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On the effects of wave steepness on higher order Stokes waves

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Abstract

The effect of wave steepness on higher order finite amplitude Stokes waves is investigated analytically and numerically. It is shown that the phase speed increases as the wave steepness increases thereby initiating the wave instabilities. As the order increases, the phase speed also increases .However, the impact of wave steepness on phase speed is more pronounced for the wave steepness parameter $\varepsilon \geq 0.09$, where $0.02 \leq \varepsilon \leq 0.33$

Beyond the upper bound, instability – wave breaking comes in.

1.0 Introduction:

Wave steepness is the ratio of the wave height to wave wavelength, $\varepsilon = \frac{H}{L}$ where *H* is the wave height and *L* the wavelength. The wave steepness factor ε ranges from 0.02 to 0.14 for normal incidence wave. According to Stokes (Kinsman, 1965) [6], the maximum value of $\varepsilon = \frac{1}{7}$. From the recent study, the upper bound of ε is $\frac{1}{3}$ (Constantin, 2001, Leblanc, 2004) [2].

In oceanographic environment, sailors are worried whenever they run into steep waves even though their amplitudes may be relatively small.

From old times, the wave motion of the ocean waves extremely attracted the attention of mankind. Occurrence of extremely large and steep ocean breaking waves imposes a hazard to fishing boats, ships, offshore and onshore structures. Small amplitude waves are sinusoidal. As the wave amplitude grows, the crest becomes steeper and sharper whilst the troughs flatten (Lukomsky et al, 2003) [8].

Enhanced evidence and description of the wave kinematics during steep wave events at sea are often utilized by the offshore and ocean engineering industry. The velocities in steep waves are required for subsequent analysis of loads on ships, offshore platforms, tension legs and risers (Grue, 2004) [5].

Waves are one of the most important phenomena to be considered among the environmental phenomena controlling maritime structures. The presence of waves makes the design procedure for maritime structures quite different from identical structures on land.

2.0 Wave steepness and stability

An essential part in the study of water waves is their stability. Instabilities arise as the wave steepness increases. It is well known that deep-water gravity waves are unstable to side band perturbations (Whitham, 1974 [14]; Stansell et al, 2003 [13]). Stokes waves are stable at $kh < 1.363$ or $h < \frac{1.363L}{\pi}$ and are unstable if $kh > 1.363$ where k is the wave number and h is

the mean water depth (Benjamin and Feir, 1967 [1]; Sedlestsky, 2005 [12]).

The stability conditions for waves in shallow water differ significantly from those for waves in deep water (Mansour et al, 1999 [10]). In Stokes waves, local ε cannot grow exponentially. Stokes waves are unstable when the steepness parameter ε exceeds $\frac{1}{3}$. Following Leblanc (2004) [7], the critical steepness of the waves is $\frac{1}{3}$. The lengthening of the waves is a necessary condition for wave energy to grow. Otherwise, the waves will be too steep to be stable (Stansell et al, 2003) [13].

Stokes calculated theoretically that wave would remain stable only if the water particle velocity at the crest is less than the wave celerity (phase velocity). If the waves heights were to become so large that the water particle velocity at the crest exceeds the wave celerity, the waves would become unstable and break. Stokes found that a wave having a crest angle less than 120 degree would break (angle between two lines tangent to the surface profile at the wave crest).

3.0 To express Stokes waves in terms of wave steepness

Following Kinsman (1965) [6], Fenton (1985) [4], Oyetunde et al (2004) [11], the expressions for the third, fourth and fifth order Stokes waves height can be expressed in terms of the wave amplitude and steepness.

$$H = 2a + \frac{3}{4}k^2a^3 \qquad (3.1)$$

and multiplying through by k, we have

$$kH = 2ak + \frac{3}{4}k^3a^3 \qquad (3.2)$$

$$2ak = kH - \frac{3}{4}k^3a^3 \qquad (3.3)$$

$$ak = \frac{1}{2}\left[kH - \frac{3}{4}k^3a^3\right] \qquad (3.4)$$

$$ka = \frac{1}{2}kH - \frac{3}{8}k^3a^3 \qquad (3.5)$$

But $\frac{1}{2}kH = \frac{1}{2}\left(\frac{2\pi}{L}\right)H$ (since $k = \frac{2\pi}{L}$) $= \pi\left(\frac{H}{L}\right) = \pi$. Also,

$$k^3a^3 = (ka)^3 = \left(\frac{2\pi}{L}\right)^3 a^3 = \frac{8\pi^3a^3}{L^3} = \left(\frac{2a}{L}\right)^3\pi^3 = \left(\frac{H}{L}\right)^3\pi^3 = \pi^3\varepsilon^3$$

therefore

$$\mathrm{ka} = \pi\varepsilon - \frac{3}{8}\pi^3\varepsilon^3 \qquad (3.6)$$

Thus, $H = 2a$ to a first order, $H = 2a + O(a^3)$ to order higher than first where a is the wave amplitude.

Following Kinsman (1965) [6], the third order Stokes waves is of the form,

$$\eta = -a\cos kx + \frac{1}{2}ka^2\cos 2kx - \frac{3}{8}k^2a^3\cos 3kx \qquad (3.7)$$

Expressing the profile of third order Stokes waves in terms of wave steepness, we have

$$\frac{\eta}{a} = -\cos kx + \frac{1}{2}\pi\varepsilon\left(1 - \frac{3}{8}\pi^2\varepsilon^3\right)\cos 2kx - \frac{3}{8}\pi^2\varepsilon^2\cos 3kx \tag{3.8}$$

while the phase speed is $$c^2 = \frac{g}{k}(1 + a^2k^2) \tag{3.9}$$

and in term of wave steepness, $$c^2 = \frac{g}{k}(1 + \pi^2\varepsilon^2) \tag{3.10}$$

To fourth order,

$$\eta = -a\cos kx + \left(\frac{1}{2}ka^2 + \frac{17}{24}k^3a^4\right)\cos 2kx - \frac{3}{8}k^2a^3\cos 3kx + \frac{1}{3}k^3a^4\cos 4kx \tag{3.11}$$

$$\frac{\eta}{a} = -\cos kx + \left(\frac{1}{2}ka + \frac{17}{24}k^3a^3\right)\cos 2kx - \frac{3}{8}k^2a^2\cos 3kx + \frac{1}{3}k^3a^3\cos 4kx \tag{3.12}$$

$$= -\cos kx + \frac{1}{2}ka\left(1 + \frac{17}{12}k^2a^2\right)\cos 2kx - \frac{3}{8}k^2a^2\cos 3kx + \frac{1}{3}k^3a^3\cos 4kx \tag{3.13}$$

$$= -\cos kx + \frac{1}{2}\left(\pi\varepsilon - \frac{3}{8}\pi^3\varepsilon^3\right)\left[1 + \frac{17}{12}\left(\pi\varepsilon - \frac{3}{8}\pi^3\varepsilon^3\right)^2\right]\cos 2kx - \frac{3}{8}\left(\pi\varepsilon - \frac{3}{8}\pi^3\varepsilon^3\right)^2\cos 3kx + \frac{1}{3}\left(\pi\varepsilon - \frac{3}{8}\pi^3\varepsilon^3\right)^3\cos 4kx \tag{3.14}$$

$$= -\cos kx + \frac{1}{2}\left(\pi\varepsilon - \frac{3}{8}\pi^3\varepsilon^3\right)\left[1 + \frac{17}{12}\pi^2\varepsilon^2 - ....\right]\cos 2kx - \frac{3}{8}\left[\pi^2\varepsilon^2 - .....\right]\cos 3kx + \frac{1}{3}\left[\pi^3\varepsilon^3 - .....\right]\cos 4kx \tag{3.15}$$

$$= -\cos kx + \frac{1}{2}\left(\pi\varepsilon + \frac{17}{12}\pi^3\varepsilon^3 - \frac{3}{8}\pi^3\varepsilon^3\right)\cos 2kx - \frac{3}{8}\pi^2\varepsilon^2\cos 3kx + \frac{1}{3}\pi^3\varepsilon^3\cos 4kx$$
$$= -\cos kx + \frac{1}{2}\pi\varepsilon\left(1 + \frac{25}{24}\pi^2\varepsilon^3\right)\cos 2kx - \frac{3}{8}\pi^2\varepsilon^2\left(1 - \frac{3}{4}\pi^2\varepsilon^2\right)\cos 3kx + \frac{1}{3}\pi^3\varepsilon^3\cos 4kx \tag{3.16}$$

Following Kinsman (1965) [6], fourth order phase speed is

$$c^2 = \frac{g}{k}\left(1 + \beta^2k^2 + \frac{1}{2}\beta^4k^4\right) = \frac{g}{k}\left(1 + \pi^2\varepsilon^2 + \frac{1}{2}\pi^4\varepsilon^4\right) \tag{3.17}$$

To fifth order,

$$\eta = -a\cos kx + \left(\frac{1}{2}a^2k + \frac{11}{6}a^4k^3\right)\cos 2kx - \left(\frac{3}{8}a^3k^2 + \frac{235}{96}a^5k^4\right)\cos 3kx + \frac{1}{3}k^3a^4\cos 4kx - \frac{31}{96}a^5k^4\cos 5kx \tag{3.18}$$

$$\frac{\eta}{a} = -\cos kx + \left(\frac{1}{2}ak + \frac{11}{6}a^3k^3\right)\cos 2kx - \left(\frac{3}{8}a^2k^2 + \frac{235}{96}a^4k^4\right)\cos 3kx + \frac{1}{3}k^3a^3\cos 4kx - \frac{31}{96}a^4k^4\cos 5kx \tag{3.19}$$

$$= -\cos kx + \frac{1}{2}ak\left(1 + \frac{11}{3}a^2k^2\right)\cos 2kx - a^2k^2\left(\frac{3}{8} + \frac{235}{96}a^2k^2\right)\cos 3kx + \frac{1}{3}k^3a^3\cos 4kx - \frac{31}{96}a^4k^4\cos 5kx$$

$$= -\cos kx + \frac{1}{2}(\pi\pi\varepsilon\frac{3}{8}\pi^3\varepsilon^3)\left[1+\frac{11}{3}\pi^2\varepsilon^2 - ...\right]\cos 2kx - \pi^2\varepsilon^2\left[\frac{3}{8}+\frac{235}{96}\pi^2\varepsilon^2 + ...\right]\cos 3kx$$
$$+ \frac{1}{3}\pi^3\varepsilon^3\cos 4kx - \frac{31}{96}\pi^4\varepsilon^4\cos 5kx \qquad (3.20)$$

$$= -\cos kx + \frac{1}{2}\left[\pi\varepsilon + \frac{79}{24}\pi^3\varepsilon^3\right]\cos 2kx - \left[\frac{3}{8}\pi^2\varepsilon^2 + \frac{235}{96}\pi^4\varepsilon^4\right]\cos 3kx$$
$$+ \frac{1}{3}\pi^3\varepsilon^3\cos 4kx - \frac{31}{96}\pi^4\varepsilon^4\cos 5kx \qquad (3.21)$$

Following Oyetunde et al (2004) [11], the fifth order phase speed is

$$c^2 = \frac{g}{k}(1 + \beta^2k^2 + \beta^4k^4 \;) = \frac{g}{k}(1+ \pi^2\varepsilon^2 + \pi^4\varepsilon^4) \qquad (3.22)$$

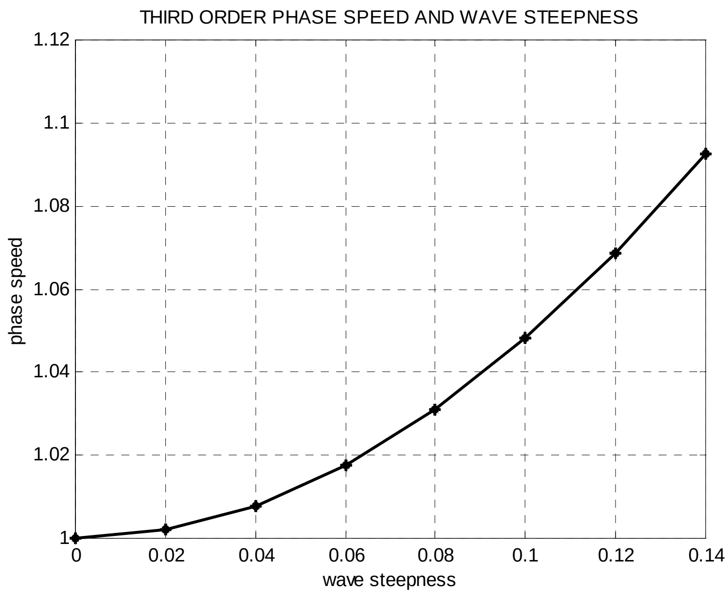

Figure 3.1: Third order Phase speed and wave steepness

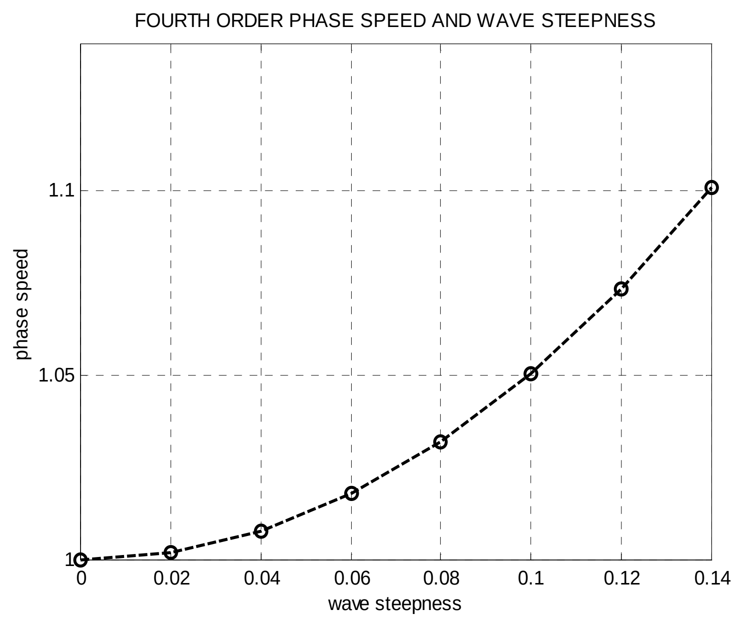

Figure3.2: Fourth order phase speed and wave steepness

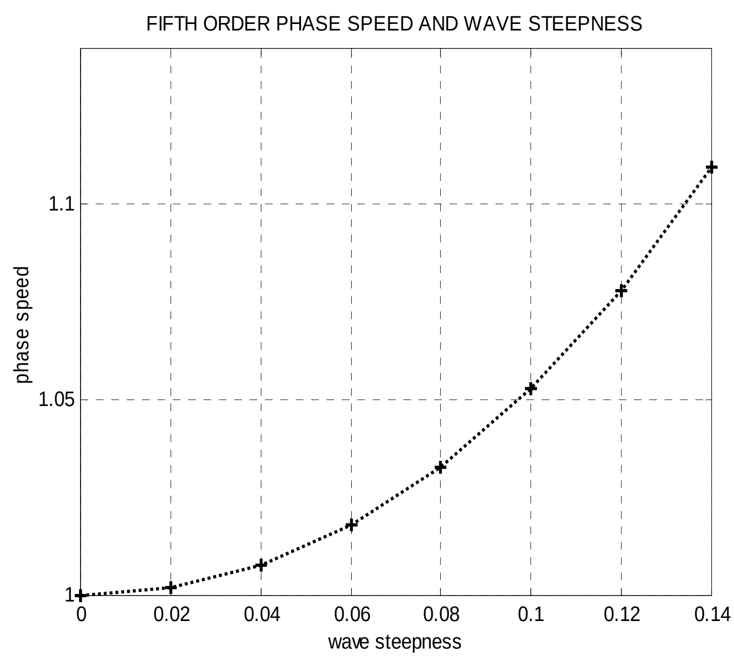

Figure 3.3: Fifth order phase speed and wave steepness.

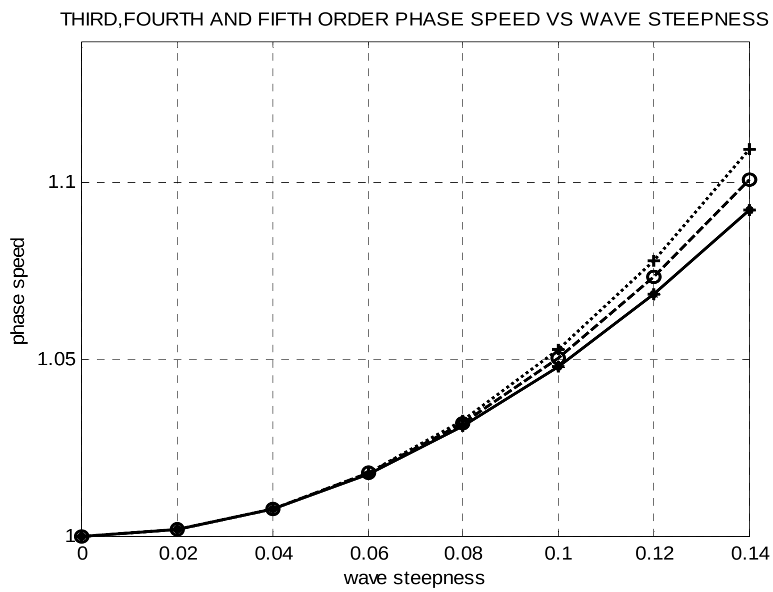

Figure 3.4: Third, fourth and fifth order phase speed and wave steepness in increasing order with the third order having the lowest value.

4.0 Conclusion

It can be observed that increase in wave steepness in Stokes waves often found in deep water has positive impact on the phase speed. The phase speed increases due to increase in wave steepness which equally increases the wave possibility of instabilities. The increase however is apparent for steeper waves where the wave steepness parameter $\varepsilon \geq 0.09$. The apparent increase in phase speed and wave instabilities has impacts on fishing boats, ships, offshore oil facilities, coastal and ocean engineering generally. It is also observed that the higher the order of Stokes waves, the higher the phase speed.

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