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Constructing an automorphism with discrete spectrum

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Abstract

This work is a desire to construct an automorphism with discrete spectrum using a numerical example. We briefly discuss how some of the definitions and theorems about its behaviour can be implemented and verified numerically. While it is not intended as a complete introduction to measure theory, only the definitions relevant to the discussion in this work are included. It went further to show that a necessary and sufficient condition for a measure-preserving transformation c on a unit circle S^1 to be invertible is that it must both be one-one and onto and concludes that it is an automorphism if the real number, α, is one.

Keywords: Measure spaces, Lebesgue measure, unit circle, automorphism, discrete spectrum.

1.0 Introduction

In ergodic theory, measure-preserving transformations are sometimes called homomorphisms. Homomorphisms with identical measures are endomorphisms. Bijective endomorphisms are called automorphisms.

But for the sake of definition we say if $T: X \to X$ is a transformation of a probability space $(X, \acute{A}, \mu)$ such that T is an invertible-measure preserving transformation of the space then T is called an automorphism. Bedford, Kaene and Caroline (1991) [1] and Petersen (1983) [8]. These are always stated theoretically.

Interestingly, Gewitz (2004) [2] constructed a homomorphism – an ergordic transformation T_α on the unit circle S^1. In other words, he constructed a transformation that preserves lebesgue measure and showed that the transformation is ergodic using some rigorous iteration. However, in this work we present an improvement using a simple interval computation method to show that T_α is not only a homomorphism (ergodic) but T_α is an automorphism by summarising it as a lemma with its accompanying numerical verification.

***Definition* 1.1**

A collection $\acute{A}$ of subsets of X is called a σ-algebra if it has the following properties:

(*i*) If $X \in \acute{A}$

(*ii*) $A^c \in \acute{A}$ if $A \in \acute{A}$

(*iii*) $\bigcup_{j=1}^{\infty} A_j \in \acute{A} \ \forall \ A_j \in \acute{A}$. See Gewitz (2004) and Isere and Osemwenkhae (2006) [[4].

This means that the union of many countable number of sets in $\acute{A}$ is again in $\acute{A}$.

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***Definition* 1.2**

A positive measure 'μ' refers to a set function defined on a σ- algebra Ấ in a non-empty set X whose range is in $[0,\infty]$ such that

(*i*) μ (ϕ_x) = 0 and

(*ii*) $\mu(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j)$ holds for every collection $\{A_j\}_{j=1}^{\infty}$ of a countable number of disjoint members A_j of Ấ, (ϕ_x being empty set of *X*). Wikipaedia (2006) [[11]

***Definition* 1.3**

X equipped with a σ-algebra Ấ is called a measurable space denoted as (X, Ấ) – See Isere and Osemwenkhae (2006) [4] and Knapp (1986) [5].

***Definition* 1.4**

A measure space is a measurable space (*X*, *Ấ*) which has a positive measure μ defined on the σ-algebra of its measurable sets. It is denoted by (*X*, *Ấ*, *μ*). See Halmos (1950) [[3]

Definition *1.5*

We wish to consider a transformation T defined on X under which the measure μ is preserved, i.e., μ (T^{-1}(A) = μ (A) for all μ- measurable sets A∈ Ấ. See Gewitz (2004) [2].

Proof

See Gewitz (2004) [2]. ■

Definition *1.6*

If $T: X \to X$ is a transformation of a probability space (*X*, *Ấ*, *μ*) such that *T* is an invertible measure preserving transformation of the space, i.e. T^{-1} exists is measurable and both *T* and T^{-1} are measure-preserving then *T* is called an automorphism – see Isere and Osemwenkhae (2006) [4].

2.0 Constructing an automorphism with discrete spectrum

Theorem *2.1*

Let (X_i, $Ấ_i$, *μ*) be measure spaces, *i* = 1, 2, and $T: X_2 \to X_2$ a transformation. Suppose A_2 is a sigma -algebra generating Ấ. Then T is a measure-preserving transformation if and only if for each set *A* ∈ $Ấ_2$, we have

$$\mu_n(T^{-1}(A)) = \mu_2(A)$$

***Remark* 2.2**

If $T^{-1}(A)$ is also measurable for each measurable set A∈ Ấ. We will refer to such a transformation T as an invertible measure-preserving transformation.

Consider the transformation T_α on the unit circle S^1. We must first verify that each transformation in question does in fact preserve lebesgue measure i.e. measure-preserving . Once that is done for each example, we can deduce that the transformation is an automorphism using definition 1.6 above.

Example *2.1*

Consider the transformation $T_\alpha : (0,1) \to (0,1)$ defined by

$$T_\alpha(x) = x + \alpha \bmod 1 \qquad (2.1)$$

which represents rotation on the unit circle by an angle of $2\pi\alpha$ radians, $\alpha \in \Re$.
We are going to use the additive form of the transformation below where "mod I" simply means that we are interested only in the fractional part of the output. Since the unit circle is normalized to have length 1, we may view the unit circle as a one-dimensional interval in Euclidean space. – see Gewitz (2004) [2].

The transformation (2.1) above can be rewritten as

$$T_\alpha(x) = \begin{cases} x+\alpha-1, & \text{if } 1-\alpha \le x \le 1 \\ x+\alpha, & \text{if } 0 \le x < 1-\alpha \end{cases} \tag{2.2}$$

Using definition 1.5 above, we compute that

$$T_\alpha^{-1}(x) = \begin{cases} x-\alpha+1, & \text{if } 0 \le x \le \alpha \\ x-\alpha, & \text{if } \alpha \le x < 1 \end{cases} \tag{2.3}$$

Using (2.2) and (2.3) we can show that the transformation preserve Lebesgue measure.

Consider an interval (a, b) $\in$ R, with this, we must verify that

$$T_\alpha^{-1}(a,b) = ((a-\alpha)+1,(b-\alpha)+1) \cup (a-\alpha, b-\alpha)$$

$$T_\alpha^{-1}(a,b) = ((a-\alpha),(b-\alpha)) \cup (a-\alpha, b-\alpha)$$

(since S^1 has length I)

Clearly, $\mu((a-\alpha),(b-\alpha)) = b-a$

And so the measurement of the interval (a,b) is unchanged under T i.e. T preserves lebesque measure.

If S = (0, 1) then consider A = (1/5, 3/5) a subinterval of S and x Є (1/5, 3/5) i.e. x = ½ say.

Next, to show that T is ergodic would have required some rigorous iterations because α is a real number, we must take into account that α can either be rational or irrational. A simple MATLAB m-file gives us insight into what happens when α is rational. See Gewitz (2004) [2]. This is rather a rigorous iteration. However, this work presents an improvement using a simple numerical computation.

Suppose $\alpha \in Q$ i.e. of the form p/q, p, $q \in Z$, q ≠ 0 and x Є (1/5, 3/5) *i.e.* ½ say

Then from definitions (1.2) and (1.3) above

$$T_\alpha(x) = \begin{cases} x+\alpha-1, & \text{if } 1-\alpha \le x \le 1 \\ x+\alpha, & \text{if } 0 \le x < 1-\alpha \end{cases}$$

Then for $\alpha = 1/7$ and $x = 1/2$

$$T_{1/7}(1/2) = \begin{cases} 1/2+1/7-1, & \text{if } 1-1/7 \le 1/2 \le 1 \\ 1/2+1/7, & \text{if } 0 \le 1/2 < 6/7 \end{cases}$$

$$T_{1/7}(1/2)=\begin{cases}-5/14, & 6/7\le 1/2\le 1\\ 9/14, & \text{if } 0\le 1/2<6/7\end{cases}$$

Thus: $T_{1/7}(1/2)=-5/14$. For $T_{1/7}(1/2)=9/14$ does not exist for $0\le 1/2<6/7$ is not true.

Similarly,

$$T_{a}^{-1}(x)=\begin{cases}x-\alpha+1, & \text{if } 0\le x\le\alpha\\ x-\alpha, & \text{if } \alpha\le x<1\end{cases}$$

For $\alpha=1/7$ and $x=1/2$, we have

$$T_{1/7}^{-1}(1/2)=\begin{cases}1/2-1/7+1, & \text{if } 0\le 1/2\le 1/7\\ 1/2-1/7, & \text{if } 1/7\le 1/2<1\end{cases}$$

$$T_{1/7}^{-1}(1/2)=\begin{cases}19/14, & \text{if } 0\le 1/2\le 1/7\\ 5/14, & \text{if } 1/7\le 1/2<1\end{cases}$$

Thus $T_{1/7}^{-1}(1/2)=5/14$. For $T_{1/7}^{-1}(1/2)=19/14$ is not true. Reason $0\le\frac{1}{2}\le\frac{1}{7}$ is not true.

Observe that $-5/14\ne 5/14$

Therefore: $T_{1/7}(1/2)\ne T_{1/7}^{-1}(1/2)$

Now, let $\alpha\notin Q$. i.e. α = 1 and $x=1/2$ (say), since its is a Lebesgue measure [0,1].

From (2.2)

$$T_1(1/2)=\begin{cases}1/2+1-1, & \text{if } 1\text{-}1<1/2\le 1\\ 1/2+1, & \text{if } 0\le 1/2<1-1\end{cases}$$

$$T_1(1/2)=\begin{cases}1/2, & \text{if } 0<1/2\le 1\\ 3/2, & \text{if } 0\le 1/2<0\end{cases}$$

Therefore $T_1(1/2)=1/2$ if 0 < ½ ≤ 1

For $T_1(1/2)=3/2$ if 0 < ½ < 0 does not hold; reasons:

(i) 0< ½ < 0 does not exist

(ii) $3/2$ > 1, we are considering a unit circle with normalized length 1 and a lebesgue measure [0,1].

From (2.3), we have,

$$T_1^{-1}\left(\tfrac{1}{2}\right)=\begin{cases}\tfrac{1}{2}-1+1 & \text{if } 0\le \tfrac{1}{2}\le 1\\ \tfrac{1}{2}-1 & \text{if } 1\le \tfrac{1}{2}<1\end{cases}$$

$$T_1^{-1}\left(\tfrac{1}{2}\right)=\begin{cases}\tfrac{1}{2} & \text{if } 0\le \tfrac{1}{2}\le 1\\ -\tfrac{1}{2} & \text{if } 1\le \tfrac{1}{2}<1\end{cases}$$

Again, $T_1^{-1}\left(\tfrac{1}{2}\right)=\tfrac{1}{2}$ if $0<\tfrac{1}{2}\le 1$. For $T_1^{-1}\left(\tfrac{1}{2}\right)=-\tfrac{1}{2}$ if $1\le \tfrac{1}{2}<1$ is not true. Thus, $T_1\left(\tfrac{1}{2}\right)=T_1^{-1}\left(\tfrac{1}{2}\right)$. Considering a Lebesgue measure [0,1] where $\mu(\phi_s)=0$ and $\tfrac{1}{2}\in A$, A = $\left(\tfrac{1}{5},\tfrac{3}{5}\right)$. Then, $1\left(T^{-1}\left(\tfrac{1}{2}\right)\right)=1\left(T\left(\tfrac{1}{2}\right)\right)$, therefore, $\mu(T^{-1}(A)) \quad =\mu(T(A))=\mu(A)$.

Next,, $T_1^{-1}\left(\tfrac{1}{5},\tfrac{3}{5}\right)=\left(\left(\tfrac{1}{5}-1\right)+1,\left(\tfrac{3}{5}-1\right)+1\right)\cup\left(\tfrac{1}{5}-1,\tfrac{3}{5}-1\right)$

$$=\left(-\tfrac{4}{5}+1,\left(-\tfrac{2}{5}+1\right)\right)\cup\left(-\tfrac{4}{5},-\tfrac{2}{5}\right)$$
$$=\left(\tfrac{1}{5},\tfrac{3}{5}\right)\cup\left(-\tfrac{4}{5},-\tfrac{2}{5}\right)$$
$$=\left(\tfrac{1}{5}-1,\tfrac{3}{5}-1\right)\cup\left(\tfrac{1}{5}-1,\tfrac{3}{5}-1\right)$$
$$=\left(-\tfrac{4}{5},-\tfrac{2}{5}\right)\cup\left(-\tfrac{4}{5},-\tfrac{2}{5}\right)$$

Clearly, $\mu\left(\tfrac{1}{5}-1,\tfrac{3}{5}-1\right)=\tfrac{3}{5}-\tfrac{1}{5}=\tfrac{2}{5}$ is unchanged under T_1. Recall that we take only the additive fraction $\frac{2}{5}$. Therefore, T_α is ergodic if and only if $\alpha\notin Q$ as shown above. This agrees with the preposition below.

Proposition **2.3**

T_α i.e. rotation on the unit circle S_1, is ergodic with respect to lesbesgue measure if and only if $\alpha\notin Q$. See Gewitz (2004) [2] and Petersen (1983) [7].

Proof

We employ Birkhoff's ergodic Theorem – see Gewitz (2004) [2] for detail. ■

Lemma **2.4**

If T_α and T_α^{-1} exist and preserve lesbesgue measure, then T_α is an automorphism with discrete spectrum whenever α is unity.

Proof

Let T_α: (0, 1) $\rightarrow$ (0, 1) be measure-preserving transformation and α lebesgue measure. The proposition above shows that T_α is ergodic.

Since the measures are identical then T_α is an endomorphism. It only remains to show that T_α is bijective (invertible). For every $\alpha\notin Q$ and using (2.2)

$$T_\alpha(x) = \begin{cases} x, & \text{if } 0 < x \le 1 \\ \\ x+1 & \text{if } 0 < x < 0 \end{cases}$$

$T_\alpha(x_1) = T_\alpha(x_2)$, iff $x_1 = x_2, \forall\, x_1, x_2 \in (0,1)$. $T_\alpha(x)$ is one-one. Obviously $T_\alpha(x)$ is onto, for every $d \in (0,1) \exists a \in (0,1)$ such that $T_\alpha(a) = d$. Therefore, $T_\alpha(x)$ is both 1 – 1 and onto. ■

The necessary and sufficient condition for T_α to be invertible is that T_α is one-one and onto i.e it must be bijective. Then $T_\alpha(x)$ is an invertible-measure-preserving Transformation on a lebesgue measure[0, 1].Hence T_α is an automorphism (compare Definition 1.6). Next, we show that the automorphism T_α has discrete spectrum.

A spectrum is any array or sequence of vectors or functions, e.g. an orthonormal basis of L_2 [G]. Let $\{f_j\}$ be an orthonormal basis of $L_2(G)$ then each f_j is an eigen vector of automorphism *T*- see Whankim and Kim (1983) [11] and Levi (2000) [7].

In measure theory any set that is finite or countable is defined to have measure zero, therefore, if we simply expand the set in question by including an arbitrarily small positive interval centered at each periodic point, then under rotation on the circle (which we have shown to be measure -preserving) if we chose the intervals to be arbitrarily small, their union clearly has measure less than unity. In the interval we will have a set of positive real numbers i.e $X = [0, \infty]$.

We will observe a sequence which forms a basis of the space $\langle a_n \rangle = \langle \ldots 1/5, 1/4, 1/3 \ldots \rangle$. Obviously the functions span the space. Hence, T_α has discrete spectrum.

3.0 Conclusion

The entire work is a construction of an automorphism with discrete spectrum. The proposition summarizes the first stage of the construction. Then the lemma is a climax of the construction process. We have been able to show that $T_\alpha : (0,1) \to (0,1)$ preserves lebesgue measure using an interval computation method. This establishes that when α is rational then T_α will not be ergodic except when α is unity. The work show further that T_α is an automorphism with discrete spectrum and that it is not only a homomorphism since it is an invertible measure preserving transformation.

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