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Mathematical modelling of uniform flow in three open channels

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Abstract

The sole aim of this work is to develop a mathematical model for dredging (excavating) three open channel sections, namely, the circular, parabolic and trapezoidal sections using the conditions for best hydraulic performance for the channels. Applying the model to a numerical example, new dimensions of the new channel for the three channel sections are determined and compared with the original ones, if the cross sectional area, bed slope and Manning's friction factor remain unchanged for each channel and if the side slopes are also stable with respect to the trapezoidal channel. Furthermore, a combination of our model with Darcy's formula can provide an alternative method for comparing the hydraulic performances of the three channel sections.

1.0 Introduction

An open channel is a conduit for flow with a free surface, e.g. canals, rivers and pipes which are not running full. The pressure at the surface is constant, usually atmospheric. Hence the flow is not due to pressure differences along the channel, but is caused by differences in the potential energy head due to the slope of the channel (Chow [1]). Various investigations have been done in open channel flows. For instance, in studying flow in a channel with a slot in the bed, Nasser et al. [2] tried to provide an insight into some aspects of spatially varied open channel flow. Other investigators include, notably, Bradley and Peterka [3], Repogle [4], Rand [5], Pnueli and Pekerlis[6],Baddour and Abbink [7],Scott-Moncrieff 8], Eyo [9], Khurmi [10], etc.

In this work the flow is assumed to be uniform and steady. Mathematical model governing the excavation of circular, parabolic and trapezoidal channels is developed and the three channels sections are compared. From the numerical results, for a channel flow problem, it is noticed that the parabolic section is hydraulically and economically the most effective section, followed by the circular and trapezoidal sections.

2.0 Conditions of hydraulic performance

2.1 Circular section of a channel

Consider a circular section of a channel shown in Figure 1 with the free surface subtending an angle 2θ at the centre O. Let h be the depth of flow and r the radius of the circular section.

The cross sectional area of the channel is

$$A_c = r^2(\theta - \tfrac{1}{2}\sin 2\theta) \quad (2.1)$$

where the subscript '*c*' refers to the circular channel

The wetted perimeter is $P = 2\,r\,\theta$ (2.2)

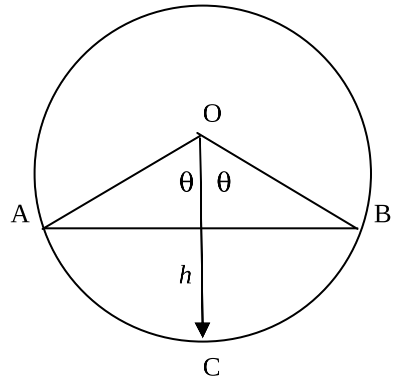

Figure 1: Circular section of an open channel

Therefore the hydraulic mean depth is

$$M = \frac{A_c}{P} = r^2\,\frac{\theta - \frac{1}{2}\sin 2\theta}{2r\theta} \quad (2.3)$$

For effective hydraulic performance the velocity u in Chezy's law [1]

$$u = C\sqrt{(MS_0)} = C\left(\frac{A_c}{P}S_0\right)^{\frac{1}{2}} \quad (2.4)$$

must be maximum where C = Chezy's coefficient, S_0 = bed slope, M, P and A_c are as above. This maximum requires

$$\frac{d}{d\theta}\left(\frac{A_c}{P}\right) = 0 \quad (2.5)$$

Thus differentiating (2.3) with respect to θ we find

$$\frac{2r\theta r^2\left(1-\cos^2\theta\right) - r^2\left(\theta - \frac{1}{2}\sin 2\theta\right)2r}{4r^2\theta^2} = 0$$

which on simplification gives $2\theta = \tan 2\theta$ (2.6)

The solution of this equation gives $2\theta = 257.5^0$ (2.7)

Therefore $\theta = 128.75^0$ (2.8)

or $\theta = 2.2474\ rad.$ (2.9)

Here the notation 'rad.' = radians, Depth of flow (see Figure 1) is

$$h = r - r\ \mathrm{os}\theta \quad (2.10)$$

substituting (2.9) into (2.2) we obtain

$$P_{effective} = 4.4948r \quad (2.11)$$

as a condition for best hydraulic performance for the circular channel. Similarly, substituting (2.7) and (2.9) into (2.3) we find $M_{effective} = 0.6085r$ (2.12)

while substitution of (2.8) into (2.10) yields $h_{effective} = 1.6259r$ (2.13) as other conditions for effective hydraulic performance

2.2 Parabolic section

The cross sectional area of flow is (see Figure 2)

$$A_p = \frac{2Bh}{3} \qquad (2.14)$$

where B = top width and h is above. The subscript 'P' denotes the parabolic channel (Figure 2 here).

The wetted perimeter is $$P = B + \frac{8h^2}{3B} \qquad (2.15)$$

The hydraulic mean depth is

$$M = \frac{A_p}{P} = \frac{\frac{2}{3}Bh}{B+\frac{8h^2}{3B}} = \frac{2B^2h}{3B^2+8h^2} \qquad (2.16)$$

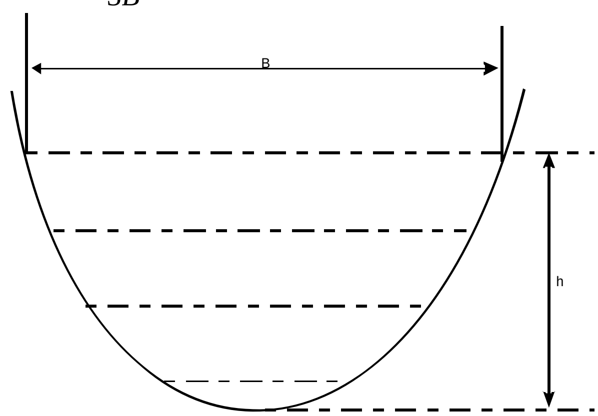

Figure 2. Parabolic section of an open channel

From (2.14) $$\mathrm{B} = \frac{3A_p}{2h} \qquad (2.17)$$

Substituting (2.17) into (2.16) we have

$$M = \frac{18A_p^2h}{27A_p^2 + 32h^4} \qquad (2.18)$$

For efficient performance the hydraulic mean depth must be maximum for a given value of A_p. This requires

$$\frac{dM}{dh} = 0 \qquad (2.19)$$

that is, $\frac{\left(27A_p^2 + 32h^4\right)18A_p^2 - 18A_p^2h128h^3 = 0}{\left(27A_p^2 + 32h^4\right)^2}$. Therefore $$A_p^2 = \frac{32h^4}{9} \qquad (2.20)$$

Substituting (2.20) into (2.18) for conditions of efficient hydraulic performance we find

$$M_{\text{efficient}} = \frac{h}{2} \qquad (2.21)$$

for parabolic channel. Substituting (2.20) into (2.17) we obtain another condition

$$\frac{B}{h} = 2\sqrt{2} \tag{2.22}$$

which determines the relationship of top width to depth of flow.

2.3 Trapezoidal section of a channel

Consider a trapezoidal section of a channel shown in Figure 3 with side slopes of 1 vertical to k horizontal. Let *h* be the depth of flow and *b* the bottom width. The cross sectional area of the trapezoidal channel is

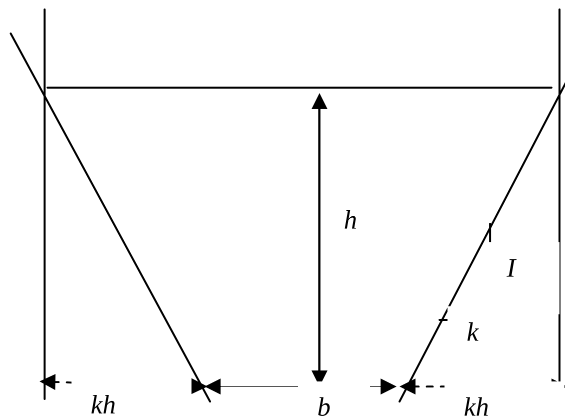

Figure 3: Trapezoidal section of an open channel

$$A_T = h(b + kh) \tag{2.23}$$

Here the subscript 'T' refers to the trapezoidal channel. The wetted perimeter is

$$P = b + 2h(1 + k^2)^{½} \tag{2.24}$$

Therefore the hydraulic mean depth is

$$M = \frac{A_T}{P} = \frac{h(b + kh)}{b + 2h(1 + k^2)^{\frac{1}{2}}} \tag{2.25}$$

From (2.23) we find $$b = \frac{A_T}{h} = kh \tag{2.26}$$

Substituting (26) into (25) we have $$M = \frac{A_T h}{A_T + \alpha h^2} \tag{2.27}$$

where $$\alpha = 2(1 + k^2)^{½} - k \tag{2.28}$$

For best hydraulic performance the hydraulic mean depth *M* must be maximum for a given value of A_T.

This requires $$\frac{dM}{dh} = 0 \tag{2.29}$$

i.e. $$\frac{A_T\left(A_T + \alpha h^2\right) - 2\alpha h\left(A_T h\right)}{\left(A_T + \alpha h^2\right)^2} = 0$$

where $$A_T = \alpha h^2 \tag{2.30}$$

Substituting (2.30) into (2.27) for conditions for best hydraulic performance we find

$$M_{\text{best}} = \frac{h}{2} \tag{2.31}$$

Combining (2.26), (2.28) and (2.30) we obtain another condition

$$\frac{b}{h} = 2(1 + k^2)^{½} - 2k \tag{2.32}$$

which determines the relationship of bottom width to depth of flow.

3.0 Development of mathematical model for dredging the three open channels

Throughout, the two systems in dredging an open channel shall be denoted by the symbols 0 and N where

System 0 = original open channel (i.e. open channel before dredging)

System N = new open channel (i.e. open channel after dredging)

3.1 Mathematical model for circular channel

(i) *Determination of the new radius* r_N

For constancy in the cross sectional area, we have

$$r_N^2\left(\theta_N\,rad - \frac{1}{2}\sin 2\theta_N^0\right) = A_c = r_0\left(\theta_0^2\,rad - \frac{1}{2}\sin 2\theta_0^0\right) \tag{3.1}$$

whence
$$r_N = \left(\frac{A_c}{\theta_N\,rad - \frac{1}{2}\sin 2\theta_N^0}\right)^{\frac{1}{2}} \tag{3.2}$$

(ii) *Determination of the new wetted perimeter* P_N

From (2.11)
$$P_N = 4.4948 r_N \tag{3.3}$$

(iii) *Determination of the new hydraulic mean depth* M_N

From (2.12)
$$M_N = 0.6086 r_N \tag{3.4}$$

(iv) Determination of the new depth h_N

From (2.13)
$$h_N = 1.6259 r_N \tag{3.5}$$

(v) *Determination of the new discharge* Q_N

From Manning's formula [1]

$$Q = \frac{1}{n}A(M)^{\frac{2}{3}} S_0^{\frac{1}{2}} \tag{3.6}$$

where Q = discharge, n = roughness factor, S_0 = bed slope, A and M as above, we find

$$Q_N = \frac{1}{n}A_c(M_N)^{\frac{2}{3}} S_0^{\frac{1}{2}} \tag{3.7}$$

(vi) *Determination of the new mean velocity* u_N

From the relation
$$Q = Au \tag{3.8}$$

we obtain
$$Q_N = A_c u_N \tag{3.9}$$

so that
$$U_N = \frac{Q_N}{A_c} \tag{3.10}$$

Alternatively, from Manning's formula (3.6), we have

$$U_N = \frac{1}{n}(M_N)^{\frac{2}{3}} S_0^{\frac{1}{2}} \tag{3.11}$$

(*vii*) *Determination of* θ_N

From (2.8) $$2\theta_N^0 = 257 \cdot 5^0 \tag{3.12}$$

or, from (2.9) $$\theta_N rad = 2.2474 rad \tag{3.13}$$

3.2 Mathematical model for parabolic channel

(*viii*) *Determination of* h_N

From (2.22) $$B_N = h_N(2\sqrt{2}) \tag{3.14}$$

Since the cross sectional area is constant, we find

$$\frac{2}{3} B_N h_N = A_p = \frac{2}{3} B_0 h_0 \tag{3.15}$$

Combining (3.14) and (3.15) and simplifying we obtain $$h_N = \left(\frac{3A_p}{4\sqrt{2}}\right)^{\frac{1}{2}} \tag{3.16}$$

(ix) *Determination of* B_N

Substituting (3.16) into (3.14) we find $$B_N = \left(\frac{3A_p}{4\sqrt{2}}\right)^{\frac{1}{2}} 2\sqrt{2} \tag{3.17}$$

(x) *Determination of* P_N

From (2.15) $$P_N = B_N + \frac{8h_N^2}{3B_N} \tag{3.18}$$

(*xi*) *Determination of* M_N

From (2.21) $$M_N = \frac{h_N}{2} \tag{3.19}$$

(xii) *Determination of* Q_N

From (3.6) $$Q_N = \frac{1}{n} A_p \left(M_N\right)^{\frac{2}{3}} S_0^{\frac{1}{2}} \tag{3.20}$$

(xiii) *Determination* u_N

From (3.8) $$U_N = \frac{Q_N}{A_p} \tag{3.21}$$

3.3 Mathematical model for trapezoidal channel

(*xiv*) *Determination of* h_N

From (2.32) $$B_N = h_N \left(2\left(1+k^2\right)^{\frac{1}{2}} - 2k\right) \tag{3.22}$$

Since the cross sectional area is constant for both channels, we find

$$h_N(b_N + kh_N) = A_T = h_0(b_0 + kh_0) \tag{3.23}$$

Combining (3.22) and (3.23) and simplifying we obtain

$$h_N = \left(\frac{A_T}{2\left(1+k^2\right)^{\frac{1}{2}} - k}\right)^{\frac{1}{2}} \tag{3.24}$$

(xv) *Determination of b_N*

Substituting (3.24) into (3.22) we find $$b_N = \left(\frac{A_T}{2\left(1+k^2\right)^{\frac{1}{2}} - k} \right)^{\frac{1}{2}} \left(2\left(1+k^2\right)^{\frac{1}{2}} - 2k \right) \quad (3.25)$$

(xvi) *Determination of P_N*

From (2.24) $$P_N = b_N + 2h_N(1 + k^2)^{½} \quad (3.26)$$

(xvii) *Determination of M_N*

From (2.31) $$M_N = \frac{h_N}{2} \quad (3.27)$$

(xviii) *Determination of Q_N*

From (3.6) $$Q_N = \frac{1}{n} A_T \left(M_N\right)^{\frac{2}{3}} S_0^{\frac{1}{2}} \quad (3.28)$$

(xix) *Determination of u_N*

From (3.8) $$U_N = \frac{Q_N}{A_T} \quad (3.29)$$

The expressions (3.2), (3.3), (3.4), (3.5), (3.7), (3.10) or (3.11), (3.12) and (3.13) constitute the mathematical model for dredging the circular channel. For the parabolic channel the model is given by the expressions (3.16) – (3.21) while the expressions (3.23) – (3.29) constitute the model in respect of the trapezoidal channel.

4.0 Alternative method for comparing hydraulic performances of the channels

From Darcy's formular [1] the head loss h_f due to friction in an open channel is

$$h_f = \frac{fLu^2}{M2g} \quad (4.1)$$

Thus

(xx) *Head loss in the original channel $(h_f)_0$:*

$$\left(h_f\right)_0 = \frac{fL_0\left(u_0\right)^2}{M_0 2g} \quad (4.2)$$

(xxi) *Head loss in the new channel $(h_f)_N$:*

$$\left(h_f\right)_N = \frac{fL_N\left(u_N\right)^2}{M_N 2g} \quad (4.3)$$

(xxii) *Decrease in head loss due to dredging $(h_f)_0 – (h_f)_N$:*

$$\left(\frac{fL_0\left(u_0\right)^2}{M_0 2g} \right) - \left(\frac{fL_N\left(u_N\right)^2}{M_N 2g} \right) \quad (4.4)$$

Here the relations (4.2) – (4.4) constitute the alternative model for comparing the hydraulic performances of the three channel sections.

5.0 Numerical illustration

Consider, for example, a channel with bed slope 1 in 500, bottom width 20 m and conveying water at a depth of 5m, Manning's coefficient n is 0.012. Using the model we wish to determine, after dredging, the new dimensions of a channel to give the maximum discharge, the new discharge, the new mean velocity and the percentage decrease in head loss in

(a) Circular section of radius 20*m*
(b) Parabolic section of top width 20*m*
(c) Trapezoidal section of sides 1 vertical to 2 horizontal

5.1 Solution

5.1.1 Circular section

For the original channel:

$r_0 = 20m$, $h_0 = 5m$, $\theta_0 = 41^0 = 0.7156rad.$,
$n = 0.012$, $S_0 = {}^1/_{500}$, $A_c = 88.24m^2$, $P_0 = 28.62m$
$M_0 = 3.08m$, $Q_0 = 695.597m^3/\text{s}$, $u_0 = 7.883m/s$

The dimension r_N of the new (excavated) circular channel is obtained by substituting (3.12) and (3.13) and the value of A_c above into the model expression (3.2). This yields $r_N = 5.679m$. The parameters P_N, M_N, h_N of the new channel are determined respectively from the expressions (3.3), (3.4), (3.5) using the value of r_N. Thus we find $P_N = 25.528$, $M_N = 3.456m$, $h_N = 9.234m$. Other parameters of the new circular channel, namely, Q_N, u_N are computed respectively from the relations (3.7), (3.10) or (3.11) via appropriate substitution. The result is $Q_N = 751.157m^3/s$, $u_N = 8.512m/s$

5.2 Parabolic section

Here

$B_0 = 20m$, $h_0 = 5m$, $n = 0.012$, $S_0 = {}^1/_{500}$, $A_p = 66.666m^2$
$P_0 = 23.333\text{m}$, $M_0=2.857m$, $Q_0= 499.911m^3/s$, $u_0 = 7.497m/s$

for the original channel. For the new parabolic channel the parameters h_N and B_N are obtained respectively by substituting the value of A_P above into the model expressions (3.16) and (3.17). Thus, $h_N = 5.946m$, $B_N = 16.818m$. The remaining parameters, P_N, M_N, Q_N, u_N of the new parabolic channel are determined respectively from the expressions (3.18), (3.19), (3.20), (3.21) through appropriate substitution of the above data in these relations. This yields, $P_N = 22.424m$, $M_N = 2.973m$, $Q_N = 513.323$, $u_N = 7.699m/s$

5.3 Trapezoidal section

For the original channel

$b_0 = 20m$, $h_0 = 5m$, $k = 2$, $n = 0.012$, $S_0 = 1/500$, $A_T = 150m^2$,
$P_0 = 42.361m$, $M_0 = 3.541m$, $Q_0 = 1297.611m^3/s$, $u_0 = 8.651m/s$

The dimensions h_N and b_N of the excavated trapezoidal channel are obtained respectively by substituting the appropriate data above in the model expression (3.24) and (3.25). Thus, $h_N = 7.789m$ and $b_N = 3.677m$. Other parameters P_N, M_N, Q_N, u_N of the new trapezoidal channel are determined respectively from the model expressions (3.26), (3.27), (3.28), (3.29) through appropriate substitution. Thus, we find $P_N = 38.513m$, $M_N = 3.894m$, $Q_N = 1382.561m^3/s$, $u_N = 9.217m/s$

5.4 Application of the alternative method

From the alternative method (4.4) we obtain

(a) *Decrease in head loss in circular channel due to dredging:*

$$\left(h_f\right)_0 = \left(h_f\right)_N = \left(\frac{0\cdot 012\times 20}{3\cdot 08}\right)\left(\frac{(7\cdot 883)^2}{2\times 9\cdot 81}\right) - \left(\frac{0\cdot 012\times 5\cdot 679}{3\cdot 456}\right)\left(\frac{(8\cdot 512)^2}{2\times 9\cdot 81}\right) = 0\cdot 1739$$

Therefore percentage decrease in head loss in the circular channel = 70.49%

(b) *Decrease in head loss in parabolic channel due to dredging*

$$\left(h_f\right)_0 = \left(h_f\right)_N = \left(\frac{0\cdot012\times20}{2\cdot857}\right)\left(\frac{(7\cdot498)^2}{2\times9\cdot81}\right) - \left(\frac{0\cdot012\times16\cdot818}{2\cdot973}\right)\left(\frac{(7\cdot699)^2}{2\times9\cdot81}\right) = 0\cdot0356$$

Therefore percentage decrease in head loss in the parabolic channel = 14.79%

(c) *Decrease in head loss in trapezoidal channel due to dredging:*

$$\left(h_f\right)_0 = \left(h_f\right)_N = \left(\frac{0\cdot012\times20}{3\cdot541}\right)\left(\frac{(8\cdot651)^2}{2\times9\cdot81}\right) - \left(\frac{0\cdot012\times3\cdot677}{3\cdot894}\right)\left(\frac{(9\cdot217)^2}{2\times9\cdot81}\right) = 0\cdot2094$$

Therefore percentage decrease in head loss in the trapezoidal channel = 81.19%

6.0 Discussion and conclusion

Tables 1, 2 and 3 show respectively the results (corrected to two decimal places) of the analysis of the flow problem for the original and new (excavated) circular, parabolic and trapezoidal channels. The new dimensions for the new channel sections are also shown in the three Tables. For instance, the new dimension r_N for the circular section is displayed in Table 1. Table 2 shows the new dimensions h_N and B_N for the parabolic section, while Table 3 shows the new dimensions b_N, h_N in respect of the trapezoidal section. Furthermore the three Tables show that whereas for each channel the new depth, new wetted perimeter, new hydraulic mean depth, new discharge and the new mean velocity are greater than the original ones, the new width for each channel is lower than the original one. Comparison of the wetted perimeters of the three channel sections shows that the parabolic section has the minimum perimeter for a particular cross sectional area, and hence it becomes more effective hydraulically than the circular and trapezoidal sections. Moreover, the parabolic section is economically better than the other two sections because its wetted perimeter is minimum and this therefore results in minimum excavation and lining costs. From the results obtained from (4.4) we observe that the parabolic section has the smallest percentage decrease in head loss due to friction (see the three Tables). This percentage is in direct proportion to the wetted perimeter of each channel section. Thus, in view of this minimum percentage (and hence minimum wetted perimeter) the parabolic section still becomes (of the three sections) the most effective hydraulic section.

Finally, it is clear from the three Tables that the new channel is deeper than the original one as a result of dredging, and this therefore removes the danger of a ship grounding if it sails too fast. Besides, apart from economic cost due to dredging and lining, the new channel is much more effective hydraulically than the original one with the parabolic section the most effective.

Table 1: Result for circular channel

	Circular channel	
	Original channel	**New channel**
Bed slope	1/500	1/500
Manning's *n*	0.012	0.012
Angle θ	41^0	128.75^0
Radius *r*	20*m*	5.68*m*
Depth *h*	5*m*	9.23*m*
Area of cross section A_c	$88.24m^2$	$88.24m^2$
Wetted perimeter *P*	28.62*m*	25.53*m*
Hydraulic mean depth *M*	3.08*m*	3.45*m*
Discharge *Q*	$695.60m^3/s$	$751.16m^3/s$
Mean velocity *u*	7.88*m/s*	8.51*m/s*
Decrease in head loss	0.1739	
% Decrease in head loss	70.49%	

Table 2: Result for parabolic channel

	Parabolic channel	
	Original channel	**New channel**
Bed slope	1/500	1/500
Manning's n	0.012	0.012
Tope Width B	20m	16.82m
Depth h	5m	5.95m
Area of cross section A_p	66.67m^2	66.67m^2
Wetted perimeter P	23.33m	22.42m
Hydraulic mean depth M	2.86m	2.97m
Discharge Q	499.91m^3/s	513.32m^3/s
Mean velocity u	7.50m/s	7.70m/s
Decrease in head loss	0.0356	
% Decrease in head loss	14.79%	

Table 3: Result for Trapezoidal channel

	Trapezoidal channel	
	Original channel	**New channel**
Side slope	**1vertical to 2 horizontal**	**1vertical to 2 horizontal**
Bed slope	1/500	1/500
Manning's n	0.012	0.012
Width b	20m	3.67m
Depth h	5m	7.79m
Area of cross section A_T	150m^2	150m^2
Wetted perimeter P	42.36m	38.51m
Hydraulic mean depth M	3.54m	3.89m
Discharge Q	1297.61m^3/s	1382.56m^3/s
Mean velocity u	8.65m/s	9.21m/s
Decrease in head loss	0.2094	
% Decrease in head loss	81.19%	

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