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A mathematical model for the interception of a moving target: contribution to optimal controllability theory

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Abstract

In this study, a mathematical model for the interception of a target governed by a linear ordinary control system is derived. The condition for interception is stated. The interception criterion is the intersection of certain well defined set functions. The equivalent of the condition is controllability of the linear control system. This research has made its modest contribution to mathematical modeling as well as provided example of an optimal control problem.

***Keywords*:** Attainable set, Reachable set, target set, compact set function, controllability.

1.0 Introduction

Every life problem, be it social, economic psychological and technological has a mathematical dimension. The necessity to translate a real life problem into a mathematical model, therefore cannot be over-emphasised. Most of these models are governed by differential systems whose solutions provide clues leading to break-throughs to real life problems. Little wonder immense interest is on mathematical modeling. Population, economic, disease control and technological models abound in the literature (see [6] [8] [9]), from simple population models such as $dp/dt = kp$ (describing the rateof increase of population), we have more complex models governed by control systems such as $\dot{x}(t) = A(t)x(t) + B(t)u(t)$.

With the realization that most action in life are not instantaneous, - that is, causes do no produce their effects immediately there is therefore need to incorporate time delays in our models; giving rise to delay systems. These models are found in the study of nuclear reactor dynamics and technological dynamics where the decisions in the control function are often shifted or twisted before affecting the evolution. In the study of dynamics of diseases, Yorke [10] obtained the following model for the control of measles.

$$\dot{x}(t) = - B(t)x(t)[2\tau + x(t-14) - x(t-12) + \tau]$$

where $x(t)$ denotes the number of susceptible individuals that have not yet been expose to the disease; $h_1 = 12$, $h_2 = 14$ are delays. Quite recently, Chukwu in [1] [2] [3] and Onwuatu and Iheagwam [7] have provided economic models governed by neutral differential systems for the control of the capital stock of nations.

Models of pursuit games have motivated interest in the study of capture problems and rescue operations. Markus and Sell reported in Gahl [5] have obtained conditions under which a derelict spaceship drifting in some astronomical system could be saved by a rescue ship. The operation dynamics furnished a nonlinear equation given by

$$\dot{x}(t) = K(t, x(t), u(t)) \qquad (1.4)$$

where $x(t)$ is the state of the rescue ship and $u(t)$, the engine thrust. It was found that the onditions for rescue of the derelict ship collapsed to the controllability of system (1.4).

In military quarters where the interception of enemy advances is a common feature, (interception of enemy missiles and menacing aircrafts) the questions that readily come to mind are: What are the state and energy requirements of the weapon for interception operation? In a situation requiring pursuit, what is the state trajectory?

This study is a mathematical response to these questions.

2.0 Notation and preliminaries

Let E denote the real line. For a positive integer n, E^n denotes the space of real n tuples with the usual Euclidean norm | . | and $C([a, b],E^n)$ is the Banach space of continuous functions from the interval $[a, b]$ into E^n with the topology of uniform convergence. The norm of ϕ in $C([a, b], E^n)$ is given by

$$\| \phi \| = \sup_{a \leq x \leq b} | \phi (x) |$$

In this paper, the state space will be E^n or $C([-h, 0], E^n)$ the control space will be $L_2([0,\infty), E^n)$; the control set will be a closed and bounded subset U of L_2 with values in $C^m = \{u : u \in E^m ; |u_j| \leq 1, j = 1, 2, \ldots, m\}$

The target $G(t)$ may be a moving point set or a compact set function in the appropriate space.

Consider the system of interest,

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (2.1)$$

on $J = [0, \infty)$, $x(t_o) = x_o \in E^n$ or $\phi(t_o) \in E^n$.

A is an n x n constant matrix and B is an $m \times n$ constant matrix

Consider the homogeneous part of (2.1)

$$\dot{x}(t) = Ax(t) \qquad (2.2)$$

The solution is given by $x(t, \phi, 0)$, where $x(t, \phi, 0) = X(t, s)x(t_o)$. $X(t, s)$ is the fundamental matrix solution of (2.2). That is, $X(t, s)$ satisfies the equation.

$$\frac{\partial X}{\partial t}(t,s) = AX(t,s), \quad t \succ s$$

and

$$X(t,s) = \begin{cases} 0, & t \prec s \\ \mathrm{I}, & t = s \end{cases}$$

where I is identity matrix. The variation of constant formula for system (2.1) thus becomes

$$x(t) = X(t,t_0)x(t_0) + X(t,t_0)\int_{t_0}^{t} X(t_0,s)B(s)u(s)ds \qquad (2.3)$$

From (2.3), we extract the attainable set - the set of all possible solutions of system (2.1) given as

$$A(t) = \{x(t, u): u \in U\}$$

We shall show in the next theorem, that $A(t)$ is convex and bounded.

***Theorem* 2.1**

The Attainable set $A(t)$ is convex and compact.

Proof

The convexity of $A(t)$ follows trivially from the convexity of the control set U. To show that $A(t)$ is bounded; we let the set S to be a convex and compact subset of the space C of continuous functions. Since $x(t, \phi, 0)$ is continuous, $x(t, S, 0)$ is bounded. Also, since $X(t, t_o)$

$B(t)$ is integrable and $u(t)$ ε U, $A(t)$ is bounded in E^n. From the weak compactness argument in [3] and the compactness of S, it is clear that $A(t)$ is closed in E^n. Thus, the boundedness and closedness of $A(t)$ establishes its compactness. $\blacksquare$

***Definition* 2.1**

System (2.1) is Euclidean Controllable if there exists a control u ε U which can steer the solution $x(t)$ with $x(t_o) = x_o$ to $x(t_1) = x_1$ for x_1 ε E^n in finite time interval $[t_o, t_i j_; \ \ t_1 > t_o]$.

3.0 Main results

3.1 A mathematical model for the interception of a moving target

We shall in this section formulate a mathematical model for the interception of a moving target. Models of this type are expected to represent – pursuit games. They are even more relevant in military adventures, in the interception of bombs and missiles and menacing air crafts.

We shall state the basic assumptions of the model.

Let $G(t)$ be a moving target and $x(t)$, the pursuer's position at any time t. which is often referred to as the state. Let the distance between the target and the state be $D(t)$ for any time t. Assume $D(t)$ is decreasing with increasing time. That is, if $t_o \leq t_1 \leq t_2 \leq \ldots \leq t_n$ then $D(t_o) \geq D(t_1) \geq D(t_2) \geq \ldots \geq D(t_n)$.

Assume further that the rate of change of state, $x(t)$ varies as the distance $D(t)$. that is $\dot{x}(t) \alpha D(t)$.

This implies that $$\dot{x}(t) = kD(t); \tag{3.1}$$
k is the constant of variation. Define

$$D(t) = \max_{1 \leq i \leq n} |x_i(t) - G_i(t)| \tag{3.2.1}$$

where $$x(t) = \sum_{i=1}^{n} x_i(t) \text{ and } G(t) = \sum_{i=1}^{n} G_i(t)$$

From (3.1) and (3.2), we have $$\dot{x}(t) = k(x(t) - G(t)) \tag{3.3}$$

Let $u(t)$ be the control energy requirement for the pursuit. Let the amount of control energy needed to increase the speed of the state vary between 0 and 1. *i.e* $0 \leq u(t) \leq 1$, $u = 0$, when no energy is applied. Ofcourse there may be times when there will be the need for the application of brakes in the pursuit. This reverse operation places u(t) between –1 and 0. That is, $u(t)$ lies in the interval $-1 \leq u(t) \leq 0$. Evidently, $u(t)$ is defined on $-1 \leq u(t) \leq 1$ and so is an admissible control.

Incorporating the control energy into (3.3), we have

$$\dot{x}(t) = k(x(t) - G(t))u(t) \tag{3.4}$$

Describing the configuration of the distance between the state and the target by a family of curves $f(t, x(t), u(t))$; (3.4) becomes $$\dot{x}(t) = f(t, x(t), u(t)) \tag{3.5}$$
which is a non linear dynamics. However using the method in [9] where f is such that

$$\frac{\partial f}{\partial x} = Ax(t), \text{ where } A \text{ is an } n \times n \text{ matrix}$$

$$\frac{\partial f}{\partial u} = Bu(t), \text{ where } B \text{ is an } m \times m;$$

(3.5) becomes $$\dot{x}(t) = Ax(t) + Bu(t) \tag{3.6}$$
which is the model we are interested in.

3.2 State trajectories

We have however provided a model describing the state of the act. To obtain the configuration of the trajectories of the state of (3.6), we consider the homogeneous part of (3.6) given by

$$\dot{x}(t) = Ax(t) \tag{3.7}$$

The matrix function $X(t)$ such that $\frac{\partial X}{\partial t}(t,s) = AX(t,s)$, and $X(0) = I$ (identity matrix) is called the fundamental matrix solution of (3.6), and has the following exponential form $X(t) = e^{At}$ such that the solution of (3.7) has the representation.

$$x(t) = X(t)x_o = e^{At} x_o \tag{3.8}$$

where $x(t_o) = x_o$, a given initial vector.

The origin (0,0) is the only critical point under the assumption that matrix A is non-singular that is, the determinant $|A| \neq 0$. The graph of the vector equation (3.8) is the trajectory that starts at $x_o = (y_o, z_o)$. Because e^{At} can sometimes be complicated it may not be easy to picture such a trajectory. However, by similarity transformation of A, we obtain the diagonal matrix N. That is, we can find a non-singular matrix P such that $P^{-1} AP = N$. N is simpler than A but still preserves the basic properties of A. By the same similarity transformation, we have that $e^{At} = Pe^{At}P^{-1} = Pe^{Nt} P^{-1}$. The trajectory equation $x = e^{At}x_o$ thus becomes $P^{-1} x = e^{Nt}P^{-1}$. If we set $M = P^{-1}x$. Our equation becomes

$$M = e^{Nt}M_o. \tag{3.9}$$

The equation $M = P^{-1}x$ can be viewed as representing a fairly simple transformation of the XY plane to MX- plane. This transformation maps the trajectory $x = e^{At}x_o$ onto $M = P^{Nt} M_o$. The similarity transformation does not only preserve the basic properties of A but that of the system trajectory, thereby enabling us to obtain basic facts about the phase portrait of equation (3.6) through studying the phase portrait of equation.

$$\dot{x} = Nx \tag{3.10}$$

where $N = P^{-1}AP$.

Note that the diagonal elements of N are the eigenvalues of A. For a 2×2 matrix A, let these eigenvalues be λ_1, λ_2. They are non-zero since $|A| \neq 0$ and so we can write e^{Nt} as

$$e^{Nt} = \begin{bmatrix} e^{\lambda it} & 0 \\ 0 & e^{\lambda 2t} \end{bmatrix} \tag{3.11}$$

From here, we obtain in component form equation (3.9) as

$$x = e^{\lambda 1t} x_o,\ y = e^{\lambda 2t} y_o \tag{3.12}$$

With these as parametric equations, λ being the parameter, we can obtain possible trajectories of equation (3.6) vis-a-vis equation (3.9)

For an illustration, obtain equations for the state trajectory of the system

$$\dot{x} = Ax(t) \tag{3.13}$$

where $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\dot{x} = (x_1, x_2)$. Re-writing the equation, we have

$$\dot{x}_1 = x_1 \tag{3.13a}$$

$$\dot{x}_2 = x_1 + x_2. \tag{3.13b}$$

Let $x_1 = \alpha e^{\lambda x}$ and $x_2 = \beta e^{\lambda t}$ where α, β are constants

Evidently, $\lambda = 1$ (twice repeated) and so we have the parametric equations for the state trajectory as, $x_1 = 0$, $x_2 = e^t$, where $\alpha = 0$ and β set equal to 1.

Sometimes, certain equations yield themselves to direct integration, like equation (3.13) above. Integrating (3.13a), we have $\text{In}x_1 = t + c = x_1 = Ae^t$.

Assuming the solution of system (3.13a) passes through the point (x_0, y_0) at time t_0, we have $x_1 = x_0e^t$. Substituting this result in the integration of (3.13b) we have the problem of finding the solution of the resulting linear equation $\dot{x}_2 - x_2 = x_0e^t$ whose solution is given as $e^tx_2 = \int x_0 e^{2t}\, dx + B$, where B is a constant. With the initial condition, $x_2(0) = y_{0,}$ the solution becomes $x_2(t) = x_0e^t + (y_0 - x_0^{\,t})e^{-t}$. The state trajectory therefore, becomes the pair of equations.

$$x_1 = x_0 et,\ x_2 = \frac{x_0 e^t}{2} + (y_0 - x_0 t)e^{-t}$$

3.3 The question of controllability

System (3.6) comes into play, when we want the trajectories to follow a desired pattern or to reach a certain target. In this case, we commence a search for a control function capable of transforming the initial state $x(t_o)$ of system (3.6) to some desired final state x_f in finite time. This, of course raises the question of the controllability of system (3.6).

Consider, our system of interest $\dot{x}(t) = Ax(t) + Bu(t)$, where A is an $n \times n$ constant matrix and B is also a constant m x m matrix. For any non singular matrix P of order $n \times n$ let $x = Pz$, then z is also a state vector. Equation (3.6) can be re-written as

$$P\dot{z} = APz + Bu \text{ or as } \dot{z} = P^{-1}APz + P^{-1}Bu \qquad (3.14)$$

Set $N = P^{-1}AP$ and $M = P^{-1}B$ so that (3.14) becomes $\dot{z} = Nz + Mu$, where N is a diagonal matrix and can be written as $N = \text{diag}(\lambda_1, \lambda_2, .., \lambda_n)$ where λ_i are the eigen values of A. Let us consider the simple case where $m = n = 2$, then equation (3.14) takes the form

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} + \begin{pmatrix} b_{11}^1 & b_{12}^1 \\ b_{21}^1 & b_{22}^1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Simplifying, we have

$$\left.\begin{aligned} \dot{z}_1 &= \lambda_1 z_1 + b_{1u}^1 \\ \dot{z}_2 &= \lambda_2 z_2 + b_{2u}^1 \end{aligned}\right\} \qquad (3.15)$$

where $b^1_i = \max(b_{i1}, b_{i2})$ for each i. b_{ij} are components of B. It is seen from (3.15) that if b^1_i, the ith row of M has all zero components, then $\dot{z}_i = \lambda_i z_i + 0$ and the control function $u(t)$ has no influence on the ith mode of the system, in which case the mode is said to be uncontrollable. And a system having one or more such modes is uncontrollable. On the other hand, where all the modes are controllable, the system is said to be completely state controllable.

To obtain controllability criterion for the system under study, we make B in equation (3.6) a one column matrix, b, of course the result obtained using the column vector b holds for the more general case.

Equations (3.6) and (3.14) then becomes

$$\dot{x}(t) = Ax(t) + bu \text{ and } \dot{z}(t) = Nz + b_1 u, \text{ where } b_1 = p^{-1}b$$

Define $b_1 = [\beta_1, \beta_2, \beta_3 \cdots, \beta_n]^t$, t transpose and $Q_1 = [b_1, | Nb_1, | N^2 b_1 |, \cdots, N^{n-1} b_1]$

$$= \begin{pmatrix} \beta_1 & \lambda_1\beta_1, \cdots & \lambda_1^{n-1}\beta_1 \\ \beta_2 & \lambda_2\beta_2, \cdots & \lambda_2^{n-1}\beta_2 \\ \vdots & \vdots \quad \cdots & \vdots \\ \beta_n & \lambda_n\beta_n, \cdots & \lambda_n^{n-1}\beta_n \end{pmatrix}$$

Q_1 being a vandermonde matrix has all the columns linearly independent and is non-singular.

Recall that $b_1 = p^{-1}b,\ Nb_1 = p^{-1}Ab,\ N^2 b_1 = P^{-1}A^2 b \cdots N^{n-1} b_1 = P^{-1}A^{n-1}b$, so that

$Q = P^{-1}[b, | Ab |, \cdots, | A^{n-1}b |] = P^{-1}Q$ where

$$Q = [b, | Ab |, \cdots, | A^{n-1} b |] \quad (3.16)$$

From the sequel, system (3.6) is controllable if the components of Q are not zero. This means that Q is non-singular and so has n linearly independent columns. The rank of matrix Q is therefore n.

Clearly, system (3.6) is controllable if $Rank[b, | Ab |, \cdots, | A^{n-1} b |] = n$. In the general case, where the system is multivariate (B has many columns) $Q = [B, AB, \cdots, A^{n-1} B]$

$$RankQ = rank[B, AB, \cdots, A^{n-1} B] = n \quad (3.17)$$

provides a computable criterion for the controllability of system (3.6) credited to R E. Kalman.

3.4 Conditions for interception of a moving target.

We shall now state conditions under which it will be possible to intercept a moving target where the target is either a moving point function or a compact set function.

***Theorem* 3.1**

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (3.18)$$

where A is an $n \times n$ *constant matrix and B an* $m \times n$ *constant matrix. Suppose*

(1) *system* (3.18) *is controllable,*

(2) *the set functions, reachable set* $R(t_o, t_1)$, *Attainable set* $A(t_1, t_o)$ *and target set G are compact then* $A(t) \cap G(t) \neq \phi$. (*That is, the interception of the target* $G(t)$).

Proof

Let $G(t)$ be the target. We shall prove that there exists a control u, such that the state $x(t,u)$ ε $A(t)$ can be found $G(t)$ in finite time. Let u_n be a sequence in U. Since U is compact, $\lim_{n \to \infty} u_n = u$. Now $x(t, \phi, u_n)$ ε $A(t)$ and $x(t, \phi, u_n) = x(t, \phi, 0) + X(t, t_o) \int_{t_0}^{t} X(t_0, s) B(s) u_n(s) ds$

Taking limits both sides, we have $\lim_{n \to \infty} x(t, \phi, u_n) = x(t, \phi, 0) + X(t, t_0) \int_{t_0}^{t} X(t_0, s) B(s) \lim_{n \to \infty} u_n(s) ds$

$\Rightarrow x(t, \phi, u) = x(t, \phi, 0) + X(t, t_0) \int_{t_0}^{t} X(t_0, s) B(s) u(s) ds$, since $A(t)$ is assumed compact $x(t, \phi, u) = G(t)$ ε $A(t)$.

This shows that $A(t) \cap G(t) \neq \phi$. This completes the proof. ■

***Remark* 3.2**

From theorem, (3.1), it is evident that the required control function must be able to steer the state into the attainable set as well as the target set. That is the condition that $A(t_0, t_1) \cap G(t_1) \neq 0$.

4.0 Discussion of results

This study has not only provided a model for the pursuit and interception of a moving target, but has established conditions for the interception. It raises the following questions: what is the state trajectory to the target what is the choice of appropriate control function to steer the state of the system to the target what is the duration for the journey of meeting the target.

4.1 Determination of state trajectory

For quick and vivid understanding of the analysis, we illustrate using the following optimal control problem. Consider the system.

$\dot{x} = Ax(t) + Bu(t)$, $x(0) = (x_0, y_0)$, $\{u \in C^m: \|u\| \leq 1\}$ (4.1)

C^m is a unit cube in E^m, the m-dimensional Euclidean space. Where A and B are given by

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

First of all, we test the system for controllability using Kalman's criterion

$$rank[B, AB] = rank\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2$$

Since the computated rank is the same as the dimension of the state space E^2, we conclude that the system (4.1) is controllable on a finite interval $[t_0,t_1]$. (See [6]). This shows that any target can be intercepted in a context described by the control system. The next issue to be addressed is obtaining the control capable of steering the state to the target at the shortest possible time. i.e. the optimal control $u^* = sgn\eta^T (X^{-1}B)$; $\eta \in E^2$ where $X(t)$ is the fundamental matrix of the homogeneous part of system (4.1) given by

$$\dot{x} = Ax(t) \tag{4.2}$$

and X^{-1} (t) its inverse. That is, $X(t)$ is a matrix solution of (4.2) such that $X(0) = I$ (identity). The exponential characterization of $X(t)$ is e^{At}. Evidently

$$X(t) = e^{At} = \mathrm{I} + At + \frac{A^2t^2}{2!} + \cdots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ and } X^{-1}(t) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Clearly, (4.1) is normal since B is a column matrix, (see [6]) hence there exists optimal control that is unique and Bang-bang (maximum control power) as given below:

$$u* = \begin{cases} 1, \ if \ \eta^T X^{-1}B(T) \succ 0 \\ -1, \ if \ \eta^T X^{-1}B(T) \prec 0 \end{cases}$$

T is matrix transpose.

For some non-zero vector $\eta = (\eta_1 \eta_2)^T$. We can easily calculate

$$Sgn\left((\eta_1, \eta_2) \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \mathrm{sgn}(-\eta_1 t + \eta_2) \text{, hence } u* = \begin{cases} 1, \ for \ \mathrm{sgn}[(-\eta_1 t + \eta_2) \succ 0 \\ -1, \ for \ \mathrm{sgn}[(-\eta_1 t + \eta_2) \prec 0 \end{cases}$$

and has only one switch between –1 and 1. The switch time is obtained by equating $-\eta_1 t + \eta_2$ to zero to have $t = \eta_2/\eta_1$. Evidently, the optimal control exists and is unique and Bang-bang. There is no loss of generality in assuming that the target is the origin.

To obtain the optimal trajectory therefore, we re-state system (4.1) and with $u^* = 1$, we have

$$\frac{dx_1}{dt} = x_2 \text{ and } \frac{dx_2}{dt} = 1 \Rightarrow \frac{dx_2}{dx_1} = \frac{1}{x_2} \tag{4.2}$$

By the method of separation of variables, we have $\frac{x_2^2}{2} = x_1 + c$ since the solution passes through (x_0, y_0), we have

$$x_1 = \frac{x_2^2}{2} + x_0 - \frac{y_0^2}{2} \tag{4.3}$$

which is a parabola with x_2 increasing.

Taking the other value of the control, that is **u*** = -1, we have

$$\frac{dx_1}{dt} = x_2 ; \ \frac{dx_2}{dt} = -1 \Rightarrow \frac{dx_2}{dx_1} = \frac{-1}{x_2} \tag{4.4}$$

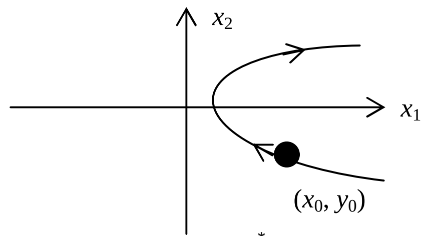

Figure 1: arc, C_1 (when u^* =1) increasing optimal trajectory.

By direct integration, we have $\frac{x_2^2}{2} = -x_1 + c$. Since this solution passes through the initial point (x_0, y_0), we have

$$x_1 = -\frac{1}{2}x_2^2 + (x_0 + \frac{1}{2}y_0^2) \qquad (4.5)$$

In this case x_2 is decreasing

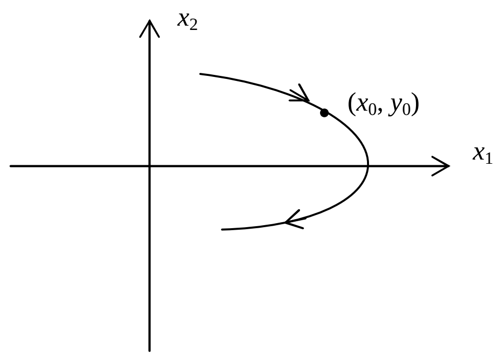

Figure 2: arc C_2 decreasing optimal trajectory

Bearing in mind that we have to reach the origin which is our target, we start a search for the optimal trajectory. We can construct the optimal trajectory as follows.

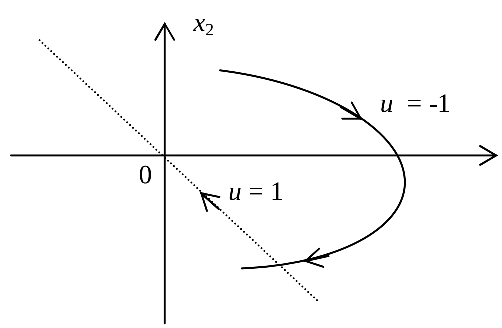

Figure 3: arc C_3 optimal trajectory

To lend concreteness to our: discussion here, let the initial point be (5, -1). That is, x_0 = 5, y_0 = -1, starting with the choice of control u^* = -1, from equation (4.4), we have $x_1 = \frac{1}{2}(11 - x_2^2)$ or $x_2 = -\sqrt{11 - 2x_1}$, since $x_2 < 0$. At the point of intersection, we change control to u = 1 and there

$x_2 = -\sqrt{2x_1}$ (since $\frac{dx_1}{dx_2} = x_2$ with the resulting curve passing through the origin) and so $-\sqrt{11-2x_1} = -\sqrt{2x_1} \Rightarrow 11-2x_1 = 2x_1 \Rightarrow 4x_1 = 11 \Rightarrow x_1 = \frac{11}{4}$ then $x_2 = -\sqrt{11-2(\frac{11}{4})} =$

$-\sqrt{\frac{22}{4}} = -\frac{1}{2}\sqrt{(22)}.$ From the point of intersection $(\frac{11}{4}, -\frac{1}{2}\sqrt{22}).$ of the two arcs, a straight line to the origin completes the pursuit.

The next and final question is: what is the total time taken for this pursuit to intercept the target? Solving the state equations (4.4), we have

$$x_2(t) = -t_1 + y_0 = \frac{1}{2}\sqrt{22} \text{ so, } t_1 = -\frac{1}{2}\sqrt{22} - 1 \text{ (since } y_0 = -1),$$

which implies $= \frac{1}{2}\sqrt{22} + 1$ (since time is positive.)

Similarly, after switching to $u = 1$ traversing the curve c_1

$$\dot{x}_1 = x_2, \ \dot{x}_2 = 1$$

from $(\frac{11}{4}, -\frac{1}{2}\sqrt{22}).$ to the origin (0, 0) and solving for x_2, we have

$$x_2(t) = t_2 - \frac{1}{2}\sqrt{22)}$$

with $x_2(t)$ now 0, we have

$$t_2 - \frac{1}{2}\sqrt{22} = 0 \Rightarrow t_2 = \frac{1}{2}\sqrt{22}.$$

The total time taken for the pursuit is

$$t_1 + t_2 = \left(\sqrt{\frac{22-2}{2}} + \frac{1}{2}\sqrt{22}\right) = \frac{\sqrt{22}+2+\sqrt{22}}{2} = \sqrt{22} + 1.$$

From the foregoing, the capture problem is an illuminating application of the theory of optimal controllability.

5.0 Conclusion

This study furnishes a model governed by an ordinary control system for the interception of a moving target. It is evident from the model that the primary concern in the pursuit for the interception of a target is the state of the weapon and the control energy requirement. The control energy should be such that has the potential of steering the state from its initial position through the phase portrait described by the moving target to reach it. This is communicated by the assumption that $A(t) \cap G(t) \neq \phi$. This condition in other words is the controllability of the system. Clearly (3.17) provides a computable criterion for the interception of a moving target in the context described by an ordinary differential autonomous system.

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