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Further results on the classes of convex functions

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Abstract

This paper presents more results on the classes of convex sets and functions. The results are follow-up to similar results obtained on the classes of convex sets and functions by* [1] *and* [9]. *These relationships were further shown among convex sets which were not considered in the above references and more results were proposed and proved.

***Keywords*:** Convex series, cs-convex sets, cs-convex functions.

1.0 Introduction

The classes of convex sets and functions have been attracting attentions of researchers because of its usefulness in optimization theory, convex optimization problems, convex analysis etc.

Prominent among researchers are Amara and Ciligot-Travain [1], Jameson [3], Jeyakumar and Wolkowicz [4], Lifsic [5], and Simons [8]. Others include Zalinescu [9].

Consider X a real topological vector space. We say that the series

$$\sum_{n\geq 1} x_n$$

is convergent (resp. Cauchy) if the sequence $(S_n)_{n\in N}$ is convergent (resp. Cauchy), where $S_n = \sum_{k=1}^{n} x_k$.

For every $n \in N$; of course any convergent sequence is Cauchy.

Let $A \subset X$; be a convex series with elements of A we mean a series of the form

$$\sum_{m\geq 1} \lambda_m x_m$$

with $(\lambda_m) \subset \Re_+, (x_m) \subset A$ and $\sum_{m\geq 1} \lambda_m = 1$.

If furthermore, the sequence (x_m) is bounded we speak about a bounded convex series. We say that A is cs-closed if any convergent convex series with elements of A has its sum in A.

A is cs-complete if any Cauchy convex series with elements of A is convergent and its sum is in A. Similarly, the set A is called ideally convex if any convergent bounded convex series with elements of A has its sum in A and A is bcs-complete if any Cauchy bounded convex series with elements of A is convergent and its sum is in A.

Relationship among these sets are as follows: Any cs-closed set is ideally convex, every ideally convex set is convex. Every cs-complete set is cs-closed and every complete set is cs-complete; if X is complete, then $A \subset X$ is cs-complete (bcs-complete if and only if A is cs-closed (ideally convex).

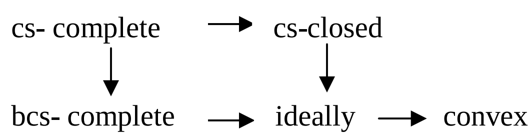


Figure 1

The notions of ideally convex sets were introduced by Lifsic [5], cs-closed sets by Jameson [3], cs-complete sets by Simons [8] while results and properties of lcs-closed set were contained in the work of Amara and Ciligot-Travain [1], Zalinescu [9] introduces several interiority notions on the sets such as ${}^{ic}A,\ riA$ etc.

***Proposition* 1.1**

Consider X a real topological vector space. Let $A \subset X$ *then*

(i) *Every cs-complete set is cs-closed and every complete convex set is cs-complete*

(ii) *If X is complete, then* $A \subset X$ *is cs-complete* (*bcs-complete*) *iff A is cs-closed* (*ideally convex*)

(i) *cs-closed set is ideally convex and every ideally convex is convex.*

Proof

(*i*) Let $\sum_{n\geq 1} \lambda_n x_n$

be a convergent (resp. Cauchy) convex series with elements of A; denote by x its sum. Let a sequence $(x_n) \subset A$ converges to $x \in A$ and has its sum in *A*. thus cs-complete. Since any convergent series is Cauchy. Then cs-complete is cs-closed.

(*ii*) The proof of (*ii*) follows from (*i*) and shall be omitted.

(*iii*) Let $A \subset X$ be a non-empty convex set and

$$\sum_{n\geq 1} \lambda_n x_n$$

be a convergent series with elements of *A*; denoted by x its sum. Suppose the set *A* is closed and fix $a \in A$, we have,

$$\sum_{k=1} \lambda_k x_k + \left[1 - \sum_{k=n+1}^{\infty} \lambda_k\right] a \in A$$

taking the limit for $n \to \infty$, we obtain that $x \in clA = A$ *i.e. A* is cs-closed. Furthermore, if the sequence (x_n) is bounded, we speak about a bounded convex series with elements of A has its sum in *A*. Therefore $x \in A$ and thus A is ideally convex (resp. convex).

The next result shows other properties of cs-closed and ideally convex sets. ■

***Proposition* 1.2 [9]**

(*i*) *If* $A_i \subset X$ *is cs-closed* (*resp. ideally convex*) *for every* $i \in I$ *then* $\bigcap_{i\in I} A_i$ *is cs-closed* (*resp. ideally convex*).

(*ii*) *If* X_i *is a topological vector space and* $A_i \subset X_i$ *is cs-closed* (*resp. ideally convex*) *for every* $i \in I$, *then* $\bigcap_{i\in I} A_i$ *is cs-closed* (res. Ideally convex) in $\prod_{i\in I} X_i$ (*which is endowed with the product topology*).

Proof

The proof of (*i*) is immediate, while for (*ii*) one must take into account that a sequence $(x_n)_{n\in N} \subset X := \prod_{i\in I} X_i$ converges to $x \in X$ (respectively is bounded) if and only if (x_n^i) converges to x^i in X_i (resp. is bounded) for every $i \in I$. ■

2.0 Classes of convex functions

In this section we introduce several classes of convex functions larger than the class of lower semicontinuous convex functions which will reveals themselves to be useful in the sequel

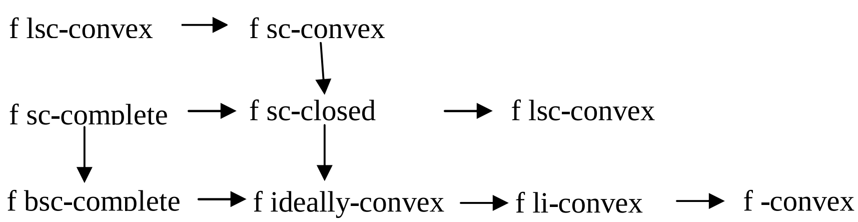
Let $f : X \to \Re$, we say f is cs-convex if

$$f(x) \leq \lim_{n\to\infty} inf \sum_{k=1}^{n} \lambda_k f(x_k)$$

where $$\sum_{n\leq 1} \lambda_n x_n$$

is a convex series with elements of X and sum $x \in X$. Also, we say that f is ideally convex, bcs-complete, cs-closed, cs-complete, li-convex or lcs-closed if epi(*f*) is ideally convex, bcs-complete, cs-closed, cs-complete, li-convex or lcs closed, respectively.

Of course, taking into account the relationship due to Zalinescu [9] for convex functions.



and the reverse implications being not true, in general.

The main result of this work is as follows:

***Theorem* 2.1**

Let $f : X \to \Re$

(i) *If f is cs-convex, then f is convex, while if f is lsc-convex then f is cs-convex.*

(ii) *If f is sc-convex then f is cs-convex, while if f is cs-convex then f is cs-closed*

(iii) *If f is cs-closed then f is lcs-closed.*

(iv) *If f is li-convex (resp. lsc-closed), then f is convex.*

Proof

(i) Suppose *f* is cs-convex. Let $f : X \to \Re$, if *f* is cs-convex $f(x) \leq \lim_{n\to\infty} inf \sum_{k=1}^{n} \lambda_k f(x_k)$

whenever $\sum_{n\geq 1} \lambda_n x_n$ is a convex series with elements X and sum $x \in X$

$$\forall n \in \aleph, x_1, \cdots, x_n \in X, \lambda_1 + \cdots + \lambda_n = 1$$

and $$X = \sum_{n\geq 1} \lambda_n x_n$$

$$f(\lambda_1 x_1 + \ldots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \ldots + \lambda_n f(x_n)$$

i.e. $$f(x) \leq \sum_{n\leq 1} \lambda_n f(x_n)$$

Hence *f* is convex. Since f is convex, *f* is lower semi continous (lsc) if and only if epi *f* is closed in $X \times \Re$, *i.e.* $[f \leq \lambda]$ is closed $\forall \lambda \in \Re$. Thus *f* is cs-convex.

(*ii*) From (*i*) *f* is lsc-convex (resp. sc-convex) $\Rightarrow$ *f* is cs-convex. To show that if *f* is cs-convex then *f* is cs-closed convex. Let $\varphi : X \to \overline{\Re}$ be a continuous affine functional (i.e. $\varphi = x^* + \alpha$ for some $x^* \in X^*$ and $\alpha \in \Re$), $f : X \to \Re$ is cs-convex if and only if $f + \alpha$ is so.

Let $f : X \to \Re$ have non-empty domains when f is bounded below by a continuous affine function and assume that $f \geq 0$, thus f is cs-closed.

(*iii*) The proof of (*iii*) follows from (*ii*) and consequently (*iv*) follows. The classes of li-convex and lsc-closed functions have good stability properties and some of these properties are shown in the following results. ■

***Proposition* 2.1. [9]**

(*i*) *If $f_n : X \to \overline{\Re}$ is li-convex (resp. lcs-closed) for every $n \in N$, then suppose $n \in N$, f_n is li-convex (resp. lcs-closed).*

(*ii*) *If $f_1, f_2, \ldots, f_n : X \to \overline{\Re}$ are li-convex (resp. lcs-closed) functions and $\lambda \in \Re_+$, then $f_1 + f_2 + \ldots + f_n$ and λf_1 are li-convex (resp. lcs-closed)*

(*iii*) *If $F : X \times Y \to \overline{\Re}$ is li-convex (resp. lcs-closed) and X is a frechet space, then $h : Y \to \overline{\Re},\ h(y) := inf_{x \in X} \ F(x, y)$ is li-convex (resp. lcs closed).*

Proof

We treat only the "li-convex" case and that of lcs-closed is immediate

(*i*) Because $epi(\sup_{n \in N} \ f_n) = \bigcap_{n \in N} epi \ f_n$, the conclusion follows

(*ii*) Taking $R_i : X \rightrightarrows \Re$, $grR_i := epi \ f_i (i = 1,2)$ we have that $epi(f_1 + f_2) = gr(R_1 + R_2)$ hence the conclusion follows. Also, $epi(\lambda f_1) = T(epi \ f_1)$ for $\lambda > 0$, where $T : X \times \Re \to X \times \Re$ is the isomorphism of topological vector spaces given by $T(x, t) = (x, \lambda t)$; hence λf_1 is li-convex in this case. If $\lambda = 0$, $\lambda f_1 = \iota_{\ dom \ f_1}$. but $dom \ f_1 = \Pr_X (epi \ f_1)$ and so $dom \ f_i$ is li-convex, whence $0\lambda_i$ is li-convex.

(*iii*) We have that $epi_S \ f = \Pr_{Y \times R}(epi_S \ F)$. Since $epi_s \ F$ if li-convex and X is a frechet space, we have that $epi_s \ f$ is li-convex ■

3.0 Concluding remarks

The paper attempts to investigate further results on the classes of convex sets and functions and shown the relationship existing between them.

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