Journal of the Nigerian Association of Mathematical Physics
Volume **11 (November 2007), 571 - 576**

Euclidean null controllability of perturbed infinite delay systems with limited control

Davies Iyai
Department of Mathematics and Computer Science,
Rivers State University of Science and Technology,
Port Harcourt, Rivers State, Nigeria.
e-mail: davsdone@yahoo.com

Abstract

Sufficient conditions for the Euclidean null controllability of perturbed infinite delay systems with limited control are developed. The results are established by placing conditions on the perturbation function which guarantee that, if the linear control base system is completely Euclidean controllable, then the perturbed system is Euclidean null controllable with limited control.

Keywords: Controllability, delay systems, perturbation, Euclidean null controllability, properness.

1.0 Introduction

Due to the fact that actions and reactions take time to take effect in real-life problems, one often introduces time delays in the variables being modeled. This often yields delay differential and delay difference equations, which are special class of differential equation called functional differential equation. The study of functional differential equation has application in population dynamics, conveyor belts, metal rolling system, urban traffic and capacity management etc. Contributors in this field of study include Hale [11], Driver [9], Lakshmikantham [15], Aiello and Freedman [1].

Controllability problems for such linear and nonlinear delay models have been the subject of many investigations. In particular relative controllability of linear and nonlinear delay systems with limited and unlimited controls has been studied by Davies [7], Decka [8], Klamka [12, 13, and 14]. Others who reported researches are on null controllability of delay systems, they include Chukwu [3, 4], Eke [10].

However, for controllability of infinite delay systems, not much has been reported. Sinha [16], developed sufficient conditions for the null controllability of nonlinear infinite delay systems with restrained control. Balachandran and Dauer [2] developed sufficient conditions for the null controllability of a nonlinear infinite delay system with time varying multiple delays in the control. Davies [6] developed sufficient condition for the Euclidean controllability of infinite delay systems with limited control.

In this paper, we shall establish sufficient conditions for the Euclidean null controllability of perturbed infinite delay systems with limited control. This will extend the work in Davies [6] of the form (2.1) to its perturbation of the form (2.2). The research aims at showing that, the uniform asymptotic stability of the uncontrolled linear base system and the properness of the controlled linear base system guarantee the Euclidean null controllability with constraints of the perturbed system under certain conditions on f.

2.0 Basic notations and preliminaries

Let E denote the real line and $J = [t_0, t_1]$ an interval in E. For a positive integer n, we denote by E^n, the space of real n-tupples with the Euclidean norm denoted by $|.|$. Let $\gamma \geq h \geq 0$ be given real numbers (γ may be $+\infty$). The function η: $[-\gamma, 0] \to [0, \infty]$ is Lebesgue integrable on $[-\gamma, 0]$ positive and non-decreasing on $[-\gamma, 0]$. Let $B = B([-\gamma, 0], E^n)$ be the Banach space of functions which are continuous and bounded on $[-\gamma, 0]$, and such that $|\phi| = \sup_{\theta\in[-h,0]} |\phi(\theta)| + \int_{-\gamma}^{0} \eta(\theta)\phi(\theta)\,d\theta < \infty$, for any $x : [t-\gamma, t] \to E^n$,

Let x_t: $[-\gamma, 0]$, E^n be defined by $x_t(\theta) = x(t+\theta),\ \theta \in [-\gamma, 0]$. Let $W_2^{(1)}$ denote the Sobolev space $W_2^{(1)}([-h,0], E^n)$ of functions $\phi : [-h,0] \to E^n$ whose derivative are square integrable.

We consider the infinite delay system given as

$$\dot{x}(t) = L(t, x_t) + C(t)u(t) + \int_{-\infty}^{0} A(\theta)x(t+\theta)\,d\theta \tag{2.1}$$

and its perturbation

$$\dot{x}(t) = L(t, x_t) + C(t)u(t) + \int_{-\infty}^{0} A(\theta)x(t+\theta)\,d\theta + f(t, x(t), x(t-\tau), u(t), u(t-\tau))$$
$$x(t) = \phi(t),\ t \in (-\infty, 0] \tag{2.2}$$

where
$$L(t,\phi) = \sum_{K=0}^{N} A_k\,\phi(-t_k) \tag{2.3}$$

satisfied almost everywhere on $[t_0, t_1]$. $L(t,\phi)$ is continuous in t, linear in ϕ. A_k is a continuous $n \times n$ matrix function for $0 \leq t_k \leq \tau$, $A(\theta)$ is an $n \times n$ matrix whose elements are square integrable on $(-\infty, 0]$ and $C(t)$ is a continuous $n \times m$ matrix function. The n-vector function f is continuous and absolutely continuous. The controls u are square integrable with values in the unit cube $C^m = \{u \in E^m : |u_j| \leq 1,\ j = 1, \ldots, m\}$ The variation of constant formula for system (2.1) by Davies [6] and all its necessary assumption is

$$x(t,u) = X(t,t_0)\phi(0) + \int_{t_0}^{t_1} X(t,s)C(s)u(s)\,ds + \int_{t_0}^{t_1} X(t,s)\int_{-\infty}^{0} A(\theta)x(t+\theta)\,d\theta\,ds \tag{2.4}$$

where X satisfy the equation $\frac{\partial}{\partial t} X(t,s) = L(t, X_t(\cdot,s)),\ t \geq s$

$$X(t,s) = \begin{cases} 0, & s-\tau \leq t \leq s \\ I, & t = s \end{cases}$$

and $X_t(\cdot,s)(\theta) = X(t+\theta, s),\ -\tau \leq \theta \leq 0$.

The corresponding result for (2.2) at $t = t_1$ is given by

$$x(t_1,u,f) = X(t,t_0)\phi(0) + \int_{t_0}^{t_1} X(t,s)C(s)u(s)\,ds$$

$$+ \int_{t_0}^{t_1} X(t,s)\int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta\, ds \quad (2.5)$$

$$+ \int_{t_0}^{t_1} X(t,s) f(s,x(s),x(s-\tau),u(s),u(s-\tau)\,ds$$

For simplicity of notation let $Y(t,s) = X(t,s)C(s)$. We now give some definition upon which our study hinges

***Definition* 2.1**

The controllability matrix of system (2.2) *is given by* $W(t_1) = \int_{t_0}^{t_1} Y(t,s)Y^T(t,s)\,ds$ *where* T *denotes the matrix transpose.*

***Definition* 2.2**

The Reachable set of system (2.1) *is given by* $R(t_1,s) = \left\{ \int_{t_0}^{t_1} X(t,s)C(s)u(s)\,ds \right\}$.

***Definition* 2.3**

System (2.2) *is Euclidean controllable if for each* $\phi \in W_2^{(1)}$, $x_1 \in E^n$, *there exists a* $t_1 > t_0$ *and an admissible control* u *such that the solution* $x(t_1,t_0,\phi,u,f)$ of (2.2) *satisfies* $x_0(t_0,\phi,u,f) = \phi$ *and* $x(t_1,t_0,\phi,u,f) = x_1$

***Definition* 2.4**

The system (2.2) *is Euclidean null controllable if* $x_1 = 0$ *in Definition* 2.3.

3.0 Controllability results

Here, we give theorems which will summarize our result on Euclidean null controllability of the system (2.2).

***Proposition* 3.1**

The control system (2.1) *is proper in* E^n *on the interval* $[t_0,t_1]$ *if and only if* $rank\ \hat{Q}_n(t_1) = n.$

Proof

This is Theorem 2.1 of Davies [6] ■

***Proposition* 3.2**

System (2.1) is proper on $[t_0,t_1]$, $t_1 > t_0$ *if and only if the origin is an interior point of* $R(t_1)$.

■

Proof

This is Theorem 2.3 of Davies [6] ■

***Proposition* 3.3**

The system (2.3) *is completely Euclidean controllable on* $[t_0,t_1]$ *if and only if* W *is nonsingular.*

Proof

The proof can be observed from proposition 3.1 of Dauer and Gahl [5] ■

4.0 Main result of this paper

Theorem 3.1

In (2.2) *assume that*

(i) (2.1) (*with* $u = 0$) *is uniformly asymptotically stable*

(ii) (2.1) *is completely Euclidean controllable*

(iii) $f(t,\phi,0,0) = f_1(t,\phi) + f_2(t,\phi)$

and $\| f_1(t,\phi) \| \le \pi(t) | D(t,\phi) |$, $\| f_2(t,\phi) \| \le \in \| D(t,\phi) \|$ where $\Pi = \int_{t_0}^{\infty} \pi(t)dt < \infty$

(iv) $f(t,0,0,0) = 0$

Then the system (2.2) i*s Euclidean null controllable with constraints*

Proof

Suppose that the solution of (2.2) with $x_{t_0}(\cdot, t_0, u, f) = \phi$ satisfies $x(t_1, u, f) = 0$ for some $u \in U$, then by (2.5)

$$0 = X(t,t_0)\phi(0) + \int_{t_0}^{t_1} Y(t,s)u(s)\,ds + \int_{t_0}^{t_1} X(t,s) \int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta\, ds$$

$$+ \int_{t_0}^{t_1} X(t,s) f(s, x(s), x(s-\tau), u(s), u(s-\tau))\, ds$$

so that

$$X(t,t_0)\phi(0) = -\int_{t_0}^{t_1} Y(t,s)u(s)\,ds - \int_{t_0}^{t_1} X(t,s) \int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta\, ds$$

$$- \int_{t_0}^{t_1} X(t,s) f(s, x(s), x(s-\tau), u(s), u(s-\tau))\, ds$$

Recall the definition of $R(t_1, s)$ and now define

$$Y(t_1,1) = \left\{ -\int_{t_0}^{t_1} X(t,s) f(s, x(s), x(s-\tau), u(s), u(s-\tau))\, ds : u \in U \right\}$$

If we now set

$$V(t_1,1) = \left\{ \begin{array}{l} -\int_{t_0}^{t_1} Y(t,s)u(s)\,ds - \int_{t_0}^{t_1} X(t,s) \int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta\, ds \\ \quad -\int_{t_0}^{t_1} X(t,s) f(s, x(s), x(s-\tau), u(s), u(s-\tau))\, ds : \ u \in U \end{array} \right\}$$

Then $$V(t_1,1) \subseteq R(t_1,1) + Y(t_1,1)$$

By definition, the domain D of null controllability of (2.2) is the collection $\phi \in C$ of all initial functions such that, there exists t_1 and $u \in U$ such that the solution of (2.2) with $x_{t_0}(t_0, \phi, u, f) = \phi$ satisfies $x(t_1, t_0, \phi, u, f) = 0$. By (ii) and Proposition 3.2, $0 \in \operatorname{int} R(t_1,1)$ and so there is an open ball S such that $0 \in S \subseteq R(t_1,1)$. Hence $S + Y(t_1,1)$ is a ball around $Y(t_1,1)$.

Therefore, $0 \in Y(t_1,1) \subseteq \operatorname{int} V(t_1,1)$, for $t > 1$, so that $0 \in \operatorname{int} D$, suppose that $0 \notin \operatorname{int} D$. Because of (iv), $0 \in D$, hence there exist a countable sequence $\{\phi_i\}_1^{\infty} \subseteq C$ such that $\phi_i \to 0$ as $i \to \infty$ and $\phi_i \in D$ for any i so that $\phi_i \neq 0$. Let $x(t_1, \phi_i, 0) = \xi_i$, then, since $\phi_i \notin D$ for any i, $Dx(t_1, \phi_i, u) \neq 0$ for any i so, by the variation of constant formula, we have a sequence

$\xi_{i1} \subseteq E^n$ no ξ_i is in $V(t_1,1)$ for any t_1, $\xi_i \to 0$ as $i \to \infty$, therefore $0 \notin \operatorname{int} V(t_1,1)$; a contradiction. This contradiction shows that $0 \in \operatorname{int} D$. Therefore there exists a ball B_2 around the origin contained in D such that $0 \subseteq B_2 \subseteq \operatorname{int} D$. By (i) and (iii), every solution of the system.

$$\dot{x}(t) = L(t, x_t) + \int_{-\infty}^{0} A(\theta)x(t+\theta)\,d\theta + f(t, x(t), x(t-\tau), 0, 0)$$

(which is a solution of (2.2) with $u = 0$) satisfies $x(t,0) \to 0$ as $t \to \infty$. Hence at some $t_2 < \infty$ we have $x_{t_2}(\cdot, 0) \in B_2$. Therefore for some $u \in U$, and some $t_3 > t_2$, the solution $x(t_3, x_{t_2}(\cdot,0), u, f)$ of (2.2) satisfies $x(t_3, x_{t_2}, u, f) = 0$, proving the theorem. ■

***Corollary* 3.1**

For system (2.2), *assume that*

(i) *The zero solution of* (2.1) *with* $u = 0$ *is uniformly asymptotically stable.*

(ii) $rank\,\hat{Q}_n(t_1) = n$

(iii) $f(t, \theta, 0, 0) = f_1(t, \phi) + f_2(t, \phi)$

and $\|f_1(t, \phi)\| \le \pi(t)\,|D(t, \phi)|$, $\|f_2(t, \phi)\| \le \in |D(t, \phi)|$ *where* $\Pi = \int_{t_0}^{\infty} \pi(t)dt < \infty$

(iv) $f(t, 0, 0, 0, 0) = 0$

Then system (2.2) *is Euclidean null controllable with constraints.* ■

Proof

Immediately from Theorem 3.1 and Proposition 3.1

4.0 Example

Consider the system.

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + C u(t) + c_o \int_{-\infty}^{0} \exp(v\theta)x(t+\theta)\,d\theta \qquad (4.1)$$

and its perturbation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + C u(t) + c_o \int_{-\infty}^{0} \exp(v\theta)x(t+\theta)\,d\theta$$
$$+ \frac{1}{1+t^2}\left[\sin(x(t) + x(t-\tau))\cos(u(t) + u(t-\tau))\right]x(t-\tau) \qquad (4.2)$$

where $A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_0 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$

$$f = \left(\frac{1}{1+t^2}\left[\sin(x(t) + x(t-\tau))\cos(u(t) + u(t-\tau))\right]x(t-\tau) \right)$$

The uniform asymptotic stability and the Euclidean null controllability of the system (3.1) have been shown by the author [6]. Moreover, $|f(t, x(t), x(t-\tau), 0, 0)| \le \pi(t)\,|x(t-h)|$ where

$\pi(t) = \dfrac{1}{1+t^2}$ and $\Pi = \int_0^{\infty} \dfrac{1}{1+t^2}dt = \left[\tan^{-1} t\right]_0^{\infty} = \dfrac{\pi}{2} < \infty$ also, $f(t, 0, 0, 0, 0) = 0$. Hence all the conditions of Theorem 3.1 are satisfied and so we conclude that system (3.2) is Euclidean null controllable.

5. 0 Conclusion

We have developed and proved computable criteria for the Euclidean null controllability of perturbed infinite delay systems with limited control. These conditions are given with respect to the stability of the free linear base system and the controllability of the linear controllable base system, with the assumption that the perturbation function satisfies some smoothness and growth conditions. An example is also given.

References

[1] Aiello. W. G and Freedman. H.I., "A time-delay model of single-spacies growth with stage structure", Maths. Biosci., 101 (1990), Pp139-153.

[2] Balachandran. K and Dauer. J. P., "Null controllability of nonlinear infinite delay systems with varying multiple delays in control", Appl. Math. Letter, Vol. 9(3) (1996), Pp 115-121

[3] Chukwu. E. N., "Null controllability in function space of nonlinear retarded systems with limited control", Journal of Mathematical Analysis and Appications , Vol. 103 (1998), Pp 198-210.

[4] Chukwu. E. N., "On the null controllability of nonlinear delay systems with restrained controls", J. Math. Anal .Appl., Vol. 78 (1980), Pp 283-299.

[5] Dauer J.P and Gahl R.D., "Controllability of nonlinear delay systems", Journal of Optimization Theory and Application , Vol.21(1) (1977), Pp 59-70

[6] Davies .I., " Euclidean controllability of infinite delay systems with limited controls",J. of Nig Maths. Society, (2006) (Accepted for publication, to appear in Vol. 25)

[7] Davies .I., "Relative controllability of nonlinear systems with delays in state and control", J. of Nig. Association of Mathematical Physics, Vol. 9 (2005), Pp 239-246

[8] Decka. C., "Relative controllability of perturbed nonlinear systems with delay in control", IEEE Trans. Auto Control, Vol. AC-27, N0.1(1982) Pp 268-270

[9] Driver R. D., "Ordinary and delay differential equations," Applied Mathematical Sciences, Springer-Verleg New York (1977).

[10] Eke. A. N., "Null controllability for linear control systems", Jour. Inst. Math. And comp. Sci. (Math. Ser.), Vol. 3(2) (1990), Pp 149-154

[11] Hale J., "Theory of functional differential equations," Springer-Verlag, New York (1977).

[12] Klamka J., "Controllability of nonlinear systems with distributed delays in control", Int. J. Control, Vol.31, No. 5 (1980), Pp 811-819

[13] Klamka J., "Relative controllability of nonlinear systems with distributed delays in control", Int. J. Control, Vol.28, No.2 (1978), Pp 307-312.

[14] Klamka J., "Relative controllability and minimum energy control of linear systems with distributed delays in control," IEEE Transaction in Automatic Control, Vol. AC - 21(1976), Pp 594-595.

[15] Lakshmikantham V., "Theory of differential equations with unbounded delays," Kluwer Academic Publishers, Dordrecht (1995).

[16] Sinha A. S. C., "Null-controllability of nonlinear infinite delay systems with restrained controls," Int. J. Control**,** Vol. 42, No.3 (1985), Pp735-741.