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On the impact of wave-current on Stokes waves

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Abstract

This study considers the impact of wave - current on Stokes waves in deep water. Using separately, the third, fourth and fifth order approximations of wave profile functions respectively and the determined expressions for wave – current speed , it is shown that the wave - current speed is more intense on the surface of the ocean, but however, the intensity reduces exponentially downwards with the depth. The present analysis also suggests that the wave – current speed increases with increase in wave steepness. It is further deduced that the wave- current speed is the same in magnitude for both fourth and fifth order; thus, showing the convergence of solutions concerning wave - current speed in Stokes waves as the order increases.

1.0 Introduction:

In the last decades, there has been significant progress in the analytical and numerical modeling of ocean surface waves (Longuet-Higgins, 1953; Phillips, 1977) [6, 7]. Two wave theories are applied in the subsequent calculations: the Stokes wave's theory and the cnoidal theory. However, the Stokes theory approach can only be used for a limited range of wave parameters. Close to the shore with diminishing water depths, the Stokes approximations are not valid. In this area, therefore, the application of cnoidal wave theory is applicable (Ostrowski et al, 2006) [7].

In coastal areas, detailed knowledge of the surface wave field is very important for the understanding of hydrodynamic and transport processes. The interaction of waves with the bottom topography and the ocean current are essential in understanding and predicting the developments in coastal seas. Coastal hydrodynamics processes are the force driving mechanism responsible for sediment transport processes. A reliable description of the wave - current field is therefore crucial for a precise determination of sediment transport rates.

2.0 Stokes waves and wave- current speed

The oceanic wind – driven current profile is difficult to observe because the velocities are small and of similar magnitude to the velocities associated with inertia oscillations and surface wave motions. The difficulties in observation are due to similarity in magnitude between the current speed and the speeds associated with other physical process. This suggests that other processes may be dynamically important. Surface waves are a ubiquitous feature of the ocean surface. The leading order water motions associated with the surface waves are periodic and, at least below the troughs, do not affect the time-averaged, mean, current profile (Polton et al 2005) [10].

Surface waves also produce a mean Lagrangian transport in their direction of propagation, that is the Stokes drift. (Longuet-Higgins, 1953) [6].

The energy transport conditions and the set–down measure are accounted for, making it possible among other things to calculate the changes in wave height and mean water surface due to varying bed topography and varying currents.

The interaction of waves and currents became fully understood when Louguet – Higgins and Stewart presented the correct equation for the energy flux of a wave superimposed on a current (Jonsson et al, 1995) [4]. Following Jonsson et al (1995) [4], Stokes theory is not assumed to be applicable to shoaling calculations for finite amplitude waves as the main part of the changes in wave form occur in the near shore line.

In wave propagation study, there is need to consider water depth and currents. For operational modeling purposes, wave currents have often been mostly ignored. The influence of currents on waves depends on local features of the current field and wave propagation relative to the current direction as reported by World Meteorological Organization (1998) [14].

Clearly, a current flowing in same direction with the waves will have the effect of increasing the speed of the waves, although the wave period (*T*) remains constant i.e. $T = \frac{L_0}{c_0} = \frac{L}{c+U}$

where

L_0 = wavelength when current speed is zero;

c_0 = wave speed when current speed is zero;

L = wavelength in the moving current;

c = wave speed in the moving current;

U = speed of the current.

Because $c+U$ is greater than c_0, then for T to remain constant L must be greater than L_0, i.e. the waves get longer. Moreover, the waves heights get correspondingly lower, because the rate of energy transfer depends upon group velocity (half wave speed in deep water) and wave height. If the rate of energy transfer is to remain constant, then as speed increases, wave height must decrease. However, in practice, not all of the wave energy is retained in the wave system; some is transferred to the current, causing wave height to decrease still further.

Conversely, if the current flow is opposed to the direction of wave propagation, then wavelength L in the moving current will decrease and the wave will get shorter and amplitude higher. Wave height will be further increased as a result of energy gained from the current (World Meteorological Organization, 1998) [14].

For opposing currents, the wave length always decreases with decreasing depth. For strong flowing currents, the wave length increases due to the increasing current velocity on the wave profile. The larger the steepness, the more non-linear the wave and the greater the effects of the added higher order terms.

3.0 Wave-current speed

3.1 Third order wave – current speed U*

Consider the flow through a vertical strip of unit width extending from the surface, $z = \eta(x, t)$, to some depth $z = -h$ (Kinsman,1965) [5]. Also, consider the notion of steady motion where the axes are moving with the phase speed c of the wave in the positive x – direction. The problem is to locate the average depth z at which the streamline, $\psi = -ch$. Starting from the third order Stokes wave expansion, thus:

$$\Psi = cz + cae^{kz} \cos kx \qquad (3.1)$$

evaluating (3.1) at $z=\bar{z}$ and solving for $\bar{z}+h$,

$\bar{z}+h = -ae^{k\bar{z}}\cos kx$ at $z = \bar{z}$, and $\Psi = -ch$. (3.2)

Let $a = a^1 e^{kh}$ (3.3)

Then $\bar{z}+h = -a^1 e^{kh} e^{k\bar{z}} \cos kx$ (3.4)

$$\bar{z}+h = -a^1 e^{k(h+\bar{z})} \cos kx \quad (3.5)$$

From the vanishing of streamline at $z=\eta(x)$, $\eta = \beta e^{k\eta}\cos kx$ (3.6)

Equations (3.5) and (3.6) are of the same form with $\bar{z}+h \approx \eta$, $-a^1 \approx \beta$ and $e^{k(z+h)} = e^{k\eta}$, $\eta = \frac{1}{2}k\beta^2 + \beta(1+\frac{9}{8}k^2\beta^2)\cos kx + \frac{1}{2}k\beta^2\cos 2kx + \frac{3}{8}k^2\beta^3\cos 3kx$ is the third order expansion for $\eta = \beta e^{k\eta}\cos kx$ and the third order expansion for $\bar{z}+h = -ae^{k(h+\bar{z})}\cos kx$ will be similar $\bar{z}+h = \frac{1}{2}ka^{1^2}\ \frac{1}{2}ka^2 e^{-2kh}$ (3.7)

The constant term of the expansion is the mean position to be obtained

$$\bar{z} = -h + \frac{1}{2}ka^2 e^{-2kh} \quad (3.8)$$

$\bar{z}$ is the depth where $\Psi = -ch$, thus, $\bar{z} = \textit{average depth}$. If the speed c of the moving axes is maintained while the wave itself is raised to the third order, $\eta = \frac{1}{2}ka^2$, the layer from the surface to

$$\bar{z} \text{ is } \quad \eta - \bar{z} = \frac{1}{2}ka^2 + h - \frac{1}{2}ka^2 e^{-2kh} \quad (3.9)$$

$$= h + \frac{1}{2}ka^2\left(1-e^{-2kh}\right) \quad (3.10)$$

The volume flux is the product $-c\left[h+\frac{1}{2}ka^2\left(1-e^{-2kh}\right)\right]$. The volume flux between the two streamlines $\Psi = 0, z=\eta\approx 0$ at surface to another streamline $\Psi = -ch, z=\bar{z}$ *is -ch.* Thus replacing *h* by *z* gives the volume flux attributable to the wave alone as a function of depth,

$$V_W = \frac{1}{2}ka^2\, c\left(1-e^{2kz}\right), \quad (-h<z<0) \quad (3.11)$$

If h is very large, total volume flux through a strip of unit width arising from the wave is approximately

$$V_W \approx \frac{1}{2}ka^2\, c \quad (3.12)$$

Differentiating (3.11) with respect to *z* gives $U^* = k^2a^2ce^{2kz} = \pi^2\delta^2ce^{2kz}$ (3.13)

where $\delta = kh = \frac{2\pi h}{L}$. For z = 0 (undisturbing free surface),

$$U_0^* = k^2a^2c = \pi^2\delta^2c \quad (3.14)$$

While U_o* may be fairly large, *U** is subject to severe exponential damping with depth. Thus, appreciable wave – induced currents which run with the waves, and therefore downwind, are on this analysis to be expected only near the surface. Wave currents arising from waves of finite amplitude can reach speeds for which we must allow in practical problems.

Wave currents must be taken into account in planning search for life rafts and in similar situations in oceanography. Volume flux and wave –current speed in shallow water are very difficult to evaluate for waves of finite amplitude.

3.2 Fourth order wave-current speed

Following Kinsman (1965) [5]

$$\frac{\Psi}{c} = z + \beta e^{kz}\cos kx + \gamma e^{2kz}\cos 2kx \tag{3.15}$$

where $\beta = a - \frac{9}{8}k^2a^3 + o(a^4), \quad \gamma = \frac{1}{2}\beta^4k^3 = \frac{1}{2}a^4k^3$, therefore

$$\frac{\Psi}{c} = z + ae^{kz}\cos kx + \frac{1}{2}a^4k^3e^{2kz}\cos 2kx \tag{3.16}$$

$$\Psi = cz + cae^{kz}\cos kx + \frac{c}{2}a^4k^3e^{2kz}\cos 2kx \tag{3.17}$$

evaluation at $z = -\bar{z}$ and solving for $\bar{z} + h$, where $\Psi(x, \bar{z}) = -ch$

$$\bar{z} + h = -a^1e^{k\bar{z}}\cos kx - \frac{1}{2}a^4k^3e^{2k\bar{z}}\cos 2kx \tag{3.18}$$

putting $a = a^1e^{kh}$

$$= -a^1e^{k(h+\overline{z})}\cos kx - \frac{1}{2}(a^1)^4k^3e^{2k(2h+\bar{z})}\cos 2kx \tag{3.19}$$

from fourth order expansion

$$\eta = \frac{1}{2}ka^2 + k^3a^4 + a\cos kx + (\frac{1}{2}ka^2 + \frac{17}{24}k^3a^4)\cos 2kx + \frac{3}{8}k^2a^3\cos 3kx + \frac{1}{3}k^3a^4\cos 4kx \tag{3.20}$$

$$\eta = \frac{1}{2}ka^2 + k^3a^4 + \text{(oscillatory term)} \tag{3.21}$$

$$\eta - \bar{z} = \frac{1}{2}ka^2 + k^3a^4 + (h - \frac{1}{2}ka^2e^{-2kh} + k^3a^4e^{-4kh}) \tag{2.22}$$

$$= h + \frac{1}{2}ka^2(1 - e^{-2kh}) + k^3a^4(1 - e^{-4kh}) \tag{2.23}$$

The volume flux is the product given by

$$V = -c\left[h + \frac{1}{2}ka^2(1 - e^{-2kh}) + k^3a^4(1 - e^{-4kh})\right] = -ch - \frac{1}{2}ka^2c(1 - e^{-2kh}) - k^3a^4c(1 - e^{-4kh}) \tag{3.24}$$

Replacing *h* by *z* gives the volume flux V_w attributable to the wave alone as a function of depth, that is

$$V_W = \frac{1}{2}ka^2c(1 - e^{-2kh}) + k^3a^4c(1 - e^{-4kh}) \tag{3.25}$$

In the case of infinite depth, h is very large, thus, total volume flux through a strip of unit width arising from the wave is approximately $V_W \approx \frac{1}{2}ka^2c + k^3a^4c$ (3.26)

Replacing *h* by $z\ (-h < z < 0)$ in (3.26) and differentiating with respect to *z*, gives

$$U^* = k^2a^2ce^{2kz} + 4k^4a^4ce^{4kz} \tag{3.27}$$

$$= k^2a^2ce^{2kz}(1 + 4k^2a^2ce^{2kz})$$

$$= \pi^2\delta^2ce^{2kz}(1 + 4k^2a^2e^{2kz}) \tag{3.28}$$

where U^* is the wave-current speed. For $z = 0$, that is, free surface undisturbed

$$U_0^* = k^2a^2c + 4k^4a^4c \quad (3.29)$$

$$= k^2a^2c(1+4k^2a^2) = \pi^2\delta^2c(1+4\pi^2\delta^2)$$

$$= \pi^2\delta^2c + 4\pi^4\delta^4c \quad (3.30)$$

The last term k^3a^4c in the right hand of (3.26) is the effect of using complete fourth expansion for $\eta(x,t)$ and is quite new and interesting. The additional term $4\pi^4\delta^4c$ in (3.30) follows subsequently.

3.3 Fifth order wave-current speed

$$\frac{\Psi}{c} = z + \beta e^{kz}\cos kx + \gamma e^{2kz}\cos 2kx \quad + \alpha e^{3kz}\cos 3kx \text{ where } \beta = a - \frac{9}{8}k^2a^3 +$$

$O(a^4)$, $\gamma = \frac{1}{2}a^4k^3$, $\alpha = \frac{1}{12}a^5k^4$

$$\Psi = cz + cae^{kz}\cos kx + \frac{c}{2}a^4k^3e^{2kz}\cos 2kx + \frac{c}{12}a^5k^4e^{3kz}\cos 3kx \quad (3.31)$$

when $\Psi(x, \bar{z}) = -ch$ at $z = \bar{z}$, putting $a = a^1e^{kh}$

$$\bar{z} + h = -a^1e^{k\bar{z}}\cos kx - \frac{1}{2}a^4k^3e^{2k\bar{z}}\cos 2kx - \frac{1}{12}a^5k^4e^{3k\bar{z}}\cos 3kx \quad (3.32)$$

$$\bar{z} + h = -a^1e^{kh}e^{k\bar{z}}\cos kx - \frac{1}{2}(a^1)^4k^3e^{4kh}e^{2k\bar{z}}\cos 2k - \frac{1}{12}(a^1)^5k^4e^{5kh}e^{3k\bar{z}}\cos 3kx \quad (3.33)$$

$$\bar{z} + h = -a^1e^{k(h+\bar{z})}\cos kx - \frac{1}{2}(a^1)^4k^3e^{2k(2h+\bar{z})4kh}\cos 2kx - \frac{1}{12}(a^1)^5k^4e^{k(5h+3\bar{z})}\cos 3kx \quad (3.34)$$

But

$$\eta = -a\cos kx + \left(\frac{1}{2}ka^2 + \frac{11}{6}k^3a^4\right)\cos 2kx - \left(\frac{3}{8}k^2a^3 + \frac{235}{96}k^4a^5\right)\cos 3kx$$

$$+\frac{1}{3}k^3a^4\cos 4kx - \frac{31}{96}k^4a^5\cos 5kx + \frac{1}{2}ka^2 + k^3a^4$$

(fifth order profile), $\eta = \frac{1}{2}ka^2 + k^3a^4$, similar to equation (3.8)

$$\eta - \bar{z} = \frac{1}{2}ka^2 + k^3a^4 + h - \frac{1}{2}ka^2e^{-2kh} + k^3a^4e^{-4kh} = h + \frac{1}{2}ka^2(1-e^{-2kh}) + k^3a^4(1-e^{-4kh})$$

The volume flux

$$V = -c\left[h + \frac{1}{2}ka^2(1-e^{-2kh}) + k^3a^4(1-e^{-4kh})\right] = -ch - \frac{1}{2}ka^2c(1-e^{-2kh}) - k^3a^4c(1-e^{-4kh}).$$

$$V_W = \frac{1}{2}ka^2c(1-e^{-2kz}) + k^3a^4c(1-e^{-4kz})$$

$$V_W \approx \frac{1}{2}ka^2c + k^3a^4c \text{ (for infinite depth).}$$

Also replace h by z in $(-h < z < 0)$ and differentiating with respect to z gives as we have in fourth order gives $U^* = k^2a^2ce^{2kz} + 4k^4a^4ce^{4kz} = \pi^2\delta^2ce^{2kz}(1+4k^2a^2e^{2kz})$.

For z = 0, that is, free surface undisturbed

$$U_0^* = k^2a^2c + 4k^4a^4c = \pi^2\delta^2c(1+4\pi^2\delta^2) = \pi^2\delta^2c + 4\pi^4\delta^4c$$

It can be observed that the wave current speed is the same for both fourth and fifth order.

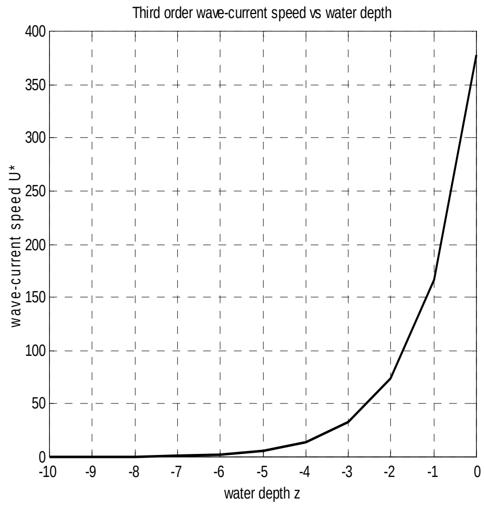

Figure 2.1: Graph of U^* against water depth for third order

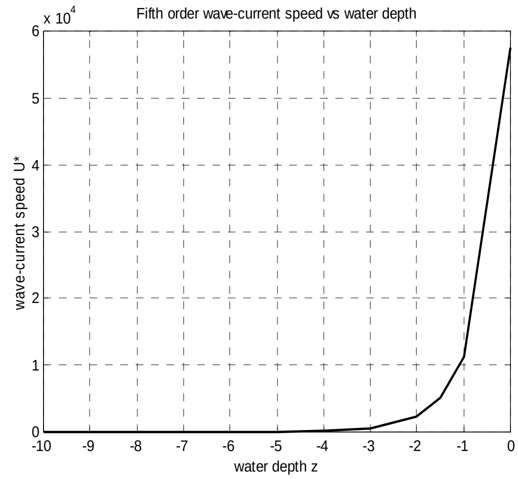

Figure 2.2: Graph of U^* against water depth for fourth and fifth order

4.0 Conclusion

It can be observed that the impact of wave-current speed has more significant effect on the surface of the ocean, but however, reduces exponentially downwards with the increasing depth increases towards the seabed. Also, the wave – current speed increases with increase in wave steepness. The wave - current speed is however the same for both fourth and fifth order in view of the additional term involved and is higher than that of third order. This seems to suggest that the wave-current is constant from fourth order expansion and beyond. Finally, it is re-emphasized that the theory of Stokes finite amplitude wave theory is mostly applicable in deep water zones.

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