\leq \left\| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} \right\|^2 \left\| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right\|^{p-2} \\ &+ \left\| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} \right\|^2 \left\| \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right\|^2 \left\| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right\|^{p-2} + \left\| \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right\|^2 \left\| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right\|^{p-2} \end{aligned} \quad (5.26)$$

$$\text{Hence } \|x_{\wedge} + y_{\wedge}\|_p^{\varphi} = \left\| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right\|_p^p \quad (5.27)$$

$$\leq \left( \left\| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} \right\|_{\varphi} + \left\| \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right\|_{\varphi} \right)^2 \left\| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right\|_{\varphi}^{p-2} \quad (5.28)$$

$$\left\| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} + \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right\|_{Lp(\varphi)} \leq \left\| \rho_{\wedge}^{\frac{1}{2p}} A_x \rho_{\wedge}^{\frac{1}{2p}} \right\|_{Lp(\varphi)} + \left\| \rho_{\wedge}^{\frac{1}{2p}} B_y \rho_{\wedge}^{\frac{1}{2p}} \right\|_{Lp(\varphi)} \quad (5.29)$$

$$\|x_{\wedge} + y_{\wedge}\|_p^{\varphi} \leq \|x_{\wedge}\|_p^{\varphi} + \|y_{\wedge}\|_p^{\varphi} \quad (5.30)$$

## 6.0 Conclusion

The  $L_p$  spaces we have constructed are quite useful in studying generalized quantum conditional expectation for such operators which will later be useful in the construction and study of their dynamics.

### References

- [1] Averton, W. (1976), An invitation to  $C^*$ -algebra, Springer-Verlage, Berlin.
- [2] Bratteli, O. and Robison, D. (1979), Operator algebra and quantum statistical mechanics I, Springer Verlage, Berlin.
- [3] Majewski, A. and Zegarlinski, B. (1995), Quantum stochastic dynamics and noncommutative  $L_p$  Space, Lett. Math. Phy.
- [4] Majewski, A. and Zegarlinski, B. (1995), Quantum stochastic dynamics I: Spin system on a lattice, MPEJ 1.
- [5] Sakai, S. (1971),  $C^*$ algebra and  $W^*$ -algebra, Springer Verlag, Berlin.
- [6] Segal, I. E. (1958) A noncommutative extension of abstract integration, Ann. of Math. 57.
- [7] Sherstnev, A. N. and Trunov, N.V. (1978), On general theory of integration on operator algebra with respect to weight I, II, IZV, VUZ. Soviet Mathematics, 22, pp 79 – 98.
- [8] Sunder, V. S. (1987), An invitation to von. Neumann algebra. Springer-Verlag, Berlin.
- [9] Takesaki, M. (1979), Theory of operator algebra I, Springer-Verlag Benin.
- [10] Trunov, N. V. (1979), On noncommutative analogue of  $L_p$  space. IZV, VUZ. Soviet Mathematics, 26 pp. 66-97.
- [11] Trunov, N. V. (1982), On the theory of normal weights on Von Neumann algebras, IZV, VUZ. Soviet Mathematics, 26 pp. 61 – 70
- [12] Yeadon, F. J. (1975), Noncommutative  $L_p$  spaces, Maths. Proc. Camb. Philos. Soc. 77, pp 91 – 102.
- [13] Zolotarev, A. A. (1978)  $L_p$  spaces with respect to state on a Von Neumann algebras, IZV, VUZ. Soviet Mathematics, 26, pp 79 – 88.