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Certain inequalities and convolution properties for multivalent and meromorphically multivalent functions

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Abstract

By means of certain extended derivative operator of Salagean type, the author introduces and investigates three new subclasses of p-valently analytic functions. The reason for this is to use generalized Salagean derivative operator to bring together many earlier inroduced subclasses of p-valently functions to become special cases of the newly defined subclasses, see **[2,8].** ***The various results obtain for these functions include coefficient inequalities, coefficient bounds and convolution properties. These results coincides with many existing results using different choices of*** n ***and*** β

***Keywords*:** Analytic function, P-valent, Meromorphic, Multivalent, Salagean, and Convolution.

1.0 Introduction

Let $T(p)$ and $M(p)$ denote the classes of functions $f(z)$ and $g(z)$ of the forms

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_k z^k,\ (p\in N=\{1,2,\cdots\}) \tag{1.1}$$

and
$$g(z)=z^{-p}+\sum_{k=p}^{\infty}a_k z^k,\ \ (p\in N) \tag{1.2}$$

which are analytic and multivalent in the unit disk $E=\{z:|z|<1\}$ and in the punctured unit disk $U=\{z:0<|z|<1\}$ respectively , see [2,5].

The author here wishes to define the following subclasses as follows:

$$T_n(p,\beta)=\{f\in T(p):\mathrm{Re}\frac{D^{n+1,p}f(z)}{D^{n,p}f(z)}>\beta,\quad n=0,1,2,\cdots,\ p\in N\} \tag{1.3}$$

$$K_n(p,\beta)=\{f\in T(p):\mathrm{Re}\frac{D^{n,p}f(z)}{z^p}>\beta,\quad n=0,1,2,\cdots,\ p\in N\} \tag{1.4}$$

$$M_n(p,\beta)=\{f\in M(p):\text{Re}-\frac{D^{n+1,p}g(z)}{D^{n,p}g(z)}>\beta,\quad n=0,1,2,\cdots,\ p\in N\} \tag{1.5}$$

where $D^{n,p}f$ is the extended Salagean derivative operator defined as

$D^{0,p}f=z^p+\sum_{k=p+1}^{\infty}a_kz^k,\ D^{1,p}f=(1-p)f(z)+zf'(z),...,D^{n,p}f(z)=z^p+\sum_{k=p+1}^{\infty}(k+1-p)^na_kz^k$ for every $f\in T_n(p,\beta)$ *and* $K_n(p,\beta)$ and

$$D^{0,p}g(z)=z^{-p}+\sum_{k=p}^{\infty}a_kz^k,...,\ D^{n,p}f=(1-2p)^nz^p+\sum(k+1-p)^na_kz^k$$

for every $g\in M_n(p,\beta)$, see [4]. These three new subclasses are the extension of the classes introduced in [2] and the result obtained for them are new. Also with different choices of n *and* β other existing results can be obtained see [8]. The Hadamard product (or convolution) of the function $f\in T(p)$ is given by

$$\prod_{j=1}^{m}f_j(z)=(f_1*...*f_m)(z)=z^p+\sum_{k=p+1}^{\infty}\left(\prod_{j=1}^{m}(a_{k,j})z^k\right)$$

and

$$\prod_{j=1}^{m}g_j(z)=(g_1*...*g_m)(z)=z^p+\sum_{k=p}^{\infty}\left(\prod_{j=1}^{m}(a_{k,j})z^k\right),$$

see [6].

To prove our main results the following shall be necessary.

***Lemma* A** [1,3], Aini et al.

If $p\in P$ *then*, $|c_k|\le 2$ *for each* k

2 0 Coefficient inequalities

In this section we shall state the following theorems:

***Theorem* 2.1**

Let $f\in T(p)$ *satisfy* $$\sum_{k=p+1}^{\infty}\alpha^n(\alpha_o+\beta)|a_k|\le 1-\beta \tag{2.1}$$

where $\alpha=(k+1-p)$ *and* $\alpha_0=(k-p-1)$ then $f\in T_n(p,\beta)$ *where* $n=0,1,2,\cdots,\ 0\le\beta$ $<1,\ p\in N$, see method of proof in [7].

***Theorem* 2.2**

Let $f\in T(p)$ *satisfies* $\sum_{k=p+1}^{\infty}\alpha^n|a_k|\le 1-\beta$ *where* $\alpha=(k+1-p)$ *then* $f\in$ $K_n(p,\beta)$, $n=0,1,2...,\ 0\le\alpha<1,\ p\in N$, see method of proof in [7]

***Theorem* 2.3**

Let $g\in M(p)$ *satisfies* $\sum_{k=p+1}^{\infty}\alpha^n(\alpha+\beta)a_k\le\rho^n(2p-1-\beta)$ *where* $\alpha=(k+1-p)$ *and* $\rho=(1-2p)$ *then* $g\in M_n(p,\beta)$

3.0 Coefficient bounds

***Theorem* 3.1**

Let $f \in T_n(p,\beta)$. *Then we have the following inequalities.*

(i) $|a_{p+1}| \le \dfrac{1-\beta}{2^{n-1}}$

(ii) $|a_{p+2}| \le \dfrac{1-\beta}{2^{n-1}} + \dfrac{(1-\beta)^2}{2^{n-2}}$

(iii) $|a_{p+3}| \le \dfrac{1-\beta}{2^{n-1}} + \dfrac{(1-\beta)^2}{2^{n-3}} + \dfrac{(1-\beta)^3}{2^{n-3}}$

Proof

Since $f \in T_n(p,\beta)$, we have
$$\frac{D^{n+1,p}f(z)}{D^{n,p}f(z)} = \beta + (1-\beta)p(z) \tag{3.1}$$

for $p \in P$. Setting $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ and comparing coefficients in (3.1) the results follow. ■

***Theorem* 3.2**

Let $f \in K_n(p,\beta)$. *Then we have*

$$a_{p+j} = \frac{(1-\beta)c_j}{2^n} \text{ or } |a_{p+j}| \le \frac{1-\beta}{2^{n-1}}, \quad j = 1,2,\ldots,m \text{ for } 0 \le \alpha < 1,\ n = 0,1,2,\ldots$$

Proof

Since $f \in T_n(p,\beta)$ we have
$$\frac{D^{n,p}f(z)}{z^p} = \beta + (1-\beta)p(z) \tag{3.2}$$
Comparing the coefficients in (3.2) the result s follow. ■

***Theorem* 3.3**

Let $g \in M_n(p,\beta)$ *then we note that the coefficient bounds for the functions in the subclass* $g \in M_n(p,\beta)$ *are zeros.*

Proof

Since $g \in M_n(p,\beta)$ ■

4.0 Convolution properties

***Theorem* 4.1.**

If $f_j(z) \in T^*_{n_j}(p,\beta), \quad (j = 1,\cdots,m),$ *then* $\prod_{j=1}^{m} f_j(z) \in T_n(p,\gamma_m)$ *where*

$$\gamma_m = 1 - \frac{\prod_{j=1}^{m}(1-\beta_j)}{\left(\frac{1}{2}\right)^{n-\sum_{j=1}^{m} n_j} \beta_1\beta_2 + \prod_{j=1}^{m}(1-\beta_j)} \tag{4.1}$$

where $n - \sum_{j=1}^{m} n_j \geq 0$

Proof

With the aid of theorem 2.1 we need to find the γ_m such that

$$\sum_{k=p+1}^{\infty} \alpha^n(\alpha_0+\gamma_m)\prod_{j=1}^{m}|a_{k,j}| \leq 1-\gamma_m .$$

Note here that if $m=1,$ then $\gamma_1=\beta_1$. Now suppose $m=2$. Then for functions $f_1(z)\in T_{n_1}^{*}(p,\beta_1)$ and $f_2(z)\in T_{n_2}^{*}(p,\beta_2)$, we have

$$\sum_{k=p+1}^{\infty} \alpha^{n_1}(\alpha_0+\beta_1)|a_{k,1}| \leq 1-\beta_1 \text{ and } \sum_{k=p+1}^{\infty} \alpha^{n_2}(\alpha_0+\beta_2)|a_{k,2}| \leq 1-\beta_2$$

so that $\sum_{k=p+1}^{\infty} \frac{\alpha^{n_1}(\alpha_0+\beta_1)}{1-\beta_1}|a_{k,1}| \leq 1$ and $\sum_{k=p+1}^{\infty} \frac{\alpha^{n_2}(\alpha_0+\beta_2)}{1-\beta_2}|a_{k,2}| \leq 1$.

Hence, by Cauchy-Schwarz inequality, we have

$$\sum_{k=p+1}^{\infty} \sqrt{\frac{\alpha^{n_1+n_2}(\alpha_0+\beta_1)(\alpha_0+\beta_2)}{(1-\beta_1)(1-\beta_2)}|a_{k,1}||a_{k,2}|} \leq 1 \tag{4.2}$$

In order to prove that $(f_1 * f_2)(z)\in T_n^{*}(p,\gamma_2)$ it is sufficient to show that

$$\sqrt{|a_{k,1}||a_{k,2}|} \leq \sqrt{\frac{\alpha^{n_1+n_2}(\alpha_0+\beta_1)(\alpha_0+\beta_2)}{(1-\beta_1)(1-\beta_2)}}\left(\frac{1-\gamma_2}{\alpha^n(\alpha_0+\gamma_2)}\right)$$

Since

$$\sum_{k=p+1}^{\infty} \alpha^n(\alpha_0+\beta)|a_{k,1}||a_{k,2}| \leq (1-\gamma_2)\sum\sqrt{\frac{\alpha^{n_1+n_2}(\alpha_0+\beta_1)(\alpha_0+\beta_2)}{(1-\beta_1)(1-\beta_2)}|a_{k,1}||a_{k,2}|}$$

$$\leq (1-\gamma_2)\left\{\left(\sum_{k=p+1}^{\infty}\frac{\alpha^{n_1}(\alpha_0+\beta_1)}{(1-\beta_1)}|a_{k,1}|\right)\left(\sum_{k=p+1}^{\infty}\frac{\alpha^{n_2}(\alpha_0+\beta_2)}{(1-\beta_2)}|a_{k,2}|\right)\right\}^{\frac{1}{2}} \leq 1-\gamma_2$$

But from (4.2) we have for all $k=p+1, \quad (p\in N=\{1,2,\cdots\})$

$$\sqrt{|a_{k,1}||a_{k,2}|} \leq \frac{(1-\beta_1)(1-\beta_2)}{\alpha^{n_1+n_2}(\alpha_0+\beta_1)(\alpha_0-\beta_2)}$$

Hence it is sufficient to find the largest γ_2 such that

$$\sqrt{\frac{(1-\beta_1)(1-\beta_2)}{\alpha^{n_1+n_2}(\alpha_0+\beta_1)(\alpha_0-\beta_2)}} \le \sqrt{\frac{\alpha^{n_1+n_2}(\alpha_0+\beta_1)(\alpha_0-\beta_2)}{(1-\beta_1)(1-\beta_2)}}\left(\frac{1-\gamma_2}{\alpha^n(\alpha_0+\gamma_2)}\right)$$

That is, we have $\dfrac{(1-\beta_1)(1-\beta_2)}{\alpha^{n_1+n_2}(\alpha_0+\beta_1)(\alpha_0+\beta_2)} \le \dfrac{1-\gamma_2}{\alpha^n(\alpha_0+\gamma_2)}$

that is, $\gamma_2 \le 1-\dfrac{(1+\alpha_0)(1-\beta_1)(1-\beta_2)}{\alpha^{n_1+n_2-n}(\alpha_0+\beta_1)(\alpha_0+\beta_2)+(1-\beta_1)(1-\beta_2)}$

It is readily seen that the right hand side of (4.3) is an increasing function of k. Hence the large value of γ_2 is given by $\gamma_2 = 1-\dfrac{(1-\beta_1)(1-\beta_2)}{2^{n_1+n_2-n}\beta_1\beta_2+(1-\beta_1)(1-\beta_2)}$.

Next we suppose that $\prod_{j=1}^{m} f_j(z) \in T_n'(p,\gamma_m,\lambda)$, $\gamma_m = 1-\dfrac{\prod_{j=1}^{m}(1-\beta_{j1})}{2^{\sum_{j=1}^{m} n_j - n}\beta_1\beta_2 + \prod_{j=1}^{m}(1-\beta_j)}$.

Then, by repeating the processes above we obtain the $\prod_{j=1}^{m+1} f_j(z) \in T_n(p,\gamma_{m+1})$ where

$$\gamma_{m+1} = 1-\frac{\prod_{j=1}^{m+1}(1-\beta_{m+1})(1-\gamma_m)}{2^{n+1}\beta_{m+1}+\prod_{j=1}^{m+1}(1-\beta_{m+1})} \text{ so that } \gamma_{m+1} = 1-\frac{\prod_{j=1}^{m+1}(1-\beta_j)}{2^{\sum_{j=1}^{m+1} n_j - n}\beta_j + \prod_{j=1}^{m+1}(1-\beta_j)}.$$

Hence conclusion follow by induction. ■

***Theorem* 4.2**

If $f_j(z) \in K_{n_j}(p,\beta_j) \quad (j=1,\ldots,m)$, *then* $\prod_{j=1}^{m} f_j(z) \in K_n(p,\gamma_m) \quad (j=1,\ldots,m)$,

where
$$\gamma_m = 1-2^{\,n-\sum_{j=1}^{m} n_j}\prod_{j=1}^{m}(1-\beta_j) \tag{4.3}$$

where $n-\sum_{j=1}^{m} n_j \ge 0$

Proof

The method of proof is similar to that of Theorem 4.1 with the use of Theorem 2.2. ■

***Theorem* 4.3**

If $g_j(z) \in M_{n_j}(p,\beta_j) \quad (j=1,\ldots,m)$, *then* $\prod_{j=1}^{m} g_j(z) \in M_n(p,\gamma_m) \quad (j=1,\ldots,m)$, *where*

$$\gamma_m = \frac{(2p-1)\prod_{j=1}^{m}(1-\beta_j) - \rho^{\sum_{j=1}^{m} n_j}\prod_{j=1}^{m}(2p-1-\beta_j)}{\rho^{\sum_{j=1}^{m} n_j}\prod_{j=1}^{m}(2p-1-\beta_j) + \prod_{j=1}^{m}(1-\beta_j)} \qquad (4.4)$$

Proof

The method of proof is the same as in Theorem 4.1 using theorem 2.3. ■

2.0 Conclusion

In conclusion, we are able to unify some existing subclasses under these three subclasses by different choices of n *and* β see [2,8].

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