Journal of the Nigerian Association of Mathematical Physics
Volume **11 (November 2007), 525 - 528**

Hamilton-Jacobi-Bellman equations for quantum control

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Abstract

The aim of this work is to study Hamilton-Jacobi-Bellman equation for quantum control driven by quantum noises. These noises are annhihilation, creation and gauge processes. We shall consider the solutions of Hamilton-Jacobi-Bellman equation via the Hamiltonian system measurable in time.

1.0 Introduction

Control systems with state constraints often deal with dynamics which are merely measurable in time together with constraints depend upon the time. Due to this, Frankowska, H. et al in [6], extended the viability theory to such case, in which the invariance problems are considered as well.

Hamilton-Jacobi equations arise in optimal control theory and various controls problems, [2], [3], [5]. Most of these problems deal with non smooth functions, hence the development of generalized gradients for non smooth functions by Clarke, F. [4]. Given a control problem, the solution to this problem can be achieved via the solution to a differential inclusion corresponding to the problem. Theory of differential inclusions and multifunctions with applications to control theory like feedback, optimal control for Mayer's problem and others have been extensively dealt with in [2], [5], [6] and [8].

In [7], Hudson and Parthasarathy introduced, Quantum stochastic differential equations driven by quantum noises; annihilation, creation and gauge operators. These operator-valued processes were shown to be stochastic processes. The control problem arising from these noises is our main focus in this work. The main result is the generalization of the classical result of Hamilton-Jacobi-Bellman equation in non commutative setting. This is a new result specifically done for the Quantum Stochastic calculus of Hudson-Parthasarathy [7].

The work shall be arranged in sequel as follows: In section 1, we shall give a brief introduction of differential inclusions and set-valued maps. In section 2, we shall consider the quantum stochastic differential equations, and the main result on Hamilton-Jacobi-Bellman equation.

1.1 Multivalued maps and differential inclusions

1.1.1 Definition

Let X, Y be sets, A map $F : X \to 2^r$ is said to be a multivalued (set-valued) map or multifunction if $F(x) \subset Y$ for all $x \in X$. By a selection of *F*, we mean a single-valued map f, $f : X \to Y$, such that $f(x) \in F(x)$ for all $x \in X$. There are various selection theorems depending on the topologies properties of F.

1.1.2 Differential inclusions

By a differential inclusion we mean a multivalue differential equation, that is an ordinary differential equation in which the right hand side *F*(*t, x*) is a multifunction (multivalued map), as stated in (1.1) below. The initial value problem involving inclusion;

$$\begin{aligned} x^t(t) &\in F(t, x(t)) \\ x(t_0) &= x_0 \end{aligned} \qquad (1.1)$$

can be solved depending on the properties of *F*. In this work we shall be considering the case of lower semicontinuous and upper semicontinuous differential inclusions. Also given an ordinary differential equation with discontinuous right hand side *F*(*t, x*), as in (1.1), such problem can be solved via differential inclusion. The solution of such O.D.E. has been shown to be a generalized solution of an upper semicontinuous differential inclusions [2].

Let $F : I \times \Re^n \to \Re^n$, a multivalued map, $I \subset \Re$, by a solution to (1.1) we mean an absolutely continuous function $x : I \to \Re^n$, such x(.) satisfies (1.1) a.e. Given a control problem $u'(t) = f(t, U(t))$, where *U*(*t*) are the state constraints we seek the space of admissible controls for the problem. We can transform the problem to a differential inclusion as $u'(t) \in F(t, y(t))$, where $F(t, u(t)) = \{f(t, u(t)) : u(t) \in U(t)\}$. Hence the set of solutions of the differential inclusions satisfying given initial conditions is the space of admissible controls for the problem.

2.0 Hamilton-Jacobi-Bellman theory

2.1 Quantum noises

The quantum noises in this work arise from annihilation, creation and gauge operators. They are infinitesimal forms of Weyl representation of Euclidean group of a Hilbert space [9]. The stochastic processes of the closure of these operators are respectively, annihilation, creation and guage processes, $\Lambda_\pi(.), A_f(,), A_g^+$. The differential equation driven by these processes is;

$$\begin{aligned} X(t) &= Ed\Lambda_\pi(t) + FdA_f(t) + GdA_g^+(t) + H(t) \\ X(0) &= x_0 \end{aligned} \qquad (2.1)$$

for almost all $t \in I$. Where E,F,G and H are square integrable, locally bounded adapted processes. In sequel we denote by $S_{[t_0,T]}(x_0)$ the solution set of (2.1) on $I = [t_0, T]$.

2.2 Mayer's problem

Let $g : \Re^n \to \Re \cup \{+\infty\}$, the Mayer's problem is the minimization problem $\min\{gx(T)) : x \in S_{[t_0,T]}(x_0)\}$. The value function, *V* is defined as $V : [0,T] \times \Re^n \to \Re \cup \{+\infty\}$ such that *V*(*t,x*) = $\min\{gx(T)) : x \in S_{[t_0,T]}(x_0)\}\ \forall (t, x) \in [0, T]$.

We assume that:

(i) *E, F, G, H,* has non-empty convex compact images.

(ii) $\forall x \in \Re^n$, $E(.,x)$, $F(.,x)$, $G(.,x)$ and $H(.,x)$ are measurable

(iii) There exists $\mu_\varepsilon, \mu_F, \mu_G, \mu_H \in L^1[0T]$ such that for almost all $t \in [0,T]$, we have $\|E(t,x)\| \le \mu_\varepsilon, \|F(t,x)\| \le \mu_F, \|G(t,x)\| \le \mu_G, \|H(t,x)\| \le \mu_H \quad \forall x \in \Re^n$ (i.e. each of them is integrably bounded)

(iv) g is lower semicontinuous. (2.2)

2.3 Proposition

If the assumption above holds true and for almost all $t \in [0,T]$, *F(t, x), is upper semicontinuous (usc), then V is lower semicontinuous (lsc) and*

(a) $\forall (t_0, x_0) \in [0,T] \text{ x } \Re^n,\ V(t_0,x_0) = \min\{g(x(T)) : x \in S_{[0,T]}(x_0)\}$

Furthermore, the set-valued map

(b) $t \mapsto P(t) = \{(x,r) \in \Re^n \text{ x } \Re : r \geq V(t,x)\}$ *is absolutely continuous.*

Proof

(a) *V* is lower semicontinuous from the definition above, also, there exists $\mu \in L^1[0,T]$ such that for almost all $t \in [0,T]$, $\forall x \in \Re^n$, $\|E(t,x)\| \leq \mu(t)$, $\|F(t,x)\| \leq \mu(t)$, $\|G(t,x)\| \leq \mu(t)$ and $\|H(t,x)\| \leq \mu(t)$. Hence the maps are upper Caratheodory and the arbitrary intersection of directionally continuous selections of *E,F,G,H* is the solution. Therefore $V(t_0,x_0) = \min\{g(x(T)) : x \in S_{[0,T]}(x_0)\}$.

(b) Graph (P) is equal to the epigraph of *V* (Epi. *V*), hence

$$g(\bar{x}) = V(T,\bar{x}) = \lim_{t \to T} \inf_{x \to x} V(t,x), \quad V(0,\bar{x}) = \lim_{t \to +0} \inf_{x \to x} V(t,x)$$

Assume E, F, G, H has non-empty compact images, we define the Hamiltonian

$$H : [0,T] \text{ x } \Re^n \text{x } \Re^n \to \Re$$

by $H(t,x,p) = \text{x } \Re^n \max_{V \in M(t,x)} < p,v >$, where $< .,. >$ is duality map and $M(t,x) = \{(E(t,x), F(t, x), G(t,x), H(t,x,.)\}$. Then *H*(*t*,*x*,.) is convex and positively homogeneous []. Furthermore, if assumption (2.2) holds then *H*(*t*,..,*p*) is use and *H*(.,*x*,*p* is measurable. ■

2.4 Solution of Hamilton-Jacobi-Bellman equation

Consider the Hamiltonian measurable in time; $H : [0,T] \text{ x } \Re^n \text{x } \Re^n \to \Re$ and the Hamilton-Jacobi-Bellman Equation

$$\frac{-\partial V(t,x)}{\partial t} + H\left(t, x \frac{-\partial V(t,x)}{\partial t}\right) = 0 \qquad (2.3)$$

We assume:

(i) *H*(*t*,*x*,*p*) is Caratheodory

(ii) *H*(*t*,*x*,*p*) is convex

(iii) *H*(*t*,.,*p*) is $c_k(t)$ –Lipschitz on KB where B denotes the closed unit ball in $\Re^n$

(iv) *H*(*t*,*x*,.) is Lipschitz continuous (2.4)

Define $M(t,x) = \bigcap_{\|p\|=1} \{v \in \Re^n :< p,v > \leq H(t,xp)\}$

We can study the solution of (2.3) via the Hamilton-Jacobi-Bellman equation with new (conjugate) Hamiltonian.

2.5 Theorem

If assumption 2.4 *holds, then M is a solution of* (2.3) *and* $\forall v \in \Re^n$, $\sup_{v \in M(t,x)}$ *<p,v>* = $= H(t,x,p)$

Proof

To prove the theorem it suffices to show that M satisfies assumption 2.2 and also $c_k(t)$ – Lipschitzian. Fix $x \in \Re^n$ and consider a dense subset $\{P_i\}_{i \geq 1}$ of the unit sphere in $\Re^n$. For every $i \geq 1$, define the set-value map $P_i : [0,T] \to \Re^n$ $P_i(t) = \{v \in \Re^n : < [p_i, v > \leq H(t,x,p_i)\}$ by $P_i(t) = \{v \in \Re^n : < [p_i, v > \leq H(t,x,p_i)\}$. From the separation theorem and the continuity of $H(t,x,.)$ it follows that $M(t,x) = \bigcap_{i \geq 1} P_i(t)$. P_i is measurable. Hence M is use and measurable, M is also integrably bounded, hence the multivalued map $M(t,x)$ is upper Caratheodory. M is $c_k(t)$ –Lipschitz on KB, follows from the definition. Furthermore, for all $v \in M(t,x); < p, v > \leq H(t,x,p)$, therefore $\sup_{v \in M(t,x)} < p, v > \leq H(t,x,p)$ and conversely, $H(t,x,p) := \max_{v \in M(t,x)} < p, v > \leq \sup_{v \in M(t,x)} < p, v >$, therefore $\sup_{v \in M(t,x)} < p, v > = H(t,x,p)$. ■

3.0 Conclusion

The Quantum control problem driven by quantum field operators, annihilation, creation and gauge operators is stochastic. It has been shown in this work that, the value function for the Hamiton-Jacobi-Bellman equation for the problem is the viscosity solution to the equation.

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