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On a two–small–parameter dynamic stability of a lightly damped spherical shell pressurized by a harmonic excitation

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Abstract

This paper is concerned with asymptotic solution, using multi-timing technique, of a nonlinear coupled elastic system in a dynamical setting where the structure investigated is a discretized imperfect spherical shell .The normal displacement at a point on the shell surface is assumed to be partly in the form of a symmetric pre-buckling mode, and partly in the form of buckling modes that have both axisymmetric and non-axisymmetric components. The geometric imperfection is assumed to be in the shape of the buckling modes. The explicitly time-dependent load function is assumed harmonic (or periodic) and the dynamic buckling load is obtained nontrivially with specializations of the results made. The results show, among other things, that (i) the only condition under which the effects of any coupling is felt is if none of the imperfections in the shapes of the modes coupling is neglected and (ii) neglecting an imperfection automatically nullifies the effects of the nonlinearity that is in the shape of the neglected imperfection.

1.0 Introduction

The subject of dynamic buckling has been a thriving area of research for the past forty years and there seems to be no letting down of the zeal, steam and even, impetus that the subject matter has so far generated. In most of the analyses frequently encountered, the ensuing time dependent loading histories are usually implicit in the time variable with the result that the resultant Mathematical formulation becomes time-autonomous. In this way, simple

Mathematical techniques, including phase plane analysis, are easily evoked to solve the problem. In this investigation, we however encounter a loading history that is not merely explicitly time–dependent, but is also dynamically periodic or harmonic. In this case, the application of phase plane analysis is inappropriate and recourse must therefore be had to a much more superlative technique to address the problem analytically. We stress that a knowledge of the dynamic buckling load of structures is of tremendous structural and engineering essence particularly in determining the state of dynamic stability of such structures for purposes of practical applications .Needless stressing that it is a failure criterion in places such as nuclear assemblies and Aeronautical and other Aerospace industries where thin-walled elastic materials are frequently used and where the interplay of the academic resourcefulness of the Applied Mathematicians and Structural Engineers, engaged in designing and manufacturing of modern day space vehicles and parts there of, is a professional calling.

2.0 Mathematical Formulation.

Using step loading as a test case, the undamped version of the structure investigated here was originally investigated by Danielson [1] using the technique of Mathieu–type of instability .The inherent interest in the use of Mathieu-type instability lies in the fact that there are already in existence ,tables for Mathieu–type of instability where readings could be easily read off without much mental labour. We shall however avoid the use of Mathieu-type of instability , for as noted by Budiansky [2, page 100] , Mathieu-type of instability is usually associated with many cycles of oscillations as opposed to just one shot of oscillation that normally triggers off dynamic buckling. Danielson however assumed the normal displacement $W(x,y,\hat{t})$ at a point on the shell surface to be given by

$$W(x,y,\hat{t}) = \xi_0(\hat{t})\mathrm{W}_0(x,y) + \xi_1(\hat{t})\mathrm{W}_1(x,y) + \xi_2(\hat{t})\mathrm{W}_2(x,y) \qquad (2.1)$$

where W_0 ,W_1 and W_2, which depend on the spatial variables x and y , are respectively the pre-buckling symmetric mode, the axisymmetric buckling mode and the non-axisymmetric buckling mode while $\xi_0(\hat{t})$,$\xi_1(\hat{t})$ and $\xi_2(\hat{t})$ are their respective time–dependent amplitudes. Danielson also assumed the imperfection $\overline{W}(x,y)$ in the shapes of the buckling modes, namely

$$\overline{W}(x,y) = \bar{\xi}_1 \,\mathrm{W}_1(x,y) + \bar{\xi}_2 \,\mathrm{W}_2(x,y) \qquad (2.2)$$

where $\bar{\xi}_1$ and $\bar{\xi}_2$ are the respective amplitudes of $\mathrm{W}_1(\mathrm{x},\mathrm{y})$ and $\mathrm{W}_2(x,y)$ in (2.2). In most similar investigations, it is customary to assume the inequalities $0<\bar{\xi}_1<1,\ \ 0<\bar{\xi}_2<1$. This restriction, though valid, is optional in this investigation .On substituting equations (2.1) and (2.2) into the associated compatibility and equilibrium equations characterizing a spherical shell, Danielson obtained the following coupled equations for the imperfect and undamped spherical shell :

$$\frac{1}{\omega_0^2}\frac{d^2\xi_0}{d\hat{t}^2} + \xi_0 = \lambda f(\hat{t}), \hat{t} > 0 \qquad (2.3)$$

$$\frac{1}{\omega_1^2}\frac{d^2\xi_1}{d\hat{t}^2} + \xi_1(1-\xi_0) - k_1\xi_1^2 + k_2\xi_2^2 = \bar{\xi}_1\xi_0 \text{ , } \hat{\mathrm{t}} > 0 \qquad (2.4)$$

$$\frac{1}{\omega_2^2}\frac{d^2\xi_2}{d\hat{t}^2} + \xi_2(1-\xi_0) + \xi_1\xi_2 = \bar{\xi}_2\xi_0 \text{ , } \hat{t} > 0 \qquad (2.5)$$

where ω_0 ,ω_1 and ω_2 are the circular frequencies of the associated modes and $k_1 > 0$, $k_2 > 0$ are constants. The parameter λ is a nondimensional load parameter (nondimensionalized with respect to the classical buckling load λ_C) and $f(\hat{t})$ is the actual explicitly time-dependent load function , which, in our case , takes the form

$$f(\hat{t}) = \cos\varphi_O\hat{t} \qquad (2.6)$$

Danielson investigated the step loading case whereby $f(\hat{t}) \equiv 1$ .To arrive at equations (2.3) - (2.5), all geometric nonlinearities greater than the quadratic were disregarded .To analytically determine the dynamic buckling load λ_D , which is here defined as the largest load parameter for which the solution of the problem (2.3) - (2.5) has a bounded solution for all time $\hat{t} > 0$, Danielson adopted the following assumptions , which , in addition to our determining the value of λ_D, are also subject of intense scrutiny in this investigation:

(a) Quantities of the order of shell thickness can be neglected .
(b) Tangential inertia and boundary effects are negligible.

(c) $\bar{\xi}_1$ can be set equal to zero, assuming that unsymmetric imperfections are the main cause of the buckling phenomenon.

(d) The effects of the quadratic term $k_1\xi_1^2$ may be neglected compared to the coupling between the buckling modes for initial post buckling behaviour.

(e) The ratio of subsequent frequencies, namely, $\frac{\omega_i}{\omega_{i-1}}, i = 1,2$ may be taken as $(1-\nu)^{\frac{1}{2}}$, where ν is the Poisson's ratio.

Since the authenticity of assumptions (c) -(e) is also a subject of investigation, we shall , at least for now, ignore these assumptions , but however note , abinitio, that such a disregard of assumptions (c) -(e) necessarily adds to the apparent computational complexities that seem to characterize the solution .The dynamic buckling load λ_D satisfies the inequality $0 < \lambda_D < \lambda_S < \lambda_C \le 1$, where λ_S is the associated static buckling load of the structure .According to Budiansky [2] ,the dynamic buckling load λ_D is calculated from the condition

$$\frac{d\lambda}{d\xi_a} = 0 \tag{2.7a}$$

where, in our case , ξ_a is the sum of the maximum buckling modes , namely

$$\xi_a = \xi_{1a} + \xi_{2c} \tag{2.7b}$$

and where ξ_{1a} and ξ_{2c} are the maxima of $\xi_1(\hat{t})$ and $\xi_2(\hat{t})$ respectively. Our initial pre-occupation will then be to analytically determine, using multi-timing scaling technique, uniformly valid asymptotic expressions for $\xi_1(\hat{t})$ and $\xi_2(\hat{t})$ subsequent upon which the dynamic buckling load λ_D is determined using the maximization (2.7a).The analysis enunciated here is related, in spirit, to those by Ette [3], Wang and Tian [4-6] , Wei et al [7], Batra and Wei [8], Zhang et al [9] and Ette [10] of which the present study is a direct extension.

3 0 Solution of the problem

We let $t = \omega_0\hat{t}$ and substitute same, together with (2.6), into (2.3) and get

$$\frac{d^2\xi_0}{dt^2} + \xi_0 = \lambda\cos\varphi t \ ; \quad \xi_0(0) = \frac{d\xi_0(0)}{dt} = 0 , \left(\varphi = \frac{\varphi_0}{\omega_0}\right) \tag{3.1a}$$

For simplicity of the analysis, we have not introduced damping on the pre-buckling mode $\xi_0(t)$. On solving (3.1a), we get

$$\xi_0(t) = \frac{\lambda}{1-\varphi^2}(\cos\varphi\,\mathrm{t} - \cos\mathrm{t}) \ ; \ 1 \ne \varphi^2 \tag{3.1b}$$

We next substitute (3.1b) into (2.4) and (2.5), simplify the resultant equations and after, introduce the viscous damping terms $2\delta\frac{d\xi_1}{dt}$ and $2\delta\frac{d\xi_2}{dt}$ to the simplified equations corresponding to equations (2.4) and (2.5) and get respectively

$$\frac{d^2\xi_1}{dt^2} + 2\delta\frac{d\xi_1}{dt} + Q^2\xi_1 - h\in\xi_1(\cos\varphi\,\mathrm{t} - \cos\mathrm{t}) - Q^2k_1\xi_1^2 + Q^2k_1\xi_2^2 = h\in\bar{\xi}_1(\cos\varphi\,\mathrm{t} - \cos\mathrm{t}) \tag{3.2}$$

$$\frac{d^2\xi_2}{dt^2} + 2\delta\frac{d\xi_2}{dt} + R^2\xi_2 - S\in\xi_2(\cos\varphi\,\mathrm{t} - \cos\mathrm{t}) + R^2\xi_1\xi_2 = S\in\bar{\xi}_2(\cos\varphi\,\mathrm{t} - \cos\mathrm{t}) \tag{3.3}$$

$$\xi_i(0)=0\,,\ \frac{d\xi_i(0)}{dt}=0\,,\ i=1,2 \qquad (3.4a)$$

where
$$Q=\left(\frac{\omega_1}{\omega_0}\right),\ R=\left(\frac{\omega_2}{\omega_0}\right),\in=\lambda\left(\frac{\omega_1}{\omega_0}\right)^2,\ S=\frac{\left(\frac{\omega_2}{\omega_1}\right)^2}{1-\varphi^2},\ h=\frac{1}{1-\varphi^2} \qquad (3.4b)$$

In general , we consider $0<\lambda<1,\ 0<Q<1,\ 0<R<1,\ 0<\in<<1,\ 0<\delta<1$ and emphasize that the small viscous damping is taken, for simplicity of analysis, only on the buckling modes. We consider the two small parameters $\in$ and δ to be Mathematically independent and now let $\tau=\delta\,t$ so that we have

$$\frac{d\xi_\alpha}{dt}=\xi_{\alpha,t}+\delta\xi_{\alpha,\tau}\ ;\ \frac{d^2\xi_\alpha}{dt^2}=\xi_{\alpha,tt}+2\delta\xi_{\alpha,t\tau}+\delta^2\xi_{\alpha,\tau\tau},\ \alpha=1,2 \qquad (3.5)$$

where a subscript following a comma indicates partial differentiation .We let

$$\xi_1(t)=\xi_1(t,\tau)=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\zeta^{ij}(t,\tau)\in^i\delta^j\ ;\ \xi_2(t)=\xi_2(t,\tau)=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\eta^{ij}(t,\tau)\in^i\delta^j \qquad (3.6)$$

where the symbols ij as in ζ^{ij} and η^{ij} are superscripts and not powers .We now substitute (3.5) and (3.6) into (3.2) - (3.4a) and equate the coefficients of $\in^i\delta^j$, $i=1,2,3,\cdots;\ j=0,1,2,3,...$ and get the following equations:

$$M\zeta^{10}\equiv\zeta^{10}_{,tt}+Q^2\zeta^{10}=h\bar{\xi}_1(\cos\varphi\,t-\cos t) \qquad (3.7a)$$

$$M\zeta^{11}=-2\zeta^{10}_{,t}-2\zeta^{10}_{,t\tau} \qquad (3.7b)$$

$$M\zeta^{12}=-2\zeta^{11}_{,t}-2\zeta^{11}_{,t\tau}-2\zeta^{10}_{,\tau} \qquad (3.7c)$$

$$M\zeta^{20}=h\zeta^{10}(\cos\varphi\,t-\cos t)+k_1Q^2\left(\zeta^{10}\right)^2-k_2Q^2\left(\eta^{10}\right)^2 \qquad (3.8a)$$

$$M\zeta^{21}=h\zeta^{11}(\cos\varphi\,t-\cos t)+2k_1Q^2\zeta^{10}\zeta^{11}-2k_2Q^2\eta^{10}\eta^{11}-2\zeta^{20}_{,t}-2\zeta^{20}_{,t\tau} \qquad (3.8b)$$

$$M\zeta^{22}=h\zeta^{12}(\cos\varphi\,t-\cos t)+k_1Q^2\left\{2\zeta^{10}\zeta^{12}+\left(\zeta^{11}\right)^2\right\}-k_2Q^2\left\{2\eta^{10}\eta^{11}+\left(\eta^{11}\right)^2\right\}-2\zeta^{21}_{,t}-2\zeta^{21}_{,t\tau}-2\zeta^{20}_{,\tau} \qquad (3.8c)$$

$$N\eta^{10}\equiv\eta^{10}_{,tt}+R^2\eta^{10} \qquad (3.9a)$$

$$N\eta^{11}=-2\eta^{10}_{,t}-2\eta^{10}_{,t\tau} \qquad (3.9b)$$

$$N\eta^{12}=-2\eta^{11}_{,t}-2\eta^{11}_{,t\tau}-2\eta^{10}_{,\tau} \qquad (3.9c)$$

$$N\eta^{20}=S\eta^{10}(\cos\varphi\,t-\cos t)-R^2\eta^{10}\zeta^{10} \qquad (3.10a)$$

$$N\eta^{21}=S\eta^{11}(\cos\varphi\,t-\cos t)-R^2\left(\zeta^{10}\eta^{11}+\zeta^{11}\eta^{10}\right)-2\eta^{20}_{,t}-2\eta^{20}_{,t\tau} \qquad (3.10b)$$

$$N\eta^{22}=S\eta^{12}(\cos\varphi\,t-\cos t)-R^2\left(\zeta^{10}\eta^{12}+\zeta^{11}\eta^{11}+\zeta^{12}\eta^{10}\right)-2\eta^{21}_{,t}-2\eta^{21}_{,t\tau}-2\eta^{20}_{,\tau} \qquad (3.10c)$$

The initial conditions are evaluated at $(t,\tau)=(0,0)$ and are given by

$$\zeta^{ij}=0,\ i=1,2,3,\cdots;\ j=0,1,2,3\cdots;\ \zeta^{10}_{,t}=0;\ \zeta^{1k}_{,t}+\zeta^{1s}_{,\tau}=0,\ \zeta^{20}_{,t}=0,\ \zeta^{2k}_{,t}+\zeta^{2s}_{,\tau}=0; \qquad (3.11a)$$

$$\eta^{i\,j} = 0, i = 1,2,3,\cdots;\ j = 0,1,2,3\cdots;\ \eta^{10}_{,t} = 0;\ \eta^{1k}_{,t} + \eta^{1s}_{,\tau} = 0, \eta^{20}_{,t} = 0, \tag{3.11b}$$

$$\eta^{2k}_{,t} + \eta^{2s}_{,\tau} = 0,\ s = k-1, k = 1,2,3,\cdots \tag{3.11c}$$

The solution of (3.7a), using the initial conditions in (3.11a) relevant to ζ^{10} ,is

$$\zeta^{10} = \alpha_{10}(\tau)\cos Qt + \beta_{10}(\tau)\sin Qt + h\bar{\xi}_1\left\{\frac{\cos\varphi t}{Q^2-\varphi^2} - \frac{\cos t}{Q^2-1}\right\}; Q \neq 1, \varphi \tag{3.12a}$$

$$\alpha_{10}(0) = -h\bar{\xi}_1 q_0;\ q_0 = \left\{\frac{1}{Q^2-\varphi^2} - \frac{1}{Q^2-1}\right\}; \beta_{10}(0) = 0 \tag{3.12b}$$

We substitute (3.12a) into (3.7b) and ensure a uniformly valid asymptotic solution in the time scale *t* by equating to zero the coefficients of cosQt and sinQt and getting respectively

$$\beta'_{10} + \beta_{10} = 0 \text{ and } \alpha'_{10} + \alpha_{10} = 0 \tag{3.13a}$$

where $\frac{d(\ )}{d\tau} = (\ )'$.The solutions of (3.13a) subject to (3.12b) are

$$\beta_{10}(\tau) = 0,\ \alpha_{10}(\tau) = \alpha_{10}(0)e^{-\tau};\ \alpha'_{10}(0) = -\alpha_{10}(0) \tag{3.13b}$$

On solving the remaining equation in (3.7b), we get

$$\zeta^{11}(t,\tau) = \alpha_{11}(\tau)\cos Qt + \beta_{11}(\tau)\sin Qt - 2h\bar{\xi}_1\left[\frac{\sin t}{\left(Q^2-1\right)^2} - \frac{\varphi\sin\varphi t}{\left(Q^2-\varphi^2\right)^2}\right] \tag{3.13c}$$

$$\alpha_{11}(0) = 0;\ \beta_{11}(0) = h\bar{\xi}_1 q_2;\ q_2 = \left[2\left\{\frac{1}{\left(Q^2-1\right)^2} - \frac{\varphi}{\left(Q^2-\varphi^2\right)^2}\right\} - q_0\right] \tag{3.13d}$$

If we substitute into (3.7c) and ensure a uniformly valid solution in *t* by equating to zero the coefficients of cosQt and sinQt , we solve the resultant equations to get respectively

$$\alpha_{11}(\tau) = 0 \text{ and } \beta_{11}(\tau) = e^{-\tau}\left[\beta_{11}(0) - \int_0^\tau\left(\frac{e^s\alpha'_{10}}{Q}\right)ds\right] \tag{3.13e}$$

On solving (3.9a) subject to the relevant initial conditions as in (3.11b), we get

$$\eta^{10}(t,\tau) = \gamma_{10}(\tau)\cos Rt + \theta_{10}(\tau)\sin Rt + S\bar{\xi}_2\left[\frac{\cos\varphi t}{R^2-\varphi^2} - \frac{\cos t}{R^2-1}\right], R \neq \varphi, 1 \tag{3.14a}$$

$$\gamma_{10}(0) = -S\bar{\xi}_2 q_3;\ q_3 = \left[\frac{1}{R^2-\varphi^2} - \frac{1}{R^2-1}\right]; \theta_{10}(0) = 0 \tag{3.14b}$$

We next substitute the relevant terms into (3.9b) and to ensure a uniformly valid solution in *t* ,equate to zero the coefficients of cosRt and sinRt and get respectively

$$\theta'_{10} + \theta_{10} = 0 \text{ and } \gamma'_{10} + \gamma_{10} = 0 \tag{3.14c}$$

On solving (3.14c) subject to (3.14b), we get

$$\theta_{10}(\tau) = 0, \gamma_{10}(\tau) = \gamma_{10}(0)e^{-\tau};\ \gamma'_{10}(0) = -\gamma_{10}(0) \tag{3.14d}$$

The remaining equation in (3.9b) is solved to get

$$\eta^{11}(t,\tau)=\gamma_{11}(\tau)\cos Rt+\theta_{11}(\tau)\sin Rt-2S\bar{\xi}_2\left[\frac{\sin t}{\left(R^2-1\right)^2}-\frac{\varphi\sin\varphi\,\mathrm{t}}{\left(R^2-\varphi^2\right)^2}\right] \quad (3.15a)$$

$$\gamma_{11}(0)=0\,;\theta_{11}(0)=S\bar{\xi}_2 q_4\,;\,\mathrm{q}_4=\left[2\left\{\frac{1}{\left(R^2-1\right)^2}-\frac{\varphi^2}{\left(R^2-\varphi^2\right)^2}\right\}-q_3\right] \quad (3.15b)$$

We substitute into (3.9c) and ensure a uniformly valid solution in the time scale t by equating to zero the coefficients of cosRt and sinRt and solve the resultant equations to get respectively

$$\gamma_{11}(\tau)=0 \quad \text{and} \quad \theta_{11}(\tau)=e^{-\tau}\left[\theta_{11}(0)-\int_0^{\tau}\left(\frac{e^{s}\gamma_{10}'}{R}\right)\mathrm{ds}\right] \quad (3.15c)$$

We need the following simplifications in the substitution into (3.8a) which follows shortly:

$$h\zeta^{10}(\cos\varphi\,\mathrm{t}\text{ - cost})=\frac{h}{2}[\alpha_{10}\{\cos(Q+\varphi)\mathrm{t}+\cos(Q-\varphi)\mathrm{t}+\cos(Q+1)\mathrm{t}+\cos(Q-1)\mathrm{t}\}$$
$$+\bar{\xi}_1\{r_0+r_1\cos 2\varphi t+r_2\cos 2t-r_3\cos(\varphi+1)\mathrm{t}+\mathrm{r}_4\cos(\varphi-1)\mathrm{t}\}] \quad (3.16a)$$

where $$r_0=r_3=r_4=\frac{1}{Q^2-\varphi^2}+\frac{1}{Q^2-1}\,;\,\mathrm{r}_1=\frac{1}{Q^2-\varphi^2}\,,\,\mathrm{r}_2=\frac{1}{Q^2-1} \quad (3.16b)$$

$$k_1Q^2\left(\zeta^{10}\right)^2=k_1Q^2\left[\frac{\alpha_{10}^2}{2}+\frac{(h\bar{\xi}_1)^2}{2}\left\{\frac{1}{\left(Q^2-\varphi^2\right)^2}+\frac{1}{\left(Q^2-1\right)^2}\right\}+\frac{\alpha_{10}^2\cos 2Qt}{2}\right.$$
$$+\frac{(h\bar{\xi}_1)^2}{2}\left\{\frac{\cos 2\varphi\,\mathrm{t}}{\left(Q^2-\varphi^2\right)^2}+\frac{\cos 2t}{\left(Q^2-1\right)^2}\right\}+\left(\frac{h\bar{\xi}_1\alpha_{10}}{Q^2-\varphi^2}\right)\{\cos(Q+\varphi)\mathrm{t}+\cos(Q-\varphi)\mathrm{t}\} \quad (3.17a)$$
$$\left.-\left(\frac{h\bar{\xi}_1\alpha_{10}}{Q^2-1}\right)\{\cos(Q+1)\mathrm{t}+\cos(Q-1)\mathrm{t}\}-\frac{(h\bar{\xi}_1)^2}{\left(Q^2-\varphi^2\right)(Q^2-1)}\{\cos(\varphi+1)\mathrm{t}-\cos(\varphi-1)\mathrm{t}\}\right]$$

We similarly need

$$k_2Q^2\left(\eta^{10}\right)^2=k_2Q^2\left[\frac{\gamma_{10}^2}{2}+\frac{(S\bar{\xi}_2)^2}{2}\left\{\frac{1}{\left(R^2-\varphi^2\right)^2}+\frac{1}{\left(R^2-1\right)^2}\right\}+\frac{\gamma_{10}^2\cos 2Rt}{2}\right.$$
$$+\frac{(S\bar{\xi}_1)^2}{2}\left\{\frac{\cos 2\varphi\,\mathrm{t}}{\left(R^2-\varphi^2\right)^2}+\frac{\cos 2t}{\left(R^2-1\right)^2}\right\}+\left(\frac{S\bar{\xi}_2\gamma_{10}}{R^2-\varphi^2}\right)\{\cos(R+\varphi)\mathrm{t}+\cos(R-\varphi)\mathrm{t}\}$$

$$-\left(\frac{S\bar{\xi}_2\gamma_{10}}{R^2-1}\right)\{\cos(R+1)\mathrm{t}+\cos(R-1)\mathrm{t}\}-\frac{(S\bar{\xi}_2)^2}{\left(R^2-\varphi^2\right)(R^2-1)}\{\cos(\varphi+1)\mathrm{t}-\cos(\varphi-1)\mathrm{t}\}\Bigg] \quad (3.17b)$$

We now substitute relevant terms into (3.8a), using (3.16a,b) and (3.17a,b),and simplify to get

$$M\zeta^{20}=[r_5+r_6\cos 2Qt+r_7\{\cos(Q+\varphi)\mathrm{t}+\cos(Q-\varphi)\mathrm{t}+\}+r_8\cos(Q-1)\mathrm{t}+r_9\cos(Q+1)\mathrm{t}$$
$$+r_{10}\cos 2\varphi\mathrm{t}+r_{11}\cos 2t-r_{12}\{\cos(\varphi+1)\mathrm{t}+\cos(\varphi-1)\mathrm{t}\}]-Q^2k_2[r_{13}\cos 2Rt \quad (3.18a)$$
$$+r_{14}\{\cos(R+\varphi)\mathrm{t}+\cos(R-\varphi)\mathrm{t}\}-r_{15}\{\cos(R+1)\mathrm{t}+\cos(R-1)\mathrm{t}\}]$$

$$\zeta^{20}(0,0)=\zeta^{20}_{,t}(0,0)=0 \quad (3.18b)$$

$$r_5(\tau)=Q^2k_1\left[\frac{\alpha_{10}^2}{2}+\frac{(h\bar{\xi}_1)^2}{2}\left\{\frac{1}{\left(Q^2-\varphi^2\right)^2}+\frac{1}{\left(Q^2-1\right)^2}\right\}\right]-Q^2k_2\left[\frac{\gamma_{10}^2}{2}+\frac{(S\bar{\xi}_2)^2}{2}\left\{\frac{1}{\left(R^2-\varphi^2\right)^2}+\frac{1}{\left(R^2-1\right)^2}\right\}\right]$$
$$+\frac{hr_0\bar{\xi}_1}{2} \quad (3.18c)$$

$$r_5(0)=h\bar{\xi}_1R_{51}+k_1\left(Qh\bar{\xi}_1\right)^2R_{52}-k_2\left(QS\bar{\xi}_2\right)^2R_{53}\,; \quad (3.18d)$$

$$R_{51}=\frac{r_0}{2};\ \mathrm{R}_{52}=\left\{\frac{q_0^2}{2}+\frac{1}{2\left(Q^2-\varphi^2\right)^2}+\frac{1}{2\left(Q^2-1\right)^2}\right\};R_{53}=\left\{\frac{q_3^2}{2}+\frac{1}{2\left(R^2-\varphi^2\right)^2}+\frac{1}{2\left(R^2-1\right)^2}\right\} \quad (3.18e)$$

$$r_6(\tau)=\frac{Q^2k_1\alpha_{10}^2}{2},\ \mathrm{r}_6(0)=\frac{k_1\left(h\bar{\xi}_1q_0Q\right)^2}{2}\ ;r_7(\tau)=\alpha_{10}\left(\frac{\mathrm{h}}{2}+\frac{Q^2k_1h\bar{\xi}_1}{Q^2-\varphi^2}\right), \quad (3.18f)$$

$$r_7(0)=h\bar{\xi}_1R_{71}-k_1\left(Qh\bar{\xi}_1\right)^2R_{72},R_{71}=\frac{q_0}{2},\mathrm{R}_{72}=\frac{q_0}{Q^2-\varphi^2} \quad (3.18g)$$

$$r_8(\tau)=\frac{h}{2}\left(\bar{\xi}_1r_4-\alpha_{10}\right)-\frac{Q^2k_1h\bar{\xi}_1\alpha_{10}}{Q^2-1};r_8(0)=h\bar{\xi}_1R_{81}+k_1\left(Qh\bar{\xi}_1\right)^2R_{82}\ ,\mathrm{R}_{81}=\left(\frac{r_4+q_0}{2}\right) \quad (3.18h)$$

$$R_{82}=\frac{q_0}{Q^2-1}\ ;\mathrm{r}_9(\tau)=-\left\{\frac{h}{2}\left(\alpha_{10}+r_3\bar{\xi}_1\right)+\frac{Q^2k_1h\bar{\xi}_1}{Q^2-1}\right\},\mathrm{r}_9(0)=-h\bar{\xi}_1R_{91}+k_1\left(Qh\bar{\xi}_1\right)^2R_{92} \quad (3.18i)$$

$$R_{91}=\frac{1}{2}\left(r_3-hq_0\right),R_{92}=\frac{q_0}{Q^2-1}=\mathrm{R}_{82}\ ,\mathrm{r}_{10}(\tau)=\left\{\frac{hr_1\bar{\xi}_1}{2}+\frac{Q^2k_1(h\bar{\xi}_1)^2}{2(Q^2-\varphi^2)^2}-\frac{Q^2k_1(S\bar{\xi}_2)^2}{2(R^2-\varphi^2)^2}\right\} \quad (3.18j)$$

$$r_{10}(0)=h\bar{\xi}_1R_{101}+k_1\left(Qh\bar{\xi}_1\right)^2R_{102}-k_2\left(\bar{\xi}_2QS\right)^2R_{103}\ ;\ \mathrm{R}_{101}=\frac{r_1}{2}\ ,\mathrm{R}_{102}=\frac{1}{2\left(Q^2-\varphi^2\right)^2}, \quad (3.18k)$$

$$\mathrm{R}_{103}=\frac{1}{2\left(R^2-\varphi^2\right)^2}\ ;\mathrm{r}_{11}(\tau)=\left\{\frac{hr_2\bar{\xi}_2}{2}+\frac{Q^2k_1(h\bar{\xi}_1)^2}{2(Q^2-1)^2}-\frac{Q^2k_1(S\bar{\xi}_2)^2}{2(R^2-1)^2}\right\} \quad (3.18l)$$

$$r_{11}(0)=h\bar{\xi}_1R_{111}+k_1\left(Qh\bar{\xi}_1\right)^2R_{112}-k_2\left(\bar{\xi}_2QS\right)^2R_{113}$$
$$\mathrm{R}_{111}=\frac{r_2}{2}\ ,\mathrm{R}_{112}=\frac{1}{2\left(Q^2-\varphi^2\right)^2},\ \mathrm{R}_{113}=\frac{1}{2\left(R^2-\varphi^2\right)^2} \quad (3.18m)$$

$$r_{12}(\tau) = Q^2\left\{\frac{k_1(h\bar{\xi}_1)^2}{(Q^2-1)\left(Q^2-\varphi^2\right)} + \frac{k_2(S\bar{\xi}_2)^2}{(R^2-1)\left(R^2-\varphi^2\right)}\right\} \quad (3.18n)$$

On solving (3.18a,b), we obtain

$$\zeta^{20}(t,\tau) = \alpha_{20}(\tau)\cos Qt + \beta_{20}\sin Qt + \frac{r_5}{Q^2} - \frac{r_6\cos 2Qt}{3Q^2} + \frac{r_7}{\varphi}\left\{\frac{\cos(Q-\varphi)\mathrm{t}}{2Q-\varphi} - \frac{\cos(Q+\varphi)\mathrm{t}}{2Q+\varphi}\right\}$$

$$+\frac{r_8\cos(Q-1)\mathrm{t}}{2Q-1} - \frac{r_9\cos(Q+1)\mathrm{t}}{2Q+1} + \frac{r_{10}\cos 2\varphi\,\mathrm{t}}{Q^2-4\varphi^2} + \frac{r_{11}\cos 2\,\mathrm{t}}{Q^2-4} - r_{12}\left\{\frac{\cos(\varphi+1)\mathrm{t}}{Q^2-(\varphi+1)^2} + \frac{\cos(\varphi-1)\mathrm{t}}{Q^2-(\varphi-1)^2}\right\}$$

$$-Q^2k_2\left[\frac{r_{13}\cos 2Rt}{Q^2-4R^2} + r_{14}\left\{\frac{\cos(R+\varphi)\mathrm{t}}{Q^2-(R+\varphi)^2} + \frac{\cos(R-\varphi)\mathrm{t}}{Q^2-(R-\varphi)^2}\right\} - r_{15}\left\{\frac{\cos(R+1)\mathrm{t}}{Q^2-(R+1)^2} + \frac{\cos(R-1)\mathrm{t}}{Q^2-(R-1)^2}\right\}\right]$$

$$Q \neq \frac{1}{2}, \frac{\varphi}{2}, 2, 2\varphi, 2R, (R\pm\varphi), (\mathrm{R}\pm 1), (\varphi\pm 1) \quad (3.19a)$$

$$\alpha_{20}(0) = \bar{\xi}_1 hR_{16} + \left(h\bar{\xi}_1 Q\right)^2 k_1R_{17} + \left(SQ\bar{\xi}_2\right)^2 k_2R_{18}\,,\ \beta_{20}(0) = 0 \quad (3.19b)$$

$$R_{16} = \left[\frac{R_{71}}{\varphi}\left\{\frac{1}{2Q-\varphi} - \frac{1}{2Q+\varphi}\right\} - \frac{R_{51}}{Q^2} - \frac{R_{81}}{2Q-1} - \frac{R_{91}}{2Q+1} - \frac{R_{101}}{Q^2-\varphi^2} - \frac{R_{111}}{Q^2-1}\right] \quad (3.19c)$$

$$R_{17} = \left[-\frac{R_{52}}{Q^2} + \frac{q_0^2}{6Q^2} + \frac{R_{72}}{\varphi}\left\{\frac{1}{2Q-\varphi} - \frac{1}{2Q+\varphi}\right\} - \frac{R_{82}}{2Q-1\text{`}} - \frac{R_{92}}{2Q+1} - \frac{R_{102}}{Q^2-\varphi^2}\right.$$

$$\left. - \frac{R_{112}}{Q^2-1} + R_{121}\left\{\frac{1}{Q^2-(\varphi+1)^2} + \frac{1}{Q^2-(\varphi-1)^2}\right\}\right] \quad (3.19d)$$

$$R_{18} = \left[\frac{R_{103}}{Q^2-\varphi^2} + \frac{R_{53}}{Q^2} + \frac{R_{113}}{Q^2-1} + R_{122}\left\{\frac{1}{Q^2-(\varphi+1)^2} + \frac{1}{Q^2-(\varphi-1)^2}\right\} + \frac{R_{131}}{Q^2-1}\right.$$

$$\left. + R_{14}\left\{\frac{1}{Q^2-(R+\phi)^2} + \frac{1}{Q^2-(R-\phi)^2}\right\} - R_{15}\left\{\frac{1}{Q^2-(R+1)^2} + \frac{1}{Q^2-(R-1)^2}\right\}\right] \quad (3.19e)$$

We need the following simplification in the substitution into (3.8b) which follows shortly

$$h\zeta^{11}(\cos\varphi\,\mathrm{t} - \cos\mathrm{t}) = r_{16}\{\sin(Q+\varphi)\mathrm{t} + \sin(Q-\varphi)\mathrm{t}\} - r_{17}\{\sin(Q+1)\mathrm{t} + \sin(Q-1)\mathrm{t}\}$$
$$+ h\bar{\xi}_1\{r_{18}\sin 2t + r_{19}\sin 2\varphi t + r_{20}\sin(\varphi-1)\mathrm{t} - \sin(\varphi+1)\mathrm{t}\} \quad (3.20a)$$

where $r_{16} = r_{17} = \dfrac{\beta_{11}}{2}$; $r_{16}(0) = r_{17}(0) = \dfrac{h\bar{\xi}_1 q_2}{2}$; $r_{18} = \dfrac{1}{\left(Q^2-1\right)^2}$, $r_{19} = \dfrac{\varphi}{\left(Q^2-\varphi\right)^2}$ (3.20b)

$$r_{20} = -\frac{1}{\left(Q^2-1\right)^2} - \frac{\varphi}{\left(Q^2-\varphi\right)^2}\,,\ r_{21} = \frac{1}{\left(Q^2-1\right)^2} + \frac{\varphi}{\left(Q^2-\varphi\right)^2} \quad (3.20c)$$

We also have

$$2k_1Q^2\zeta^{10}\zeta^{11} = 2k_1Q^2[r_{22}\sin 2Qt + r_{23}\sin 2t + r_{24}\sin(Q+1)\mathrm{t} + \mathrm{r}_{25}\sin(Q-1)\mathrm{t}$$
$$+\,\mathrm{r}_{26}\sin(Q+\varphi)\mathrm{t} + \mathrm{r}_{27}\sin(Q-\varphi)\mathrm{t} + \mathrm{r}_{28}\sin 2\varphi t + r_{29}\sin(\varphi+1)\mathrm{t} + \mathrm{r}_{30}\sin(\varphi-1)\mathrm{t}] \quad (3.21a)$$

where

$$r_{22} = \frac{\alpha_{10}\beta_{11}}{2}, \mathrm{r}_{22}(0) = (h\bar{\xi}_1)^2 R_{22}, \mathrm{R}_{22} = -\frac{q_0 q_2}{2}; \mathrm{r}_{23} = \frac{(h\bar{\xi}_1)^2}{(Q^2-1)^3}, \mathrm{r}_{23}(0) = (h\bar{\xi}_1)^2 R_{23} \quad (3.21b)$$

$$R_{23} = \frac{1}{(Q^2-1)^3}; r_{24} = -\frac{h\bar{\xi}_1\alpha_{10}}{(Q^2-1)^2} - \frac{h\bar{\xi}_1\beta_{11}}{2(Q^2-1)}, \mathrm{r}_{24}(0) = (h\bar{\xi}_1)^2 R_{24} \quad (3.21c)$$

$$\mathrm{R}_{24} = \frac{q_0}{(Q^2-1)^2} - \frac{q_2}{2(Q^2-1)} \quad (3.21d)$$

$$r_{25} = \frac{h\bar{\xi}_1\alpha_{10}}{(Q^2-1)^2} - \frac{h\bar{\xi}_1\beta_{11}}{2(Q^2-1)}, \mathrm{r}_{25}(0) = (h\bar{\xi}_1)^2 R_{25}, \mathrm{R}_{25} = -\frac{q_0}{(Q^2-1)^2} - \frac{q_2}{2(Q^2-1)} \quad (3.21e)$$

$$r_{26} = \frac{h\bar{\xi}_1\alpha_{10}\varphi}{(Q^2-\varphi^2)^2} + \frac{h\bar{\xi}_1\beta_{11}}{2(Q^2-\varphi^2)}, \mathrm{r}_{26} = (h\bar{\xi}_1)^2 R_{26}; \mathrm{R}_{26} = -\frac{\varphi q_0}{(Q^2-\varphi^2)^2} + \frac{q_2}{2(Q^2-\varphi^2)} \quad (3.21f)$$

$$r_{27} = -\frac{h\bar{\xi}_1\alpha_{10}\varphi}{(Q^2-\varphi^2)^2} + \frac{h\bar{\xi}_1\beta_{11}}{2(Q^2-\varphi^2)}, \mathrm{r}_{27} = (h\bar{\xi}_1)^2 R_{27}; \mathrm{R}_{27} = \frac{\varphi q_0}{(Q^2-\varphi^2)^2} + \frac{q_2}{2(Q^2-\varphi^2)} \quad (3.21g)$$

$$r_{28} = \frac{(h\bar{\xi}_1)^2\varphi}{(Q^2-\varphi^2)^3}, \mathrm{r}_{28}(0) = (h\bar{\xi}_1)^2 R_{28}, \mathrm{R}_{28} = \frac{\varphi}{(Q^2-\varphi^2)^3}, \ \mathrm{r}_{29} = -(h\bar{\xi}_1)^2\left[\frac{1}{(Q^2-\varphi^2)(Q^2-1)^2}\right.$$
$$\left.+\frac{\varphi}{(Q^2-\varphi^2)^2(Q^2-1)}\right], \mathrm{r}_{29}(0) = -(h\bar{\xi}_1)^2 R_{29}, \mathrm{R}_{29} = \left[\frac{1}{(Q^2-\varphi^2)(Q^2-1)^2} + \frac{\varphi}{(Q^2-\varphi^2)^2(Q^2-1)}\right] \quad (3.21h)$$

We also have

$$\mathrm{r}_{30} = -(h\bar{\xi}_1)^2\left[\frac{\varphi}{(Q^2-\varphi^2)^2(Q^2-1)} - \frac{1}{(Q^2-\varphi^2)(Q^2-1)^2}\right], \mathrm{r}_{29}(0) = -(h\bar{\xi}_1)^2 R_{30} \quad (3.21i)$$

$$\mathrm{R}_{30} = \left[\frac{\varphi}{(Q^2-\varphi^2)^2(Q^2-1)} - \frac{1}{(Q^2-\varphi^2)(Q^2-1)^2}\right] \quad (3.21j)$$

We shall similarly need

$$2k_2Q^2\eta^{10}\eta^{11} = 2k_2Q^2[r_{31}\sin 2Rt + r_{32}\sin 2t + r_{33}\sin(R+1)\mathrm{t} + \mathrm{r}_{34}\sin(R-1)\mathrm{t}$$
$$+\,r_{35}\sin(R+\varphi)\mathrm{t} + r_{36}\sin(R-\varphi)\mathrm{t} + \mathrm{r}_{37}\sin 2\varphi\,\mathrm{t} + \mathrm{r}_{38}\sin(\varphi+1)\mathrm{t} + \mathrm{r}_{39}\sin(\varphi-1)\mathrm{t}] \quad (3.22a)$$

where

$$r_{31}=\frac{\gamma_{10}\theta_{11}}{2},\ r_{31}(0)=(S\bar{\xi}_2)^2R_{31},\ R_{31}=-\frac{q_3q_4}{2}\ ;\ r_{32}=\frac{(S\bar{\xi}_2)^2}{\left(R^2-1\right)^3},\ r_{32}(0)=(S\bar{\xi}_2)^2R_{32} \quad (3.22b)$$

$$R_{32}=\frac{1}{\left(R^2-1\right)^3},\ r_{33}=-\frac{(\gamma_{10}S\bar{\xi}_2)}{\left(R^2-1\right)^2},\ r_{33}(0)=(S\bar{\xi}_2)^2R_{33},\ R_{33}=\frac{q_3}{\left(R^2-1\right)^2}-\frac{q_4}{2\left(R^2-1\right)}, \quad (3.22c)$$

$$r_{34}=\frac{S\bar{\xi}_2\gamma_{10}}{\left(R^2-1\right)^2}-\frac{S\bar{\xi}_2\theta_{11}}{2\left(R^2-1\right)},\ r_{34}(0)=(S\bar{\xi}_2)^2R_{34},\ R_{34}=-\frac{q_3}{\left(R^2-1\right)^2}-\frac{q_4}{2\left(R^2-1\right)} \quad (3.22d)$$

$$r_{35}=\frac{S\bar{\xi}_2\varphi\gamma_{10}}{\left(R^2-\varphi^2\right)^2}+\frac{S\bar{\xi}_2\theta_{11}}{2\left(R^2-\varphi^2\right)},\ r_{35}(0)=(S\bar{\xi}_2)^2R_{35},\ R_{35}=-\frac{\varphi q_3}{\left(R^2-\varphi^2\right)^2}+\frac{q_4}{2\left(R^2-\varphi^2\right)} \quad (3.22e)$$

$$r_{36}=-\frac{S\bar{\xi}_2\varphi\gamma_{10}}{\left(R^2-\varphi^2\right)^2}+\frac{S\bar{\xi}_2\theta_{11}}{2\left(R^2-\varphi^2\right)},\ r_{36}(0)=(S\bar{\xi}_2)^2R_{36},\ R_{36}=\frac{\varphi q_3}{\left(R^2-\varphi^2\right)^2}+\frac{q_4}{\left(R^2-\varphi^2\right)} \quad (3.22f)$$

$$r_{37}=\frac{(S\bar{\xi}_2)^2\varphi}{\left(R^2-\varphi^2\right)^3},\ r_{37}(0)=(S\bar{\xi}_2)^2R_{37},\ R_{37}=\frac{\varphi}{\left(R^2-\varphi^2\right)^3} \quad (3.22g)$$

$$r_{38}=-(S\bar{\xi}_2)^2R_{38},\ R_{38}=\left\{\frac{1}{\left(R^2-\varphi^2\right)\left(R^2-1\right)^2}+\frac{\varphi}{\left(R^2-\varphi^2\right)^2\left(R^2-1\right)}\right\} \quad (3.22h)$$

If we substitute the relevant terms into (3.8b) from (3.19a, 3.20a) and (3.22a), we ensure a uniformly valid solution in t by equating to zero the coefficients of cosQt and sinQt and getting respectively

$$\beta_{20}'+\beta_{20}=0 \text{ and } \alpha_{20}'+\alpha_{20}=0 \quad (3.23a)$$

The solutions of (3.23a) subject to (3.19b) are

$$\beta_{20}(\tau)\equiv 0;\ \alpha_{20}(\tau)=\alpha_{20}(0)e^{-\tau}\ ;\ \alpha_{20}'(0)=-\alpha_{20}(0) \quad (3.23b)$$

On re-arranging terms in the remaining equation in (3.8b), we have

$$\begin{aligned} M\zeta^{21}&=r_{40}\sin(Q+\varphi)t+r_{41}\sin(Q-\varphi)t+r_{42}\sin(Q+1)t+r_{43}\sin(Q-1)t+r_{44}\sin 2\varphi t\\ &+r_{45}\sin(\varphi-1)t+r_{46}\sin(\varphi+1)t+r_{47}\sin 2Qt+r_{48}\sin 2t-2k_2Q^2[r_{49}\sin 2Rt+r_{50}\sin(R+1)t\\ &+r_{51}\sin(R-1)t+r_{52}\sin(R+\varphi)t+r_{53}\sin(R-\varphi)t] \end{aligned} \quad (3.24a)$$

$$\zeta^{21}(0,0)=0\ ;\ \zeta^{21}_{,t}(0,0)+\zeta^{20}_{,\tau}(0,0)=0 \quad (3.24b)$$

where $r_{40}=\left\{r_{16}+2k_1Q^2r_{26}-\frac{2r_7(Q+\varphi)}{(2Q+\varphi)}\right\}\ ;\ r_4(0)=\bar{\xi}_1hR_{401}+2k_1(\bar{\xi}_1Qh)^2R_{402}$ (3.24c)

$$R_{401}=\frac{q_2}{2}+\frac{2(Q+\varphi)R_{71}}{2Q+\varphi},\ R_{402}=R_{26}+\frac{(Q+\varphi)R_{42}}{2Q+\varphi}, \quad (3.24d)$$

$$r_{41}=r_{16}+2k_1Q^2r_{27}+\frac{2r_7(q-\omega)}{2Q-\varphi},r_{41}(0)=(\bar{\xi}_1h)R_{411}+k_1(\bar{\xi}_1Qh)^2R_{412} \quad (3.34e)$$

$$R_{411}=\frac{q_2}{2}-\frac{2(Q-\varphi)}{2Q-\varphi}R_{71},\ R_{412}=R_{27}-\frac{(Q-\varphi)R_{72}}{(2Q-\varphi)} \quad (3.24f)$$

$$r_{42} = -r_{17} + 2k_1Q^2r_{24} - \frac{2r_9(Q+1)}{2Q+1} - \frac{2r_9'(Q+1)}{(2Q+1)},\ r_{42}(0) = (h\bar{\xi}_1)R_{421} + k_i(\bar{\xi}_1 hQ)^2 R_{422}\ (31g)$$

$$R_{421} = -\frac{q_2}{2} + \frac{2(Q+1)R_9}{2Q+1},\ R_{422} = -R_{24} - \frac{(Q+1)R_{92}}{2Q+1} \qquad (3.24h)$$

$$r_{43} = \left\{-r_{17} + 2k_1Q^2r_{25} + \frac{2r_8(Q-1)}{2Q-1} + \frac{2r_8'(Q-1)}{2Q-1}\right\} \qquad (3.24i)$$

$$r_{43}(0) = \bar{\xi}_1 hR_{431} + k_1(Q\bar{\xi}_1 h)^2 R_{432},\ R_{431} = -\frac{q_2}{2} - \frac{(Q-1)q_0}{2Q} + \frac{2(Q-1)}{2Q-1}R_{81} \qquad (3.24j)$$

$$R_{432} = R_{26} + \frac{2(Q-1)q_0}{(2Q-1)(Q^2-1)} + \frac{2(Q-1)}{2Q-1}R_{82},\ r_{44} = h\bar{\xi}_1 r_{19} + 2k_1Q^2r_{28} - 2k_2Q^2r_{37} + \frac{4\varphi r_{10}}{Q^2-4\varphi^2} \qquad (3.24k)$$

$$r_{44}(0) = (\bar{\xi}h)R_{441} + k_1(Q\bar{\xi}_1 h)^2 R_{442} + k_2(Q\bar{\xi}_2 S)^2 R_{448} \qquad (3.24l)$$

$$R_{441} = r_{19} + \frac{4\varphi R_{101}}{Q^2-4\varphi^2},\ R_{442} = 2R_{28} + \frac{4\varphi R_{102}}{Q^2-4\varphi^2},\ R_{443} = -\left(2R_{37} + \frac{4\varphi R_{103}}{Q^2-4\varphi^2}\right) \qquad (3.25a)$$

$$r_{45} = h\bar{\xi}_1 r_{20} + 2k_1Q^2r_{30} - \frac{2r_{12}(\varphi-1)}{Q^2-(\varphi-1)^2} - 2k_2Q^2r_{39} \qquad (3.25b)$$

$$r_{45}(0) = \bar{\xi}_1 hR_{451} + k_1(\bar{\xi}_1 Qh)^2 R_{452} + k_2(QS\bar{\xi}_2)^2 R_{453} \qquad (3.25c)$$

$$R_{451} = r_{20},\ R_{452} = -2R_{30} - \frac{2(\varphi-1)R_{121}}{Q^2-(\varphi-1)^2},\ R_{453} = 2R_{39} - \frac{2(\varphi-1)R_{122}}{Q^2-(\varphi-1)^2} \qquad (3.25d)$$

$$r_{46} = -h\bar{\xi}_1 r_{21} + 2kQ^2r_{29} - \frac{2r_{12}(\varphi+1)}{Q^2-(\varphi+1)^2} - 2Q^2k_2r_{38} \qquad (3.25e)$$

$$r_{46}(0) = (h\bar{\xi}_1)R_{461} + k_1(Q\bar{\xi}_1 h)^2 R_{462} + k_2(Q\bar{\xi}_2 S)^2 R_{463},\ R_{461} = -r_{21} \qquad (3.25f)$$

$$R_{462} = -2R_{30} - \frac{2(\varphi+1)R_{121}}{Q^2-(\varphi+1)^2};\ r_{47} = 2k_1Q^2r_{22} - \frac{4r_6}{3Q} - \frac{4r_6'}{3Q},\ r_{47}(0) = k_1(\bar{\xi}_1 Qh)^2 R_{47},\ R_{47} = 2R_{22} \qquad (3.26a)$$

$$r_{48} = h\bar{\xi}_1 r_{18} + 2k_1Q^2r_{25} - 2k_2Q^2r_{32} + \frac{2r_{11}}{Q^2-4} + \frac{2r_{11}'}{Q^2-4} \qquad (3.26b)$$

$$r_{48}(0) = h\bar{\xi}_1 R_{481} + k_1(Q\bar{\xi}_1 h)^2 R_{482} + k_2(Q\bar{\xi}_2 S)^2 R_{483},\ R_{481} = r_{18} + R_{111},\ R_{482} = 2R_{23} + R_{112} \qquad (3.26c)$$

$$R_{483} = R_{49}\ ;\ r_{49} = 2k_2Q^2r_{31} - Q^2k_2\left(\frac{4Rr_{13}}{Q^2-4R^2} + \frac{4Rr_{13}'}{Q^2-4R^2}\right) \qquad (3.26d)$$

$$r_{49}(0) = k_2(Q\bar{\xi}_2 S)^2 R_{49},\ R_{49} = -2R_{31} - \left\{\frac{4RR_{13}}{Q^2-4R^2} - \frac{4Rq_3^2}{Q^2-4R^2}\right\} \qquad (3.26e)$$

$$r_{50} = 2k_2Q^2\left\{-r_{33} + \frac{r_{15}(R+1)}{Q^2-(R+1)^2} + \frac{r_{15}'(R+1)}{Q^2-(R+1)^2}\right\},\ r_{50}(0) = k_2(Q\bar{\xi}_2 S)^2 R_{50} \qquad (3.26f)$$

$$R_{50} = -2R_{33} + \frac{(R+1)}{Q^2-(R+1)^2}\left\{R_{15} + \frac{q_3}{R^2-1}\right\} \qquad (3.26g)$$

$$r_{51} = 2k_2Q^2\left\{-r_{34} + \frac{r_{15}(R-1)}{Q^2-(R-1)^2} + \frac{r_{15}'(R-1)}{Q^2-(R-1)^2}\right\},\ r_{51}(0) = k_2(Q\bar{\xi}_2 S)^2 R_{511} \qquad (3.26h)$$

$$R_{511} = -2R_{34} + \frac{(R-1)}{Q^2-(R-1)^2}\left\{R_{15} + \frac{q_3}{R^2-1}\right\} \qquad (3.26i)$$

$$r_{52} = 2k_2Q^2\left\{-r_{35} - \frac{r_{14}(R+\varphi)}{Q^2-(R+\varphi)^2} - \frac{r'_{14}(R+\varphi)}{Q^2-(R+\varphi)^2}\right\}, \ \mathrm{r}_{52}(0) = k_2\left(Q\bar{\xi}_2 S\right)^2 R_{521} \qquad (3.26j)$$

$$R_{521} = -2R_{35} - \frac{(R+\varphi)}{Q^2-(R+\varphi)^2}\left\{R_{14} + \frac{q_3}{R^2-\varphi^2}\right\} \qquad (3.26k)$$

$$r_{53} = 2k_2Q^2\left\{-r_{36} - \frac{r_{14}(R-\varphi)}{Q^2-(R-\varphi)^2} - \frac{r'_{14}(R-\varphi)}{Q^2-(R-\varphi)^2}\right\}, \ \mathrm{r}_{53}(0) = k_2\left(Q\bar{\xi}_2 S\right)^2 R_{531} \qquad (3.26l)$$

$$R_{531} = 2R_{36} - \frac{(R-\varphi)}{Q^2-(R-\varphi)^2}\left\{R_{14} + \frac{q_3}{R^2-\varphi^2}\right\} \qquad (3.26m)$$

On solving (3.24a,b),we obtain

$$\zeta^{21} = \alpha_{21}(\tau)\cos Q t + \beta_{21}(\tau)\sin Q\, t - \frac{\mathrm{r}_{40}\sin(Q+\varphi)t}{\varphi(2Q+\varphi)} + \frac{\mathrm{r}_{41}\sin(Q-\varphi)t}{\varphi(2Q-\varphi)} - \frac{\mathrm{r}_{42}\sin(Q+1)t}{(2Q+1)}$$
$$+ \frac{\mathrm{r}_{43}\sin(Q-1)t}{(2Q-1)} + \frac{r_{44}\sin 2\varphi t}{Q^2-4} - 2k_2\left[\frac{r_{49}\sin 2R t}{Q^2-4R^2} + \frac{r_{50}\sin(R+1)t}{Q^2-(R+1)^2} + \frac{r_{51}\sin(R-1)t}{Q^2-(R-1)^2}\right. \qquad (3.27a)$$
$$\left. + \frac{r_{52}\sin(R+\varphi)t}{Q^2-(R+\varphi)^2} + \frac{r_{53}\sin(R-\varphi)t}{Q^2-(R-\varphi)^2}\right]$$

where $\alpha_{21}(0) = 0\,;\ \beta_{21}(0) = h\bar{\xi}_1 R_{871} + k_1\left(Q\bar{\xi}_1 h\right)^2 R_{872} + K_2\left(Q\bar{\xi}_2 S\right)^2 R_{873}$ (3.27b)

$$R_{871} = \frac{1}{Q}\left[\frac{(Q+\varphi)R_{401}}{\varphi(2Q+\varphi)} - \frac{(Q-\varphi)R_{411}}{\varphi(2Q-\varphi)} + \frac{(Q+1)R_{421}}{2Q+1} - \frac{(Q-1)R_{431}}{2Q-1} - \frac{2\varphi R_{441}}{Q^2-4\varphi^2} - \frac{(\varphi-1)R_{451}}{Q^2-(\varphi-1)^2}\right.$$
$$\left. - \frac{(\varphi+1)R_{461}}{Q^2-(\varphi+1)^2} + R_{16} - \frac{hq_0}{2\varphi}\left\{\frac{1}{2Q-\varphi} - \frac{1}{2Q+\varphi}\right\} + \frac{hq_0}{2(2Q-1)} - \frac{hq_0}{2(2Q+1)}\right] \qquad (3.27c)$$

$$R_{872} = \frac{1}{Q}\left[\frac{(Q+\varphi)R_{402}}{\varphi(2Q+\varphi)} - \frac{(Q-\varphi)R_{412}}{\varphi(2Q-\varphi)} + \frac{(Q+1)R_{422}}{2Q+1} - \frac{(Q-1)R_{432}}{2Q-1} - \frac{2\varphi R_{442}}{Q^2-4\varphi^2} + \frac{2R_{47}}{3Q} - \frac{(\varphi-1)R_{452}}{Q^2-(\varphi-1)^2}\right.$$

$$\left. - \frac{2q_0^2}{3Q^2} - \frac{(\varphi+1)R_{462}}{Q^2-(\varphi+1)^2} + R_{17} - \frac{q_0}{\varphi\left(Q^2-\varphi^2\right)}\left\{\frac{1}{2Q-\varphi} - \frac{1}{2Q+\varphi}\right\} + \frac{q_0}{(2Q-1)\left(Q^2-1\right)} - \frac{q_0}{(2Q+1)\left(Q^2-1\right)}\right] \qquad (3.27d)$$

We also have

$$R_{873} = \frac{1}{Q}\left[\frac{(\varphi+1)R_{463}}{Q^2-(\varphi+1)^2} - \frac{(\varphi-1)R_{453}}{Q^2-(\varphi-1)^2} - \frac{2\varphi R_{443}}{Q^2-4\varphi^2} + \frac{2R_{482}}{Q^2-4} + 2\left\{\left\{\frac{2R_{49}}{Q^2-4R^2} + \frac{(R+1)R_{50}}{Q^2-(R+1)^2}\right.\right.\right.$$
$$+ \frac{(R-1)R_{511}}{Q^2-(R-1)^2} + \frac{(R+\varphi)R_{521}}{Q^2-(R+\varphi)^2} + \frac{(R-\varphi)R_{531}}{Q^2-(R-\varphi)^2} + R_{18} - \frac{q_3^2}{Q^2} - \frac{q_3}{Q^2-4R^2}$$
$$\left.\left.\left. + \frac{q_3}{R^2-\varphi^2}\left\{\frac{1}{Q^2-(R+\varphi)^2} + \frac{1}{Q^2-(R-\varphi)^2}\right\} - \frac{q_3}{R^2-1}\left\{\frac{1}{Q^2-(R+1)^2} + \frac{1}{Q^2-(R-1)^2}\right\}\right\}\right\}\right] \qquad (3.27e)$$

We shall need the following simplifications in the substitution into (3.10a) which follows shortly:

$$S(\cos\varphi\,\mathrm{t} - \cos\mathrm{t})\eta^{10} = \frac{S}{2}\left[\gamma_{10}\{\cos(R+\varphi)\,\mathrm{t} + \cos(\mathrm{R}-\varphi)\,\mathrm{t} - \cos(R+1)\,\mathrm{t} - \cos(\mathrm{R}-1)\,\mathrm{t}\}\right.$$

$$\left.+ S\bar{\xi}_2\{r_{54} + r_{55}\cos 2\varphi\,\mathrm{t} + r_{56}\cos 2t - r_{57}\cos(\varphi+1)\,\mathrm{t} + r_{58}\cos(\varphi-1)\,\mathrm{t}\}\right] \quad (3.28a)$$

where $r_{54} = \frac{1}{R^2-\varphi^2} + \frac{1}{R^2-1} = r_{58} = r_{57}$; $r_{55} = \frac{1}{R^2-\varphi^2}$; $r_{56} = \frac{1}{R^2-1}$ (3.28b)

We also need

$$R^2\eta^{10}\zeta^{10} = \frac{R^2}{2}\left[-\alpha_{10}\bar{\xi}_2 S\left\{\left\{\frac{1}{(R^2-\varphi^2)}\{\cos(Q+\varphi)\,\mathrm{t} + \cos(\mathrm{Q}-\varphi)\,\mathrm{t}\} - \frac{1}{(R^2-1)}\{\cos(Q+1)\,\mathrm{t} + \cos(\mathrm{Q}-1)\,\mathrm{t}\}\right\}\right\}\right.$$

$$+ \alpha_{10}\gamma_{10}\{\cos(Q+R)\,\mathrm{t} + \cos(Q-R)\,\mathrm{t}\} + h\bar{\xi}_1\left\{\left\{\frac{\gamma_{10}}{Q^2-\varphi^2}\{\cos(R+\varphi)\,\mathrm{t} + \cos(\mathrm{R}-\varphi)\,\mathrm{t}\}\right\}\right\}$$

$$+ \frac{\bar{\xi}_2 S\cos 2\varphi\,\mathrm{t}}{(Q^2-\varphi^2)(R^2-\varphi^2)} + \frac{\bar{\xi}_2 S\cos 2\,\mathrm{t}}{(Q^2-1)(R^2-1)} + \bar{\xi}_2 S\left\{\frac{1}{(Q^2-\varphi^2)(R^2-\varphi^2)} + \frac{1}{(Q^2-1)(R^2-1)}\right\}$$

$$- \bar{\xi}_2 S\left\{\frac{1}{(Q^2-\varphi^2)(R^2-1)} + \frac{1}{(Q^2-1)(R^2-\varphi^2)}\right\}\cos(\varphi-1)\,\mathrm{t} - \frac{\gamma_{10}}{\mathrm{Q}^2-1}\{\cos(R+1)\,\mathrm{t} + \cos(\mathrm{R}-1)\,\mathrm{t}\}$$

$$\left.- \bar{\xi}_2 S\left\{\frac{1}{(Q^2-1)(R^2-\varphi^2)} + \frac{1}{(Q^2-\varphi^2)(R^2-1)}\right\}\cos(\varphi+1)\,\mathrm{t}\right] \quad (3.28c)$$

We now substitute into (3.10a), using (3.28a-c), and simplify to get

$$N\eta^{20} = r_{59}\{\cos(R+\varphi)\,\mathrm{t} + \cos(R-\varphi)\,\mathrm{t}\} + r_{60}\cos(R+1)\,\mathrm{t} + r_{61}\cos(R-1)\,\mathrm{t} + r_{62} + r_{63}\cos 2\varphi t$$

$$+ r_{64}\cos 2t - r_{65}\cos(\varphi+1)\,\mathrm{t} - r_{66}\{\cos(R+Q)\,\mathrm{t} + \cos(Q-R)\,\mathrm{t}\} + [r_{67}\{\cos(Q+1)\,\mathrm{t} + \cos(Q-1)\,\mathrm{t}\}$$

$$- r_{68}\cos(\varphi-1)\,\mathrm{t}] \quad (3.29a)$$

$$\eta^{20}(0,0) = 0 \;;\; \eta^{20}_{,\mathrm{t}}(0,0) = 0 \quad (3.29b)$$

where $r_{59} = \frac{S\gamma_{10}}{2} - \frac{R^2 h\bar{\xi}_1\gamma_{10}}{2(Q^2-\varphi^2)}$ (3.30a)

$$r_{59}(0) = -S^2\bar{\xi}_2 R_{591} + S\bar{\xi}_1\bar{\xi}_2 R_{592} \;;\; \mathrm{R}_{591} = \frac{q_3}{2},\; \mathrm{R}_{592} = \frac{R^2 h\,\mathrm{q}_3}{2(\mathrm{Q}^2-\varphi^2)} \quad (3.30b)$$

$$r_{60} = -\left(\frac{S\gamma_{10}}{2} - \frac{h\bar{\xi}_1 R^2\gamma_{10}}{2(Q^2-1)}\right),\; r_{60}(0) = S^2\bar{\xi}_2 R_{601} - S\bar{\xi}_1\bar{\xi}_2 R_{602},\; \mathrm{R}_{601} = \frac{q_3}{2},\; R_{602} = \frac{R^2 h\,\mathrm{q}_3}{2(\mathrm{Q}^2-1)} \quad (3.30c)$$

$$r_{61} = -\left(\frac{S\gamma_{10}}{2} - \frac{h\bar{\xi}_1 R^2\gamma_{10}}{2(Q^2-1)}\right) = r_{60} \;;\; r_{61}(0) = S^2\bar{\xi}_2 R_{611} - S\bar{\xi}_1\bar{\xi}_2 R_{612},\; R_{611} = R_{601},\; R_{612} = R_{602} \quad (3.30d)$$

$$r_{62} = \frac{S^2 \bar{\xi}_2 r_{54}}{2} - \frac{R^2 h \bar{\xi}_1 \bar{\xi}_2}{2} \left\{ \frac{1}{\left(R^2 - \varphi^2\right)\left(Q^2 - \varphi^2\right)} + \frac{1}{\left(R^2 - 1\right)\left(Q^2 - 1\right)} \right\} \quad (3.30e)$$

$$r_{62}(0) = S^2 \bar{\xi}_2 R_{621} - S \bar{\xi}_1 \bar{\xi}_2 R_{622} \;; \mathrm{R}_{621} = \frac{r_{54}}{2}, \mathrm{R}_{622} = \frac{R^2}{2} \left\{ \frac{1}{\left(R^2 - \varphi^2\right)\left(Q^2 - \varphi^2\right)} + \frac{1}{\left(R^2 - 1\right)\left(Q^2 - 1\right)} \right\} \quad (3.30f)$$

$$r_{63} = \frac{S^2 \bar{\xi} r_{55}}{2} - \frac{R^2 \bar{\xi}_1 \xi_2 h S}{\left(Q^2 - \varphi^2\right)\left(R^2 - \varphi^2\right)}, r_{63}(0) = S^2 \bar{\xi}_2 R_{631} - S \bar{\xi}_1 \bar{\xi}_2 R_{632}, \quad (3.30g)$$

$$R_{631} = \frac{r_{55}}{2},\; R_{632} = \frac{R^2 h}{\left(Q^2 - \varphi^2\right)\left(R^2 - \varphi^2\right)}, \quad (3.30h)$$

$$r_{64} = \frac{S^2 \bar{\xi}^2 r_{56}}{2} - \frac{R^2 h S \bar{\xi}_1 \bar{\xi}_2}{2\left(Q^2 - 1\right)\left(R^2 - 1\right)};\; r_{64}(0) = S^2 \bar{\xi}_2 R_{641} - S^2 \bar{\xi}_1 \bar{\xi}_2 R_{642} \quad (3.30i)$$

$$R_{641} = \frac{r_{56}}{2},\; R_{642} = \frac{R^2 h}{2\left(Q^2 - 1\right)\left(R^2 - 1\right)} \quad (3.30j)$$

$$r_{65} = -\frac{S^2 \bar{\xi}_2 r_{51}}{2} + \frac{R^2 h S \bar{\xi}_1 \bar{\xi}_2}{2} \left\{ \frac{1}{\left(Q^2 - 1\right)\left(R^2 - \varphi^2\right)} + \frac{1}{\left(Q^2 - \varphi^2\right)\left(R^2 - 1\right)} \right\} \quad (3.30k)$$

$$r_{65}(0) = -S^2 \bar{\xi}_2 R_{651} - S \bar{\xi}_1 \bar{\xi}_2 R_{652} \quad (3.30l)$$

$$R_{651} = \frac{r_{57}}{2},\; R_{652} = -\frac{R^2 h}{2} \left\{ \frac{1}{\left(Q^2 - 1\right)\left(R^2 - \varphi^2\right)} + \frac{1}{\left(Q^2 - \varphi^2\right)\left(R^2 - 1\right)} \right\} \quad (3.30m)$$

$$r_{66} = -\frac{R^2 \alpha_{10} \gamma_{10}}{2},\; r_{66}(0) = -S \bar{\xi}_1 \bar{\xi}_2 R_{66},\; R_{66} = \frac{R^2 h q_0 q_3}{2} \quad (3.30n)$$

$$r_{67} = -\frac{R^2 \alpha_{10} S \bar{\xi}_2}{2} \left\{ \frac{1}{R^2 - \varphi^2} - \frac{1}{R^2 - 1} \right\}, \quad (3.30o)$$

$$r_{67}(0) = S \bar{\xi}_1 \bar{\xi}_2 R_{67},\; R_{67} = -\frac{R^2 h q_0}{2} \left\{ \frac{1}{R^2 - \varphi^2} - \frac{1}{R^2 - 1} \right\} \quad (3.30p)$$

$$r_{68} = -\frac{S^2 r_{58} \bar{\xi}_2}{2} + \frac{h S \bar{\xi}_1 \bar{\xi}_2 R^2}{2} \left[\frac{1}{\left(Q^2 - \varphi^2\right)\left(R^2 - 1\right)} + \frac{1}{\left(Q^2 - 1\right)\left(R^2 - \varphi^2\right)} \right] \quad (3.30q)$$

$$r_{68}(0) = -S^2 \bar{\xi}_2 R_{681} + S \bar{\xi}_1 \bar{\xi}_2 R_{682} \quad (3.30r)$$

$$R_{681} = -\frac{r_{58}}{2},\; R_{682} = \frac{R^2}{2} \left[\frac{1}{\left(Q^2 - \varphi^2\right)\left(R^2 - 1\right)} + \frac{1}{\left(Q^2 - 1\right)\left(R^2 - \varphi^2\right)} \right] \quad (3.30s)$$

On solving (3.2a,b) we get

$$\eta^{20} = \Bigg[\gamma_{20}(\tau) \cos Rt + \theta_{20}(\tau) \sin Rt + \frac{r_{59}}{\varphi} \left\{ \frac{\cos(R - \varphi)t}{2R - \varphi} - \frac{\cos(R + \varphi)\mathrm{t}}{2R + \varphi} \right\}$$
$$- \frac{r_{60} \cos(R + 1)t}{2R + 1} + \frac{r_{61} \cos(R - 1)t}{2R - 1} + \frac{r_{62}}{R^2} + \frac{r_{63} \cos 2\varphi t}{R^2 - 4\varphi^2} + \frac{r_{64} \cos 2t}{R^2 - 4} - \frac{r_{65} \cos(\varphi + 1)t}{R^2 - (\varphi + 1)^2} \quad (3.31a)$$

$$\gamma_{20}(0) = S^2 \bar{\xi}_2 R_{6901} + S^2 \bar{\xi}_1 \bar{\xi}_2 R_{6902},\; \theta_{20}(0) = 0$$

$$-\frac{r_{66}}{Q}\left\{\frac{\cos(Q-R)t}{2R-Q}-\frac{\cos(Q+R)t}{2R+Q}\right\}+r_{67}\left\{\frac{\cos(Q-1)t}{R^2-(Q-1)^2}+\frac{\cos(Q+1)t}{R^2-(Q+1)^2}\right\}$$

$$-\frac{r_{68}\cos(\varphi-1)t}{R^2-(\varphi-1)^2}\Bigg] \tag{3.31b}$$

$$R_{6901}=\left[\frac{R_{591}}{\varphi}\left\{\frac{1}{2R-\varphi}-\frac{1}{2R+\varphi}\right\}+\frac{R_{601}}{2R+1}-\frac{R_{611}}{2R-1}-\frac{R_{621}}{R^2}-\frac{R_{631}}{R^2-4\varphi^2}\right.$$

$$\left.-\frac{R_{641}}{R^2-4}-\frac{R_{651}}{R^2-(\varphi+1)^2}+\frac{R_{681}}{R^2-(\varphi-1)^2}\right] \tag{3.31c}$$

$$R_{6902}=\left[-\frac{R_{592}}{\varphi}\left\{\frac{1}{2R-\varphi}-\frac{1}{2R+\varphi}\right\}-\frac{R_{602}}{2R+1}+\frac{R_{612}}{2R-1}+\frac{R_{622}}{R^2}+\frac{R_{632}}{R^2-4\varphi^2}\right.$$

$$+\frac{R_{642}}{R^2-4}-\frac{R_{652}}{R^2-(\varphi+1)^2}+\frac{R_{682}}{R^2-(\varphi-1)^2}-\frac{R_{66}}{Q}\left\{\frac{1}{2R-Q}-\frac{1}{2R+Q}\right\}$$

$$\left.-R_{67}\left\{\frac{1}{R^2-(Q-1)^2}+\frac{1}{R^2-(Q+1)^2}\right\}\right] \tag{3.31d}$$

We now perform the following simplifications which are needed in the next substitution into (3.10b)

$$-R^2\zeta^{11}\eta^{10}=-\frac{R^2}{2}\left[\beta_{11}\gamma_{10}\{\sin(R+Q)t+\sin(Q-R)t\}-2\gamma_{10}h\bar{\xi}_1\left\{\left\{\frac{\{\sin(R+1)t-\sin(R-1)t\}}{(Q^2-1)^2}\right.\right.\right.$$

$$\left.\left.-\frac{\varphi}{(Q^2-\varphi^2)^2}\{\sin(R+\varphi)t-\sin(R-\varphi)t\}\right\}\right\}+\beta_{11}S\bar{\xi}_2\left\{\left(\frac{\sin(Q+\varphi)t+\sin(Q-\varphi)t}{R^2-\varphi^2}\right)\right.$$

$$\left.-\left(\frac{\sin(Q+1)t+\sin(Q-1)t}{R^2-1}\right)\right\}-2h\bar{\xi}_1\bar{\xi}_2S\left\{\left\{-\frac{2\sin 2t}{(Q^2-1)^2(R^2-\varphi^2)}-\frac{\varphi\sin 2\varphi t}{(Q^2-\varphi^2)^2(R^2-\varphi^2)}\right.\right.$$

$$+\left\{\frac{1}{(R^2-\varphi^2)(Q^2-1)^2}+\frac{\varphi}{(Q^2-\varphi^2)^2(R^2-1)}\right\}\sin(1+\varphi)t+\left\{\frac{\varphi}{(Q^2-\varphi^2)^2(R^2-1)}\right.$$

$$\left.\left.\left.\left.-\frac{1}{(R^2-\varphi^2)(Q^2-1)^2}\right\}\sin(\varphi-1)t\right\}\right\}\right] \tag{3.32a}$$

$$-R^2\zeta^{10}\eta^{11}=-\frac{R^2}{2}\left[\alpha_{10}\theta_{11}\{\sin(R+Q)t+\sin(Q-R)t\}+\theta_{11}h\bar{\xi}_1\left\{\left(\frac{\sin(R+\varphi)t+\sin(R-\varphi)t}{Q^2-\varphi^2}\right)\right.\right.$$

$$-\left(\frac{\sin(R+1)t+\sin(R-1)t}{Q^2-1}\right)-2h\bar{\xi}_1\bar{\xi}_2 S\left\{\left\{-\frac{\sin 2t}{\left(Q^2-1\right)\left(R^2-1\right)^2}-\frac{\varphi\sin 2\varphi\,t}{\left(Q^2-\varphi^2\right)\left(R^2-\varphi^2\right)^2}\right.\right.$$

$$+\frac{\{\sin(\varphi+1)t-\sin(\varphi-1)t\}}{\left(Q^2-\varphi^2\right)\left(R^2-1\right)^2}+\frac{\varphi\{\sin(\varphi+1)t+\sin(\varphi-1)t\}}{\left(R^2-\varphi^2\right)^2\left(Q^2-1\right)}-2\alpha_{10}S\bar{\xi}_2\left\{\frac{\sin(Q+1)t-\sin(Q-1)t}{\left(R^2-1\right)^2}\right.$$

$$\left.\left.\left.-\frac{\varphi}{\left(R^2-\varphi^2\right)}\left(\sin(Q+\varphi)t-\sin(Q-\varphi)t\right)\right\}\right\}\right\}\Bigg] \qquad (3.32b)$$

We now substitute (3.31a,b) and (3.32a,b) into (3.10b) and to ensure a uniformly valid solution in the time scale t, equate to zero, the coefficient of cos Rt and sinRt and get respectively and

$$\theta_{20}' + \theta_{20} = 0 \quad and \quad \gamma_{20}' + \gamma_{20} = 0 \qquad (3.33a)$$

On volving (3.33a), we obtain

$$\theta_{20}(\tau) \equiv 0,\ \gamma_{20}(\tau) = \gamma_{20}'(0)e^{-\tau};\ \gamma' = -\gamma_{20}(0) \qquad (3.33b)$$

After re-arranging the remaining terms in the substitution into (3.10b), we get

$$\begin{aligned} N\eta^{21} &= r_{69}\sin(R+\varphi)t + r_{70}\sin(R-\varphi)t + r_{71}\{\cos(R-1)t-\cos(R+1)t\} \\ &+ r_{72}\{\cos(\varphi-1)t-\cos(\varphi+1)t\} + r_{73}\sin 2\varphi t + r_{74}\sin 2t + r_{75}\sin(\varphi+1)t \\ & r_{76}\sin(\varphi-1)t + r_{77}\sin(Q+R)t + r_{78}\sin(R-Q)t + r_{79}\sin(R+1)t + r_{80}\sin(R-1)t \\ & r_{81}\sin(Q+\varphi)t + r_{82}\sin(Q-\varphi)t + r_{83}\sin(Q+1)t + r_{84}\sin(Q-1)t \end{aligned} \qquad (3.34a)$$

$$\eta^{21}(0,0)=0,\ \eta,_t^{21}(0,0)+\eta,_\tau^{20}(0,0)=0 \qquad (3.34b)$$

where $$r_{69} = \left[\frac{S\theta_{11}}{2} - \frac{R^2\varphi\gamma_{10}h\bar{\xi}_1}{\left(Q^2-\varphi^2\right)^2} - \frac{\theta_{11}h\bar{\xi}_1R^2}{2\left(Q^2-\varphi^2\right)} - \frac{2r_{59}(R+\varphi)}{\varphi(2R+\varphi)} - \frac{2r_{59}'(R+\varphi)}{\varphi(2R+\varphi)}\right] \qquad 3.35a)$$

$$r_{70} = \left[\frac{S\theta_{11}}{2} + \frac{R^2\varphi\gamma_{10}h\bar{\xi}_1}{\left(Q^2-\varphi^2\right)^2} - \frac{\theta_{11}R^2h\bar{\xi}_1}{2\left(Q^2-\varphi^2\right)} + \frac{2(R-\varphi)}{\varphi(2R-\varphi)}\left(r_{59}+r_{59}'\right)\right] \qquad (3.35b)$$

$$r_{71} = -\frac{\theta_{11}S}{2};\ r_{72} = -\frac{S^2\bar{\xi}_2\varphi}{R^2-\varphi^2} \qquad (3.35c)$$

$$r_{73} = \left[\frac{2S^2\bar{\xi}_2\varphi}{\left(R^2-\varphi^2\right)^2} - \frac{R^2\varphi h\bar{\xi}_1\bar{\xi}_2S}{\left(Q^2-\varphi^2\right)^2\left(R^2-\varphi^2\right)} - \frac{R^2\varphi Sh\bar{\xi}_1\bar{\xi}_2}{\left(Q^2-\varphi^2\right)\left(R^2-\varphi^2\right)^2} + \frac{4\varphi r_{63}}{R^2-4\varphi^2}\right] \qquad (3.35d)$$

$$r_{74} = \left[\frac{2S^2\bar{\xi}_2}{\left(R^2-1\right)^2} - \frac{R^2h\bar{\xi}_1\bar{\xi}_2S}{\left(Q^2-1\right)^2\left(R^2-\varphi^2\right)} - \frac{R^2Sh\bar{\xi}_1\bar{\xi}_2}{\left(Q^2-1\right)\left(R^2-1\right)^2} + \frac{4r_{64}}{R^2-4}\right] \qquad (3.35e)$$

$$r_{75}=\left[-\frac{S^2\bar{\xi}_2}{\left(R^2-1\right)^2}+R^2hS\bar{\xi}_1\bar{\xi}_2\left\{\frac{1}{R^2-\varphi^2\left(Q^2-1\right)^2}+\frac{\varphi}{(Q^2-\varphi^2)^2\left(R^2-1\right)}\right\}\right. \tag{3.35f}$$

$$\left.+R^2hS\bar{\xi}_1\bar{\xi}_2\left\{\frac{1}{(Q^2-\varphi^2)\left(R^2-1\right)^2}+\frac{\varphi}{(Q^2-1)\left(R^2-\varphi^2\right)^2}\right\}-\frac{2(\varphi+1)\mathrm{r}_{65}}{R^2-(\varphi+1)^2}\right]$$

$$r_{76}=\left[\frac{S^2\bar{\xi}_2}{\left(R^2-1\right)^2}+R^2hS\bar{\xi}_1\bar{\xi}_2\left\{\frac{\varphi}{(Q^2-\varphi^2)^2\left(R^2-1\right)}-\frac{1}{R^2-\varphi^2\left(Q^2-1\right)^2}\right\}\right.$$

$$\left.+R^2hS\bar{\xi}_1\bar{\xi}_2\left\{\frac{\varphi}{(Q^2-1)\left(R^2-\varphi^2\right)^2}-\frac{1}{(Q^2-\varphi^2)\left(R^2-1\right)^2}\right\}+\frac{2(\varphi-1)\mathrm{r}_{68}}{R^2-(\varphi-1)^2}\right] \tag{3.35g}$$

$$r_{77}=\left[-\frac{R^2\beta_{11}\gamma_{10}}{2}-\frac{R^2\alpha_{10}\theta_{11}}{2}+\frac{2(Q+R)}{Q(2R+Q)}\left(r_{66}+r'_{66}\right)\right] \tag{3.35h}$$

$$r_{78}=\left[\frac{R^2\beta_{11}\gamma_{10}}{2}-\frac{R^2\alpha_{10}\theta_{11}}{2}-\frac{2(Q-R)}{Q(2R-Q)}\left(r_{66}+r'_{66}\right)\right] \tag{3.35i}$$

$$r_{79}=\left[\frac{\gamma_{10}h\bar{\xi}_1}{\left(Q^2-1\right)^2}+\frac{\theta_{11}h\bar{\xi}_1\bar{\xi}_2}{2\left(Q^2-1\right)}-\frac{2(R+1)}{(2R+1)}\left(r_{60}+r'_{60}\right)\right] \tag{3.35j}$$

$$r_{80}=\left[-\frac{\gamma_{10}h\bar{\xi}_1}{\left(Q^2-1\right)^2}+\frac{R^2\theta_{11}h\bar{\xi}_1}{2\left(Q^2-1\right)}+\frac{2(R-1)}{(2R-1)}\left(r_{61}+r'_{61}\right)\right],\ \mathrm{r}_{81}=-\left[\frac{R^2hS\beta_{11}\bar{\xi}_1}{2\left(R^2-\varphi^2\right)}+\frac{R^2\alpha_{10}S\varphi\bar{\xi}_2}{R^2-\varphi^2}\right] \tag{3.35k}$$

$$\mathrm{r}_{82}=\left[\frac{R^2\alpha_{10}S\varphi\bar{\xi}_2}{R^2-\varphi^2}-\frac{R^2S\beta_{11}\bar{\xi}_1}{2\left(R^2-\varphi^2\right)}\right],\ \mathrm{r}_{83}=\left[\frac{R^2hS\beta_{11}\bar{\xi}_1}{2\left(R^2-1\right)}+\frac{R^2\alpha_{10}S\bar{\xi}_2}{(R^2-1)^2}+\frac{2(Q+1)}{R^2-(Q+1)^2}\left(r_{67}+r'_{67}\right)\right] \tag{3.35l}$$

$$\mathrm{r}_{84}=\left[\frac{R^2hS\beta_{11}\bar{\xi}_1}{2\left(R^2-1\right)}-\frac{R^2\alpha_{10}S\bar{\xi}_2}{(R^2-1)^2}-\frac{2(Q-1)}{R^2-(Q-1)^2}\left(r_{67}+r'_{67}\right)\right] \tag{3.35m}$$

$$r_{69}(0)=S^2\bar{\xi}_2R_{691}+S\bar{\xi}_1\bar{\xi}_2R_{692}\ ;\ \mathrm{R}_{691}=\frac{q_4}{2}+\frac{(R+\varphi)}{\varphi(2R+\varphi)}\left(2R_{591}-q_3\right) \tag{3.35n}$$

$$R_{692}=\left[\frac{R^2\varphi hq_3}{\left(Q^2-\varphi^2\right)^2}-\frac{R^2h\,q_4}{2\left(Q^2-\varphi^2\right)}-\frac{2(R+\varphi)R_{592}}{\varphi(2R+\varphi)}+\frac{2R^2h(R+\varphi)}{\varphi(2R+\varphi)\left(Q^2-\varphi^2\right)}\right] \tag{3.35o}$$

$$r_{70}(0)=S^2\bar{\xi}_2R_{701}+S\bar{\xi}_1\bar{\xi}_2R_{702}\ ;\ \mathrm{R}_{701}=\frac{q_4}{2}+\frac{2(R+\varphi)}{\varphi(2R+\varphi)}\left(q_3-R_{591}\right) \tag{3.35p}$$

$$R_{702}=\left[-\frac{R^2\varphi hq_3}{\left(Q^2-\varphi^2\right)^2}-\frac{q_4hR^2}{2\left(Q^2-\varphi^2\right)}+\frac{2(R+\varphi)}{\varphi(2R+\varphi)}\left\{R_{592}-\frac{q_3}{Q^2-\varphi^2}\right\}\right] \quad (3.35q)$$

$$r_{71}(0)=S^2\bar{\xi}_2R_{711}\,;\ \mathrm{R}_{711}=-\frac{q_4}{2};\ \mathrm{r}_{72}(0)=S^2\bar{\xi}_2R_{721}\ ,\ \mathrm{R}_{721}=-\frac{\varphi}{R^2-\varphi^2} \quad (3.35r)$$

$$r_{73}(0)=S^2\bar{\xi}_2R_{731}+S\bar{\xi}_1\bar{\xi}_2R_{732}\ ;\ \mathrm{R}_{731}=\frac{2\varphi}{(R^2-\varphi^2)^2}+\frac{4\varphi R_{681}}{R^2-4\varphi^2} \quad (3.35s)$$

$$R_{732}=-\left[\frac{R^2\varphi\,\mathrm{h}}{\left(\mathrm{Q}^2-\varphi^2\right)^2\left(R^2-\varphi^2\right)}+\frac{R^2\varphi\,\mathrm{h}}{\left(\mathrm{Q}^2-\varphi^2\right)\left(R^2-\varphi^2\right)^2}+\frac{4\varphi\,\mathrm{R}_{632}}{\mathrm{R}^2-4\varphi^2}\right] \quad (3.36a)$$

$$r_{74}(0)=S^2\bar{\xi}_2R_{741}+S\bar{\xi}_1\bar{\xi}_2R_{742}\ ;\ \mathrm{R}_{741}=\frac{S}{(R^2-1)^2}+\frac{4\varphi\,R_{641}}{R^2-4} \quad (3.36b)$$

$$R_{742}=-\left[\frac{R^2\mathrm{h}}{\left(\mathrm{Q}^2-1\right)^2\left(R^2-\varphi^2\right)}+\frac{R^2\,\mathrm{h}}{\left(\mathrm{Q}^2-1\right)\left(R^2-1\right)^2}+\frac{4\,\mathrm{R}_{642}}{\mathrm{R}^2-4}\right] \quad (3.36c)$$

$$r_{75}(0)=S^2\bar{\xi}_2R_{751}+S\bar{\xi}_1\bar{\xi}_2R_{752}\ ;\ \mathrm{R}_{751}=\left[\frac{1}{(R^2-1)}+\frac{2(\varphi+1)R_{651}}{R^2-(\varphi+1)^2}\right] \quad (3.36d)$$

$$R_{752}=\left[R^2h\left\{\frac{1}{\left(R^2-\varphi^2\right)\left(Q^2-1\right)^2}+\frac{\varphi}{\left(Q^2-\varphi^2\right)^2\left(R^2-1\right)}+\frac{1}{\left(Q^2-\varphi^2\right)\left(R^2-1\right)^2}\right.\right.$$

$$\left.\left.+\frac{\varphi}{\left(Q^2-1\right)\left(R^2-\varphi^2\right)^2}\right\}-\frac{2(\varphi+1)R_{652}}{R^2-(\varphi+1)^2}\right] \quad (3.36e)$$

$$r_{76}(0)=S^2\bar{\xi}_2R_{761}+S\bar{\xi}_1\bar{\xi}_2R_{762}\ ;\ \mathrm{R}_{761}=\left[\frac{1}{(R^2-1)^2}+\frac{2(\varphi-1)R_{681}}{R^2-(\varphi-1)^2}\right] \quad (3.36f)$$

$$R_{762}=\left[R^2h\left\{\frac{\varphi}{\left(Q^2-\varphi^2\right)^2\left(R^2-1\right)}-\frac{1}{\left(R^2-\varphi^2\right)\left(Q^2-\varphi^2\right)^2}-\frac{1}{\left(Q^2-\varphi^2\right)\left(R^2-\varphi^2\right)^2}\right.\right. \quad (3.36g)$$

$$\left.\left.+\frac{\varphi}{\left(Q^2-1\right)\left(R^2-\varphi^2\right)^2}\right\}+\frac{2(\varphi-1)R_{682}}{R^2-(\varphi-1)^2}\right]$$

$$r_{77}(0)=S\bar{\xi}_1\bar{\xi}_2R_{77}\ ,\mathrm{R}_{77}=\left[\frac{R^2q_2q_3h}{2}+\frac{hq_0q_4R^2}{2}+\frac{(Q+R)}{Q(2R+Q)}\left(q_0q_3R^2h-R_{66}\right)\right] \quad (3.36h)$$

$$r_{78}(0)=S\bar{\xi}_1\bar{\xi}_2R_{78}\ ,\mathrm{R}_{78}=\left[-\frac{R^2q_2q_3h}{2}+\frac{hq_0q_4R^2}{2}+\frac{(Q-R)}{Q(2R-Q)}\left(q_0q_3R^2h-R_{66}\right)\right] \quad (3.36i)$$

$$r_{79}(0)=S^2\bar{\xi}_2R_{791}+S\bar{\xi}_1\bar{\xi}_2R_{792}\ ,\mathrm{R}_{791}=-\frac{2(R+1)}{2R+1}\left(R_{601}+q_3\right) \quad (3.36j)$$

$$R_{792}=\left[-\frac{hq_3}{\left(Q^2-1\right)^2}+\frac{hR^2q_4}{\left(Q^2-1\right)}+\frac{2(R+1)}{2R+1}\left(R_{602}+\frac{R^2hq_3}{Q^2-1}\right)\right] \quad (3.36k)$$

$$r_{80}(0)=S^2\bar{\xi}_2R_{801}+S\bar{\xi}_1\bar{\xi}_2R_{802}\ ,\mathrm{R}_{801}=-\frac{2(R-1)}{2R-1}\left(R_{611}+\frac{q_3}{2}\right) \quad (3.36l)$$

$$R_{802}=\left[\frac{hq_3}{\left(Q^2-1\right)^2}+\frac{hR^2q_4}{\left(Q^2-1\right)}-\frac{2(R-1)}{2R-1}\left(R_{612}+\frac{R^2hq_3}{2(Q^2-1)}\right)\right] \quad (3.36m)$$

$$r_{81}(0)=S\bar{\xi}_1\bar{\xi}_2R_{81}\ ;\mathrm{R}_{81}=\left[\frac{R^2hq_0\varphi}{R^2-\varphi^2}-\frac{R^2h^2q_2}{2\left(R^2-\varphi^2\right)^2}\right] \quad (3.36n)$$

$$r_{82}(0)=S\bar{\xi}_1\bar{\xi}_2R_{82}\ ,\mathrm{R}_{82}=-\frac{R^2h}{\left(R^2-\varphi^2\right)}\left[\frac{hq_2}{2}+\varphi q_0\right] \quad (3.36o)$$

$$r_{83}(0)=S\bar{\xi}_1\bar{\xi}_2R_{83}\ ;\mathrm{R}_{83}=\left[\frac{R^2hq_2}{2(R^2-1)}-\frac{R^2hq_0}{\left(R^2-1\right)^2}+\frac{2(Q+1)}{R^2-(Q+1)^2}\left\{R_{67}-\left(\frac{1}{R^2-\varphi^2}-\frac{1}{R^2-1}\right)\right\}\right] \quad (3.36p)$$

$$r_{84}(0)=S\bar{\xi}_1\bar{\xi}_2R_{84}\ ;\mathrm{R}_{84}=\left[\frac{R^2hq_2}{2(R^2-1)}+\frac{R^2hq_0}{\left(R^2-1\right)^2}-\frac{2(Q-1)}{R^2-(Q-1)^2}\left\{R_{67}-\left(\frac{1}{R^2-\varphi^2}-\frac{1}{R^2-1}\right)\right\}\right] \quad (3.36q)$$

On solving the(3.34a,b), we get

$$\eta^{21}=[\gamma_{21}\cos Rt+\theta_{21}\sin Rt-\frac{r_{69}\sin(R+\varphi)t}{\varphi(2R+\varphi)}+\frac{r_{70}\sin(R-\varphi)t}{\varphi(2R-\varphi)}+r_{71}\left\{\frac{\cos(R-1)t}{2R-1}+\frac{\cos(R+1)t}{2R+1}\right\}$$

$$+r_{72}\left\{\frac{\cos(\varphi-1)t}{R^2-(\varphi-1)^2}-\frac{\cos(\varphi+1)t}{R^2-(\varphi+1)^2}\right\}+\frac{r_{73}\sin 2\varphi t}{R^2-4\varphi^2}+\frac{r_{74}\sin 2t}{R^2-4}+\frac{r_{75}\sin(\varphi+1)t}{R^2-(\varphi+1)^2}+\frac{r_{76}\sin(\varphi-1)t}{R^2-(\varphi-1)^2}$$

$$-\frac{r_{77}\sin(Q+R)t}{Q(2R+Q)}+\frac{r_{78}\sin(R-Q)t}{Q(2R-Q)}-\frac{r_{79}\sin(R+1)t}{2R+1}+\frac{r_{80}\sin(R-1)t}{2R-1}+\frac{r_{81}\sin(Q+\varphi)t}{R^2-(Q+\varphi)^2}$$

$$+\frac{r_{82}\sin(Q-\varphi)\mathrm{t}}{\mathrm{R}^2-(Q-\varphi)^2}+\frac{r_{83}\sin(Q+1)\mathrm{t}}{\mathrm{R}^2-(Q+1)^2}+\frac{r_{84}\sin(Q-1)\mathrm{t}}{\mathrm{R}^2-(Q-1)^2}\Bigg] \quad (3.37a)$$

where

$$\gamma_{21}(0)=S^2\bar{\xi}_2R_{803}\,;\ \mathrm{R}_{803}=-\left[\left\{\frac{1}{2\mathrm{R}-1}+\frac{1}{2\mathrm{R}+1}\right\}+\left\{\frac{1}{R^2-(\varphi-1)^2}-\frac{1}{R^2-(\varphi+1)^2}\right\}\right] \quad (3.37b)$$

$$\theta_{21}(0)=S^2\bar{\xi}_2R_{804}+S\bar{\xi}_1\bar{\xi}_2R_{805} \quad (3.37c)$$

$$R_{804}=\frac{1}{R}\Bigg[\frac{(R+\varphi)R_{691}}{\varphi(2R+\varphi)}-\frac{R_{701}(R-\varphi)}{\varphi(2R-\varphi)}-\frac{2\varphi R_{731}}{\mathrm{R}^2-4\varphi^2}-\frac{2R_{741}}{R^2-4}-\frac{(\varphi+1)R_{751}}{R^2-(\varphi+1)^2}$$
$$-\frac{(\varphi-1)R_{761}}{R^2-(\varphi-1)^2}+\frac{(R+1)R_{791}}{2R+1}-\frac{(R-1)R_{801}}{2R-1}-R_{6901}-\frac{q_3}{2\varphi}\left\{\frac{1}{2R+\varphi}-\frac{1}{2R-\varphi}\right\} \quad (3.37d)$$
$$+\frac{q_3}{2(2R+1)}-\frac{q_3}{2(2R-1)}\Bigg]$$

$$R_{805}=\frac{1}{R}\Bigg[\frac{(R+\varphi)R_{692}}{\varphi(2R+\varphi)}-\frac{R_{702}(R-\varphi)}{\varphi(2R-\varphi)}-\frac{2\varphi R_{732}}{\mathrm{R}^2-4\varphi^2}-\frac{2R_{742}}{R^2-4}-\frac{(\varphi+1)R_{752}}{R^2-(\varphi+1)^2}+\frac{R^2hq_0q_3}{Q}\left\{\frac{1}{2R+\varphi}\right.$$
$$\left.-\frac{1}{2R-\varphi}\right\}-\frac{(\varphi-1)R_{762}}{R^2-(\varphi-1)^2}+\frac{(R+1)R_{792}}{2R+1}-\frac{(R-1)R_{802}}{2R-1}-\frac{(Q+\varphi)R_{811}}{R^2-(Q+\varphi)^2}-\frac{(Q-\varphi)R_{821}}{R^2-(Q-\varphi)^2}$$
$$+\frac{(R+Q)R_{77}}{Q(2R+Q)}-\frac{(R-Q)R_{78}}{Q(2R-Q)}-\frac{(Q+1)R_{83}}{R^2-(Q+1)^2}-\frac{(Q-1)R_{84}}{R^2-(Q-1)^2}+R_{6902}$$
$$+\frac{R^2h}{2\varphi(Q^2-\varphi^2)}\left\{\frac{1}{2R+\varphi}-\frac{1}{2R-\varphi}\right\}-\frac{R^2hq_3}{2(2R+Q)(Q^2-1)}-\frac{R^2hq_3}{2(2R+Q)(Q^2-1)}$$
$$+\frac{R^2hq_0}{2}\left\{\frac{1}{R^2-\varphi^2}-\frac{1}{R^2-1}\right\}\left\{\frac{1}{R^2-(Q+1)^2}+\frac{1}{R^2-(Q-1)^2}\right\}\Bigg] \quad (3.37e)$$

So far, the expressions for the time dependent components of the normal displacement are

$$\xi_1(t,\tau)=\in\left(\zeta^{10}+\delta\zeta^{11}\right)+\in^2\left(\zeta^{20}+\delta\zeta^{21}\right)+O\left(\in\delta^2\right)+O\left(\in^2\delta^2\right) \quad (3.38a)$$

$$\xi_2(t,\tau)=\in\left(\eta^{10}+\delta\eta^{11}\right)+\in^2\left(\eta^{20}+\delta\eta^{21}\right)+O\left(\in\delta^2\right)+O\left(\in^2\delta^2\right) \quad (3.38b)$$

4.0 Components of maximum displacements

To determine the maximum values ξ_{1a} and ξ_{2c} of $\xi_1(t,\tau)$ and $\xi_2(t,\tau)$ respectively, we shall let t_a and τ_a be the critical values, at maximum displacement , of t and τ respectively for $\xi_1(t,\tau)$ .We shall similarly let t_c and τ_c be the corresponding values of t and τ for $\xi_2(t,\tau)$ at maximum displacement and now assume the following asymptotic series:

$$t_a=t_0+\delta t_{01}+\in(t_{10}+\delta t_{11})+\cdots;\ \tau_a=\delta\,\mathrm{t_a}=\delta\{t_0+\delta t_{01}+\in(t_{10}+\delta t_{11})+\cdots\} \quad (4.1a)$$

$$t_c=\tilde{t}_0+\delta\tilde{t}_{01}+\in(\tilde{t}_{10}+\delta\tilde{t}_{11})+\cdots;\ \tau_c=\delta\,\mathrm{t_c}=\delta\{\tilde{t}_0+\delta\tilde{t}_{01}+\in(\tilde{t}_{10}+\delta\tilde{t}_{11})+\cdots\} \quad (4.1b)$$

Using the first of (12) (for $\alpha=1$) ,we observe that the condition for maximum displacement of $\xi_1(t,\tau)$ is $$\xi_{1,t}(t_a,\tau_a)+\delta\,\xi_{1,\tau}(t_a,\tau_a)=0 \tag{4.2a}$$

If we substitute the relevant terms into (47a) , using (46a), and equate to zero the coefficients of $\in, \in\delta$ and $\in^2$, we get respectively

$$\zeta^{10}_{,t}=0\ ;\quad t_{01}\zeta^{10}_{,tt}+t_0\zeta^{10}_{,t\tau}+\zeta^{11}_{,t}=0 \quad\text{and}\quad t_{10}\zeta^{10}_{,tt}+\zeta^{20}_{,t}=0 \tag{4.2b}$$

where (4.2b) is evaluated at $(t,\tau)=(t_0,0)$. From the first of equation (4.2b) ,we have

$$Qq_0\sin Qt_0+\left[\frac{\sin t_0}{Q^2-1}-\frac{\varphi\sin\varphi t_0}{Q^2-\varphi^2}\right]=0 \tag{4.2c}$$

An approximate value of t_0 is obtained from (4.2c) by maintaining just the first three terms in each of the relevant Taylor series expansions to get

$$t_0\cong\sqrt{\frac{6\left\{\left(\dfrac{\varphi^2}{Q^2-\varphi^2}-\dfrac{1}{Q^2-1}\right)-Q^2q_0\right\}}{\left(\dfrac{\varphi^4}{Q^2-\varphi^2}+\dfrac{1}{Q^2-1}\right)-Q^4q_0}} \tag{4.2d}$$

From the second equation in (4.2b), we have $$t_{01}=-\frac{\zeta^{11}_{,t}(t_0,0)}{\zeta^{10}_{,tt}(t_0,0)}=-\frac{T_0}{T_1} \tag{4.2e}$$

where $$\zeta^{10}_{,tt}(t_0,0)=T_1\bar{\xi}_1h\,;\ \mathrm{T}_1=\left[Q^2q_0\cos Qt_0+\left\{\frac{\cos t_0}{Q^2-1}-\frac{\varphi^2\cos\varphi t_0}{Q^2-\varphi^2}\right\}\right] \tag{4.2f}$$

$$\zeta^{11}_{,t}(t_0,0)=T_0\bar{\xi}_1h\,;\ \mathrm{T}_0=\left[Q^2q_2\cos Qt_0-2\left\{\frac{\cos t_0}{(Q^2-1)^2}-\frac{\varphi^2\cos\varphi t_0}{(Q^2-\varphi^2)^2}\right\}\right] \tag{4.2g}$$

From the third equation in (4.2b), we have $$t_{10}=-\frac{\zeta^{20}_{,t}(t_0,0)}{\zeta^{10}_{,tt}(t_0,0)} \tag{4.2h}$$

where $$\zeta^{20}_{,t}(t_0,0)=h\bar{\xi}_1R_{851}+k_1\left(h\bar{\xi}_1Q\right)^2R_{852}+k_2\left(QS\bar{\xi}_2\right)^2R_{853} \tag{4.2i}$$

$$\begin{aligned}R_{851}=\Bigg[&\frac{R_{71}}{\varphi}\left\{\frac{(Q+\varphi)\sin(Q+\varphi)t}{2Q+\varphi}-\frac{(Q-\varphi)\sin(Q-\varphi)t}{2Q-\varphi}\right\}-QR_{16}\sin Qt\\&-\frac{(Q-1)R_{81}\sin(Q-1)t}{2Q-1}-\frac{(Q+1)R_{91}\sin(Q+1)t}{2Q+1}-\frac{2\varphi R_{101}\sin2\varphi t}{Q^2-4\varphi^2}-\frac{2R_{111}\sin2t}{Q^2-4}\Bigg]\Bigg|_{t=t_0}\end{aligned} \tag{4.2j}$$

$$\begin{aligned}R_{852}=\Bigg[&\frac{q_0^2\sin2Qt}{3Q}-QR_{17}\sin Qt-\frac{R_{72}}{\varphi}\left\{\frac{(Q+\varphi)\sin(Q+\varphi)t}{2Q+\varphi}-\frac{(Q-\varphi)\sin(Q-\varphi)t}{2Q-\varphi}\right\}\\&-\frac{(Q-1)R_{82}\sin(Q-1)t}{2Q-1}+\frac{(Q+1)R_{92}\sin(Q+1)t}{2Q+1}-\frac{2\varphi R_{102}\sin2\varphi t}{Q^2-4\varphi^2}-\frac{2R_{112}\sin2t}{Q^2-4}\\&+R_{121}\left\{\frac{(\varphi+1)\sin(\varphi+1)t}{Q^2-(\varphi+1)^2}+\frac{(\varphi-1)\sin(\varphi-1)t}{Q^2-(\varphi-1)^2}\right\}\Bigg]\Bigg|_{t-t_0}\end{aligned} \tag{4.2k}$$

$$R_{853}=\left[\frac{2\varphi R_{103}\sin 2\varphi t}{Q^2-4\varphi^2}-Q_{18}\sin Qt+\frac{2R_{113}\sin 2t}{Q^2-4}+\frac{2R_{13}\sin 2Rt}{Q^2-4R^2}+R_{122}\left\{\frac{(\varphi+1)\sin(\varphi+1)t}{Q^2-(\varphi+1)^2}\right.\right.$$

$$\left.+\frac{(\varphi-1)\sin(\varphi-1)t}{Q^2-(\varphi-1)^2}\right\}+R_{14}\left\{\frac{(R+\varphi)\sin(R+\varphi)t}{Q^2-(R+\varphi)^2}+\frac{(R-\varphi)\sin(R-\varphi)t}{Q^2-(R-\varphi)^2}\right\}$$

$$\left.\left.-R_{15}\left\{\frac{(R+1)\sin(R+1)t}{Q^2-(R+1)^2}+\frac{(R-1)\sin(R-1)t}{Q^2-(R-1)^2}\right\}\right]\right|_{t=t_0} \qquad (4.2l)$$

As in (4.2a) ,the condition for the attainment of the maximum displacement ξ_{2c} is

$$\xi_{2,t}(t_c,\tau_c)+\delta\,\xi_{2,\tau}(t_c,\tau_c)=0 \qquad (4.3a)$$

On substituting the relevant terms into (4.3a), using (4.1b), and equating the coefficients of $\in,\in\delta$ and $\in^2$, we have the following respective equations

$$\eta^{10}_{,t}=0\ ;\quad \tilde{t}_{01}\eta^{10}_{,tt}+\tilde{t}_0\eta^{10}_{,t\tau}+\eta^{11}_{,t}=0 \ \text{ and }\ \tilde{t}_{10}\eta^{10}_{,tt}+\eta^{20}_{,t}=0 \qquad (4.3b)$$

where (48b) is evaluated at $(t,\tau)=(\tilde{t},0)$.From the first equation in (4.3b), we get

$$Rq_3\sin R\tilde{t}_0+\left[\frac{\sin\tilde{t}_0}{R^2-1}-\frac{\varphi\sin\varphi\,\tilde{t}_0}{R^2-\varphi^2}\right]=0 \qquad (4.4a)$$

An approximate value of $\tilde{t}_0$ is

$$\tilde{t}_0\cong\sqrt{\frac{6\left\{\left(\frac{\varphi^2}{R^2-\varphi^2}-\frac{1}{R^2-1}\right)-R^2q_3\right\}}{\left(\frac{\varphi^4}{R^2-\varphi^2}+\frac{1}{Q^2-1}\right)-R^4q_3}} \qquad (4.4b)$$

From the second equation in (4.3b) , we have

$$\tilde{t}_{01}=\frac{\eta^{11}_{,t}(\tilde{t}_0,0)}{\eta^{10}_{,tt}(\tilde{t}_0,0)}=-\frac{T_2}{T_3} \qquad (4.4c)$$

$$\eta^{11}_{,t}(\tilde{t}_0,0)=S\bar{\xi}_2T_2\,;\mathrm{T}_2=\left[Rq_4\cos R\tilde{t}_0-2\left\{\frac{\cos\tilde{t}_0}{\left(R^2-1\right)^2}-\frac{\varphi^2\cos\varphi\,\tilde{t}_0}{\left(R^2-\varphi^2\right)^2}\right\}\right] \qquad (4.4d)$$

$$\eta^{10}_{,tt}(\tilde{t}_0,0)=S\bar{\xi}_2T_3\,;\mathrm{T}_3=\left[R^2q_3\cos Rt_0+\left\{\frac{\cos\tilde{t}_0}{\left(R^2-1\right)}-\frac{\varphi^2\cos\varphi\,\tilde{t}_0}{\left(R^2-\varphi^2\right)}\right\}\right] \qquad (4.4e)$$

From the third equation in (4.3b) , we have

$$\tilde{t}_{10}=-\frac{\eta^{20}_{,t}(\tilde{t}_0,0)}{\eta^{10}_{,tt}(\tilde{t}_0,0)} \qquad (4.4f)$$

where $\eta^{20}_{,t}(\tilde{t}_0,0)=S^2\bar{\xi}_2R_{861}+S\bar{\xi}_1\bar{\xi}_2R_{862}$ (4.4g)

$$R_{861}=\Bigg[-\frac{R_{591}}{\varphi}\left\{\frac{(R+\varphi)\sin(R+\varphi)\mathrm{t}}{2\mathrm{R}+\varphi}-\frac{(R-\varphi)\sin(R-\varphi)\mathrm{t}}{2\mathrm{R}-\varphi}\right\}-RR_{6901}\sin Rt$$

$$+\frac{(R+1)R_{601}\sin(R+1)\mathrm{t}}{2\mathrm{R}+1}-\frac{(R-1)R_{611}\sin(R-1)\mathrm{t}}{2\mathrm{R}-1}-\frac{2\varphi R_{631}\sin 2\varphi\,\mathrm{t}}{\mathrm{R}^2-4\varphi^2}-\frac{2R_{641}\sin 2\,\mathrm{t}}{\mathrm{R}^2-4} \quad (4.4h)$$

$$-\frac{(\varphi+1)R_{651}\sin(\varphi+1)\mathrm{t}}{R^2-(\varphi+1)^2}+\frac{(\varphi-1)R_{681}\sin(\varphi-1)\mathrm{t}}{R^2-(\varphi-1)^2}\Bigg]\Bigg|_{\mathrm{t}=\tilde{\mathrm{t}}_0}$$

$$R_{862}=\Bigg[\frac{R_{592}}{\varphi}\left\{\frac{(R+\varphi)\sin(R+\varphi)\mathrm{t}}{2\mathrm{R}+\varphi}-\frac{(R-\varphi)\sin(R-\varphi)\mathrm{t}}{2\mathrm{R}-\varphi}\right\}-\frac{(R+1)R_{602}\sin(R+1)\mathrm{t}}{2\mathrm{R}+1}$$

$$+\frac{(R-1)R_{612}\sin(R-1)\mathrm{t}}{2\mathrm{R}-1}+\frac{2\varphi\,\mathrm{R}_{632}\sin 2\varphi\,\mathrm{t}}{\mathrm{R}^2-4\varphi^2}+\frac{2\mathrm{R}_{642}\sin 2\,\mathrm{t}}{\mathrm{R}^2-4}-\frac{(\varphi+1)R_{652}\sin(\varphi+1)\mathrm{t}}{\mathrm{R}^2-(\varphi+1)^2}$$

$$+\frac{R_{66}}{Q}\left\{\frac{(\mathrm{Q}+\mathrm{R})\sin(\mathrm{Q}+\mathrm{R})\mathrm{t}}{2\mathrm{R}+\mathrm{Q}}-\frac{(\mathrm{Q}-\mathrm{R})\sin(\mathrm{Q}-\mathrm{R})\mathrm{t}}{2\mathrm{R}-\mathrm{Q}}\right\}-R_{67}\left\{\frac{(\mathrm{Q}-1)\sin(\mathrm{Q}-1)\mathrm{t}}{R^2-(\mathrm{Q}-1)^2}\right.$$

$$\left.+\frac{(\mathrm{Q}+1)\sin(\mathrm{Q}+1)\mathrm{t}}{R^2-(\mathrm{Q}+1)^2}\right\}+\frac{\mathrm{R}_{682}(\varphi-1)\sin(\varphi-1)\mathrm{t}}{\mathrm{R}^2-(\varphi-1)^2}\Bigg]\Bigg|_{\mathrm{t}=\tilde{\mathrm{t}}_0} \quad (4.4i)$$

By determining (3.38a) at (t_a,τ_a), using (4.1a), we get the only nonvanishing terms of $\xi_1(t_a,\tau_a)=\xi_{1a}$ as

$$\xi_{1a}=\in\left[\zeta^{10}+\delta\left(\zeta^{11}+t_0\zeta^{10}_{,\tau}\right)\right]\Big|_{(\mathrm{t}_0,0)}+\in^2\Big[\zeta^{20}+\delta\Big\{\mathrm{t}_{10}\zeta^{10}_{,\tau}+t_{01}t_{10}\zeta^{10}_{,t\,\mathrm{t}}+\mathrm{t}_{10}\zeta^{11}_{,t}+t_{01}\zeta^{20}_{,t}$$

$$+t_0\zeta^{20}_{,\tau}+\zeta^{21}+t_0t_{10}\zeta^{10}_{,t\,\tau}\Big\}\Big]\Big|_{(\mathrm{t}_0,0)}+O\left(\in\delta^2\right)+O\left(\in^2\delta^2\right) \quad (4.5a)$$

Similarly , by determining (3.38b) at (t_c,τ_c),using (4.1b), we get

$$\xi_{2c}=\in\left[\eta^{10}+\delta\left(\eta^{11}+\tilde{t}_0\eta^{10}_{,\tau}\right)\right]\Big|_{(\tilde{\mathrm{t}}_0,0)}+\in^2\Big[\eta^{20}+\delta\Big\{\tilde{\mathrm{t}}_{10}\eta^{10}_{,\tau}+\tilde{t}_{01}\tilde{t}_{10}\eta^{10}_{,t\,\mathrm{t}}+\tilde{\mathrm{t}}_{10}\eta^{11}_{,t}+\tilde{t}_{01}\eta^{20}_{,t}$$

$$+\tilde{t}_0\eta^{20}_{,\tau}+\eta^{21}+\tilde{t}_0\tilde{t}_{10}\eta^{10}_{,t\,\tau}\Big\}\Big]\Big|_{(\tilde{\mathrm{t}}_0,0)}+O\left(\in\delta^2\right)+O\left(\in^2\delta^2\right) \quad (4.5b)$$

The following terms appearing in (4.5a,b) are easily evaluated as follows:

$$t_{01}t_{10}\zeta^{10}_{,t\,\mathrm{t}}(t_0,0)=-t_{01}\zeta^{20}_{,\,\mathrm{t}}(t_0,0)=-t_{01}\left\{h\bar{\xi}_1R_{851}+k_1\left(h\bar{\xi}_1Q\right)^2R_{852}+k_2\left(QS\bar{\xi}_2\right)R_{853}\right\} \quad (4.5c)$$

$$\tilde{t}_{01}\tilde{t}_{10}\eta^{10}_{,t\,\mathrm{t}}(\tilde{t}_0,0)=-\tilde{t}_{01}\eta^{20}_{,\,\mathrm{t}}(\tilde{t}_0,0)=-\tilde{t}_{01}\left\{S^2\bar{\xi}_2R_{861}+\bar{\xi}_1\bar{\xi}_2SR_{862}\right\} \quad (4.5d)$$

$$t_{01}\zeta^{20}_{,t}(t_0,0)=t_{01}\left[h\bar{\xi}_1R_{851}+k_1\left(h\bar{\xi}_1Q\right)^2R_{852}+k_2\left(QS\bar{\xi}_2\right)R_{853}\right] \quad (4.6a)$$

$$t_0\zeta^{20}_{,\tau}(t_0,0)=t_0\left[h\bar{\xi}_1R_{891}+k_1\left(h\bar{\xi}_1Q\right)^2R_{892}+k_2\left(QS\bar{\xi}_2\right)R_{893}\right] \quad (4.6b)$$

where

$$R_{891} = \left[\frac{hq_0}{2\varphi} \left\{ \frac{\cos(Q-\varphi)\mathrm{t}}{2Q-\varphi} - \frac{\cos(Q+\varphi)\mathrm{t}}{2Q+\varphi} \right\} - \frac{hq_0 \cos(Q-1)\mathrm{t}}{2(2\mathrm{Q}-1)} - R_{16} \cos Qt + \frac{hq_0 \cos(Q+1)\mathrm{t}}{2(2\mathrm{Q}+1)} \right] \Bigg|_{\mathrm{t}=t_0} \tag{4.6c}$$

$$R_{892} = \left[-\left(\frac{q_0}{Q}\right)^2 \left(1 - \frac{1}{3}\cos 2Qt\right) - R_{17}\cos Qt + \frac{q_0}{\varphi(Q^2-\varphi^2)} \left\{ \frac{\cos(Q-\varphi)\mathrm{t}}{2\mathrm{Q}-\varphi} - \frac{\cos(Q+\varphi)\mathrm{t}}{2\mathrm{Q}+\varphi} \right\} - \frac{q_0 \cos(Q-1)\mathrm{t}}{(2\mathrm{Q}-1)(Q^2-1)} + \frac{q_0 \cos(Q+1)\mathrm{t}}{(2\mathrm{Q}+1)(Q^2-1)} \right] \Bigg|_{\mathrm{t}=\mathrm{t}_0} \tag{4.6d}$$

$$R_{893} = \left[\left(\frac{q_3}{Q}\right)^2 + \frac{q_3^2 \cos 2Rt}{Q^2-4R^2} - R_{18}\cos Qt - \frac{q_3}{R^2-\varphi^2} \left\{ \frac{\cos(R+\varphi)\,\mathrm{t}}{\mathrm{Q}^2-(R+\varphi)^2\,\mathrm{t}} + \frac{\cos(R-\varphi)\,\mathrm{t}}{\mathrm{Q}^2-(R-\varphi)^2\,\mathrm{t}} \right\} + \frac{q_3}{R^2-1} \left\{ \frac{\cos(R+1)\,\mathrm{t}}{\mathrm{Q}^2-(R+\varphi)^2\,\mathrm{t}} + \frac{\cos(R-1\,\mathrm{t}}{\mathrm{Q}^2-(R-1)^2\,\mathrm{t}} \right\} \right] \Bigg|_{\mathrm{t}=\mathrm{t}_0} \tag{4.6e}$$

$$\zeta^{21}(t_0,0) = \bar{\xi}_1 hR_{881} + k_1(h\bar{\xi}_1 Q)^2 R_{882} + k_2(Q\bar{\xi}_2 S)R_{883} \tag{4.7a}$$

$$R_{881} = \left[-\frac{R_{401}\sin(Q+\varphi)\mathrm{t}}{\varphi(2\mathrm{Q}+\varphi)} - \frac{(Q+1)R_{421}\sin(Q+1)\mathrm{t}}{2\mathrm{Q}+1} + R_{871}\sin Qt + \frac{(Q-\varphi)R_{441}\sin(Q-\varphi)\mathrm{t}}{\varphi(2\mathrm{Q}-\varphi)} + \frac{R_{431}\sin(Q-1)\mathrm{t}}{2\mathrm{Q}-1} + \frac{R_{441}\sin 2\varphi\,\mathrm{t}}{\mathrm{Q}^2-4\varphi^2} + \frac{R_{451}\sin(\varphi-1)\mathrm{t}}{\mathrm{Q}^2-(\varphi-1)^2} + \frac{R_{461}\sin(\varphi+1)\mathrm{t}}{\mathrm{Q}^2-(\varphi+1)^2} \right] \Bigg|_{\mathrm{t}=\mathrm{t}_0} \tag{4.7b}$$

$$R_{882} = \left[-\frac{2R_{402}\sin(Q+\varphi)\mathrm{t}}{\varphi(2Q+\varphi)} - \frac{(Q+1)R_{422}\sin(Q+1)\mathrm{t}}{2\mathrm{Q}+1} + R_{872}\sin Qt + \frac{R_{432}\sin(Q-1)\,\mathrm{t}}{2Q-1} + \frac{R_{412}(Q-\varphi)\sin(Q-\varphi)\mathrm{t}}{\varphi(2\mathrm{Q}-\varphi)} + \frac{R_{442}\sin 2\varphi\,\mathrm{t}}{\mathrm{Q}^2-4\varphi^2} + \frac{R_{452}\sin(\varphi-1)\mathrm{t}}{\mathrm{Q}^2-(\varphi-1)^2} + \frac{R_{462}\sin(\varphi+1)\mathrm{t}}{\mathrm{Q}^2-(\varphi+1)^2} - \frac{R_{47}\sin 2Qt}{3Q^2} + \frac{\mathrm{R}_{48}\sin 2t}{Q^2-4} \right] \Bigg|_{\mathrm{t}=\mathrm{t}_0} \tag{4.7c}$$

$$R_{883} = \left[\frac{R_{443}\sin 2\varphi\,\mathrm{t}}{\mathrm{Q}^2-4\varphi^2} + R_{873}\sin Qt + \frac{R_{453}\sin(\varphi-1)\mathrm{t}}{\mathrm{Q}^2-(\varphi-1)^2} + \frac{R_{463}\sin(\varphi+1)\mathrm{t}}{\mathrm{Q}^2-(\varphi+1)^2} - \frac{R_{482}\sin 2t}{Q^2-4} - 2\left\{ \frac{R_{49}\sin 2Rt}{Q^2-4R^2} + \frac{R_{50}\sin(R+1)\mathrm{t}}{\mathrm{Q}^2-(R+1)^2} + \frac{R_{511}\sin(R-1)\mathrm{t}}{\mathrm{Q}^2-(R-1)^2} + \frac{R_{521}\sin(R+\varphi)\mathrm{t}}{\mathrm{Q}^2-(R+\varphi)^2} + \frac{R_{531}\sin(R-\varphi)\mathrm{t}}{\mathrm{Q}^2-(R-\varphi)^2} \right\} \right] \Bigg|_{\mathrm{t}=\mathrm{t}_0} \tag{4.7d}$$

We also have

$$\zeta^{20}(t_0,0) = \bar{\xi}_1 hR_{208} + k_1(h\bar{\xi}_1 Q)^2 R_{209} + k_2(S\bar{\xi}_2 Q)^2 R_{210} \tag{4.8a}$$

$$R_{208} = \left[\frac{R_{51}}{Q^2} - \frac{R_{71}}{\varphi} \left\{ \frac{\cos(Q-\varphi)\,t}{2Q - \varphi} - \frac{\cos(Q+\varphi)\,t}{2Q+\varphi} \right\} + R_{16} \cos Qt + \frac{R_{81}\cos(Q-1)\,t}{2Q - 1} \right.$$

$$\left. + \frac{R_{91}\cos(Q+1)\,t}{2Q+1} + \frac{R_{101}\cos 2\varphi\, t}{Q^2 - 4\varphi^2} + \frac{R_{111}\cos 2\,t}{Q^2 - 4} \right]\Bigg|_{t=t_0} \quad (4.8b)$$

$$R_{209} = \left[\frac{R_{52}}{Q^2} - \frac{q_0^2 \cos 2Qt}{6Q^2} + R_{17}\cos Qt - \frac{R_{72}}{\varphi} \left\{ \frac{\cos(Q-\varphi)\,t}{2Q - \varphi} - \frac{\cos(Q+\varphi)\,t}{2Q+\varphi} \right\} \right.$$

$$+ \frac{R_{82}\cos(Q-1)\,t}{2Q - 1} - \frac{R_{92}\cos(Q+1)\,t}{2Q+1} + \frac{R_{102}\cos 2\varphi\, t}{Q^2 - 4\varphi^2} + \frac{R_{112}\cos 2\,t}{Q^2 - 4}$$

$$\left. - R_{121}\left\{ \frac{\cos(\varphi+1)\,t}{Q^2 - (\varphi+1)^2} + \frac{\cos(\varphi-1)\,t}{Q^2 - (\varphi-1)^2} \right\} \right]\Bigg|_{t=t_0} \quad (4.8c)$$

$$R_{210} = \left[-\frac{R_{53}}{Q^2} - \frac{R_{113}\cos 2t}{Q^2 - 4} + R_{18}\cos Qt - R_{122}\left\{ \frac{\cos(\varphi+1)\,t}{Q^2 - (\varphi+1)^2} + \frac{\cos(\varphi-1)\,t}{Q^2 - (\varphi-1)^2} \right\} \right.$$

$$\left. - \frac{R_{13}\cos 2Rt}{Q^2 - 4R^2} - R_{14}\left\{ \frac{\cos(R+\varphi)\,t}{Q^2 - (R+\varphi)^2} + \frac{\cos(R-\varphi)\,t}{Q^2 - (R-\varphi)^2} \right\} + R_{15}\left\{ \frac{\cos(R+1)\,t}{Q^2 - (R+1)^2} + \frac{\cos(R-1)\,t}{Q^2 - (R-1)^2} \right\} \right]\Bigg|_{t=t_0} \quad (4.8d)$$

$$t_{10}\zeta^{11}_{,t}(t_0, 0) = -\frac{T_0}{T_1}\left[h\bar{\xi}_1 R_{851} + k_1 (h\bar{\xi}_1 Q)^2 R_{852} + k_2 (S\bar{\xi}_2 Q)^2 R_{853} \right] \quad (4.9a)$$

$$t_{10}\zeta^{10}_{,\tau}(t_0, 0) = -\frac{q_0}{T_1}\left[h\bar{\xi}_1 R_{851} + k_1 (h\bar{\xi}_1 Q)^2 R_{852} + k_2 (S\bar{\xi}_2 Q)^2 R_{853} \right] \quad (4.9b)$$

$$\zeta^{10}(t_0, 0) = h\bar{\xi}_1 T_4 \,,\; T_4 = \left[\frac{\cos\varphi\, t_0}{Q^2 - \varphi^2} - \frac{\cos t_0}{Q^2 - 1} - q_0 \cos Qt_0 \right] \quad (4.9c)$$

$$\zeta^{11}(t_0, 0) = h\bar{\xi}_1 T_5,\; T_5 = \left[q_2 \sin Qt_0 - 2\left\{ \frac{\sin t_0}{\left(Q^2 - 1\right)^2} \frac{\varphi \sin\varphi\, t_0}{(Q^2 - \varphi^2)^2} - \right\} \right] \quad (4.9d)$$

On substituting all the relevant evaluated terms into (4.5a), we get

$$\xi_{1a} = h\bar{\xi}_1 \in \left[T_4 + \delta\left(T_5 + t_0 q_0 \cos Qt_0\right)\right] + \in^2 \left[h\bar{\xi}_1 R_{208} + k_1 (Qh\bar{\xi}_1)^2 R_{209} + k_2 \left(SQ\bar{\xi}_2\right)^2 R_{210} \right.$$

$$\left. + \delta\left\{ h\bar{\xi}_1 T_6 + k_1 (Qh\bar{\xi}_1)^2 T_7 + k_2 \left(SQ\bar{\xi}_2\right)^2 T_8 \right\} \right] + O\left(\in \delta^2\right) + O\left(\in^2 \delta^2\right) \quad (4.10a)$$

$$T_6 = \left[\frac{R_{851}}{T_1} \left\{ t_0 q_0 Q \sin Qt_0 - q_0 \cos Qt_0 - T_0 \right\} + t_0 R_{891} + R_{881} \right] \quad (4.10b)$$

$$T_7 = \left[\frac{R_{852}}{T_1} \left\{ t_0 q_0 Q \sin Qt_0 - q_0 \cos Qt_0 - T_0 \right\} + t_0 R_{892} + R_{882} \right] \quad (4.10c)$$

$$T_8=\left[\frac{R_{853}}{T_1}\{t_0 q_0 Q\sin Qt_0-q_0\cos Qt_0-T_0\}+t_0R_{893}+R_{883}\right] \tag{4.10d}$$

We next evaluate relevant terms in (4.5b) as follows:

$$\eta^{10}(\tilde{t}_0,0)=S\bar{\xi}_2T_9\ ,\ \mathrm{T}_9=\left[\frac{\cos\varphi\,\tilde{t}_0}{\mathrm{R}^2-\varphi^2}-\frac{\cos\tilde{t}_0}{\mathrm{R}^2-1}-q_3\cos R\tilde{t}_0\right] \tag{4.11a}$$

$$\eta^{11}(\tilde{t}_0,0)=S\bar{\xi}_2T_{10},\ \mathrm{T}_{10}=\left[q_4\sin R\tilde{t}_0-2\left\{\frac{\sin\tilde{t}_0}{\left(\mathrm{R}^2-1\right)^2}\frac{\varphi\sin\varphi\,\tilde{t}_0}{(\mathrm{R}^2-\varphi^2)^2}-\right\}\right] \tag{4.11b}$$

$$\tilde{t}_0\eta^{10}_{,\tau}(\tilde{t}_0,0)=\tilde{t}_0Sq_3\bar{\xi}_2\cos R\tilde{t}_0\ ;\ \eta^{20}(\tilde{t}_0,0)=S^2\bar{\xi}_2R_{900}+S\bar{\xi}_1\bar{\xi}_2R_{901} \tag{4.11c}$$

$$R_{900}=\left[-\frac{R_{591}}{\varphi}\left\{\frac{\cos(R-\varphi)\mathrm{t}}{2R-\varphi}-\frac{\cos(R+\varphi)\mathrm{t}}{2R+\varphi}\right\}+R_{6901}\cos Rt-\frac{R_{601}\sin(R+1)\mathrm{t}}{2R+1}\right.$$
$$+\frac{R_{611}\sin(R-1)\mathrm{t}}{2R-1}+\frac{R_{621}}{R^2}+\frac{R_{631}\cos 2\varphi\,\mathrm{t}}{\mathrm{R}^2-4\varphi^2}+\frac{R_{641}\cos 2\,\mathrm{t}}{\mathrm{R}^2-4}+\frac{R_{651}\cos(\varphi+1)\mathrm{t}}{\mathrm{R}^2-(\varphi+1)^2}$$
$$\left.\left.-\frac{R_{681}\cos(\varphi-1)\mathrm{t}}{\mathrm{R}^2-(\varphi-1)^2}\right]\right|_{\mathrm{t}=\tilde{\mathrm{t}}_0} \tag{4.12a}$$

$$R_{901}=\left[\frac{R_{592}}{\varphi}\left\{\frac{\cos(R-\varphi)\mathrm{t}}{2R-\varphi}-\frac{\cos(R+\varphi)\mathrm{t}}{2R+\varphi}\right\}+R_{6902}\cos Rt+\frac{R_{602}\sin(R+1)\mathrm{t}}{2R+1}\right.$$
$$-\frac{R_{612}\sin(R-1)\mathrm{t}}{2R-1}-\frac{R_{622}}{R^2}-\frac{R_{632}\cos 2\varphi\,\mathrm{t}}{\mathrm{R}^2-4\varphi^2}-\frac{R_{642}\cos 2\,\mathrm{t}}{\mathrm{R}^2-4}+\frac{R_{652}\cos(\varphi+1)\mathrm{t}}{\mathrm{R}^2-(\varphi+1)^2}$$
$$-\frac{R_{682}\cos(\varphi-1)\mathrm{t}}{\mathrm{R}^2-(\varphi-1)^2}+\frac{R_{66}}{Q}\left\{\frac{\cos(Q-R)\mathrm{t}}{2\mathrm{R}-\mathrm{Q}}-\frac{\cos(Q+R)\mathrm{t}}{2\mathrm{R}+\mathrm{Q}}\right\}$$
$$\left.\left.+\mathrm{R}_{67}\left\{\frac{\cos(Q-1)\mathrm{t}}{\mathrm{R}^2-(Q-1)^2}-\frac{\cos(Q+1)\mathrm{t}}{\mathrm{R}^2-(Q+1)^2}\right\}\right]\right|_{\mathrm{t}=\tilde{\mathrm{t}}_0} \tag{4.12b}$$

We also have

$$\tilde{t}_{10}\eta^{10}_{,\tau}(\tilde{t}_0,0)=-\frac{q_3}{T_3}\left(S^2\bar{\xi}_2R_{861}+S\bar{\xi}_1\bar{\xi}_2R_{682}\right)\cos R\tilde{t}_0 \tag{4.13a}$$

$$\tilde{t}_{10}\eta^{11}_{,\tau}(\tilde{t}_0,0)=-\frac{T_{10}}{T_3}\left(S^2\bar{\xi}_2R_{861}+S\bar{\xi}_1\bar{\xi}_2R_{682}\right) \tag{4.13b}$$

$$T_{10}=\left[Rq_4\cos R\tilde{t}_0-2\left\{\frac{\cos\tilde{t}_0}{\left(R^2-1\right)^2}-\frac{\varphi^2\cos\varphi\,\tilde{t}_0}{\left(R^2-\varphi^2\right)^2}\right\}\right] \tag{4.13c}$$

$$\tilde{t}_0\eta^{20}_{,\tau}(\tilde{t}_0,0)=\tilde{t}_0\left(S^2\bar{\xi}_2R_{911}+S\bar{\xi}_1\bar{\xi}_2R_{912}\right) \tag{4.13d}$$

where

$$R_{911}=\left[\frac{q_3}{2\varphi}\left\{\frac{\cos(R-\varphi)\mathrm{t}}{2\mathrm{R}-\varphi}-\frac{\cos(R+\varphi)\mathrm{t}}{2\mathrm{R}+\varphi}\right\}-R_{6901}\cos Rt-\frac{q_3\cos(R+1)\mathrm{t}}{2(2R+1)}\right.$$
$$\left.\left.+\frac{q_3\cos(R-1)\mathrm{t}}{2(2R-1)}\right]\right|_{\mathrm{t}=\tilde{\mathrm{t}}_0} \quad (4.13e)$$

$$R_{912}=\left[-\frac{q_3R^2h}{2\varphi\left(Q^2-\varphi^2\right)}\left\{\frac{\cos(R-\varphi)\mathrm{t}}{2R-\varphi}+\frac{\cos(R+\varphi)\mathrm{t}}{2R+\varphi}\right\}+\frac{q_3R^2h\cos(R+1)\mathrm{t}}{2(2R+1)\left(Q^2-1\right)}\right. \quad (4.13f)$$
$$+\frac{q_3R^2h\cos(R-1)\mathrm{t}}{2(2R-1)\left(Q^2-1\right)}-\frac{R^2hq_0q_3}{Q}\left\{\frac{\cos(Q-R)\mathrm{t}}{2\mathrm{R}-\mathrm{Q}}-\frac{\cos(Q+R)\mathrm{t}}{2\mathrm{R}+\mathrm{Q}}\right\}-R_{6902}\cos Rt$$

$$\tilde{t}_0\tilde{t}_{10}\eta^{10}_{,\mathrm{t}\,\tau}(\tilde{t}_0,0)=\frac{\tilde{t}_0Rq_3}{T_3}\left(S^2\bar{\xi}_2R_{861}+S\bar{\xi}_1\bar{\xi}_2R_{862}\right)\sin R\tilde{t}_0 \quad (4.14a)$$

$$\eta^{21}(\tilde{t}_0,0)=\left(S^2\bar{\xi}_2R_{921}+S\bar{\xi}_1\bar{\xi}_2R_{931}\right) \quad (4.14b)$$

$$R_{921}=\left[-\frac{R_{691}\sin(R+\varphi)\mathrm{t}}{\varphi(2R+\varphi)}+R_{803}\cos Rt+R_{804}\sin Rt+\frac{R_{701}\sin(R-\varphi)\mathrm{t}}{\varphi(2R-\varphi)}\right.$$
$$+R_{711}\left\{\frac{\cos(R-1)\mathrm{t}}{2\mathrm{R}-1}+\frac{\cos(R+1)\mathrm{t}}{2\mathrm{R}+1}\right\}+R_{721}\left\{\frac{\cos(\varphi-1)\mathrm{t}}{\mathrm{R}^2-(\varphi-1)^2}+\frac{\cos(\varphi+1)\mathrm{t}}{\mathrm{R}^2-(\varphi+1)^2}\right\}+\frac{R_{731}\sin2\varphi\,\mathrm{t}}{\mathrm{R}^2-4\varphi^2} \quad (4.14c)$$
$$+\frac{R_{741}\sin2\mathrm{t}}{\mathrm{R}^2-4}+\frac{R_{751}\sin(\varphi+1)\mathrm{t}}{\mathrm{R}^2-(\varphi+1)^2}+\frac{R_{761}\sin(\varphi-1)\mathrm{t}}{\mathrm{R}^2-(\varphi-1)^2}-\frac{R_{791}\sin(R+1)\mathrm{t}}{2R+1}$$
$$\left.\left.+\frac{R_{801}\sin(R-1)\mathrm{t}}{2R-1}\right]\right|_{\mathrm{t}=\tilde{\mathrm{t}}_0}$$

$$R_{931}=\left[-\frac{R_{692}\sin(R+\varphi)\mathrm{t}}{\varphi(2R+\varphi)}+R_{805}\sin Rt+\frac{R_{702}\sin(R-\varphi)\mathrm{t}}{\varphi(2R-\varphi)}\right.$$
$$+\frac{R_{732}\sin2\varphi\,\mathrm{t}}{\mathrm{R}^2-4\varphi^2}+\frac{R_{742}\sin2\mathrm{t}}{\mathrm{R}^2-4}+\frac{R_{752}\sin(\varphi+1)\mathrm{t}}{\mathrm{R}^2-(\varphi+1)^2}+\frac{R_{762}\sin(\varphi-1)\mathrm{t}}{\mathrm{R}^2-(\varphi-1)^2}-\frac{R_{792}\sin(R+1)\mathrm{t}}{2R+1}$$
$$-\frac{R_{77}\sin(R+\varphi)\mathrm{t}}{\varphi(2R+\varphi)}+\frac{R_{78}\sin(R-\varphi)\mathrm{t}}{\varphi(2R-\varphi)}+\frac{R_{821}\sin(Q-\varphi)\mathrm{t}}{\mathrm{R}^2-(Q-\varphi)^2}+\frac{R_{83}\sin(Q+1)\mathrm{t}}{\mathrm{R}^2-(Q+1)^2} \quad (4.14d)$$
$$\left.\left.+\frac{R_{84}\sin(Q-1)\mathrm{t}}{\mathrm{R}^2-(Q-1)^2}+\frac{R_{802}\sin(R-1)\mathrm{t}}{2R-1}+\frac{R_{811}\sin(Q+\varphi)\mathrm{t}}{\mathrm{R}^2-(Q+\varphi)^2}\right]\right|_{\mathrm{t}=\tilde{\mathrm{t}}_0}$$

The maximum displacement ξ_{2c} easily follows from (4.5b) to give

$$\xi_{2c}=S\bar{\xi}_2\in\left[T_9+\delta\left(\mathrm{T}_{10}+\tilde{t}_0q_3\cos R\tilde{t}_0\right)\right]+\in^2\left[S^2\bar{\xi}_2R_{900}+S\bar{\xi}_1\bar{\xi}_2R_{901}\right.$$
$$\left.+\delta\left\{\mathrm{S}^2\bar{\xi}_2T_{11}+S\bar{\xi}_1\bar{\xi}_2T_{12}\right\}\right]+O\left(\in\delta^2\right)+O\left(\in^2\delta^2\right) \quad (4.15a)$$

where $$T_{11} = \frac{R_{861}}{T_3}\left(\tilde{t}_0 Rq_3 \sin R\tilde{t}_0 - q_3 \cos R\tilde{t}_0 - T_{10}\right) + \tilde{t}_0 R_{911} + R_{92} \quad (4.15b)$$

$$T_{12} = \frac{R_{862}}{T_3}\left(\tilde{t}_0 Rq_3 \sin R\tilde{t}_0 - q_3 \cos R\tilde{t}_0 - T_{10}\right) + R_{93} + \tilde{t}_0 R_{912} \quad (4.15c)$$

Thus, the sum of the maximum displacement, ξ_a, follows from (2.7b) thus:

$$\xi_a = \xi_{1a} + \xi_{2c} = \in C_1 + \in^2 C_2 + \cdots \quad (4.16a)$$

where $$C_1 = hT_4 + ST_9 + \delta\left\{ \mathrm{h}\left(\mathrm{T}_5 + t_0 q_0 \cos Qt_0\right) + S\left(T_{10} + \tilde{t}_0 q_3 \cos R\tilde{t}_0\right) \right\} \quad (4.16b)$$

$$C_2 = \left(\bar{\xi}_1 hR_{208} + S\bar{\xi}_2 R_{900}\right) + \left\{ k_1\left(Q\bar{\xi}_1 h\right)^2 R_{209} + k_2\left(QS\bar{\xi}_2\right)^2 R_{210} + S\bar{\xi}_1\bar{\xi}_2 R_{901}\right\}$$
$$+ \delta\left\{ \bar{\xi}_1 hT_6 + k_1\left(Q\bar{\xi}_1 h\right)^2 T_7 + k_2\left(QS\bar{\xi}_2\right)^2 T_8 + S^2\bar{\xi}_2 T_{11} + S\bar{\xi}_1\bar{\xi}_2 T_{12} \right\} \quad (4.16c)$$

5.0 Dynamic buckling load λ_D.

The dynamic buckling load λ_D easily follows from the maximization (2.7a). This is however preceded [3,10] by a reversal of the series (4.16a) in the form

$$\in = d_1\xi_a + d_2\xi_a^2 + \cdots \quad (5.1a)$$

By substituting into (5.1a) for ξ_a from (4.16a) and equating the coefficients of $\in$ and $\in^2$, we have

$$d_1 = \frac{1}{C_1}, \ \mathrm{d}_2 = -\frac{C_2}{C_1^3} \quad (5.1b)$$

The maximization (2.7a) now follows from (5.1a,b) to yields $\xi_a(\lambda_D) = -\dfrac{d_1}{2d_2} = \dfrac{C_1^2}{2C_2}$ (5.1c)

On evaluating (5.1a) at $\lambda = \lambda_D$, using (62c), we have $\in_D = \dfrac{C_1}{4C_2}$ (5.1d)

where $\in_D$ is the value of $\in$ at $\lambda = \lambda_D$.On substituting the relevant terms into (5,1d), we get

$$\lambda_D\left(\frac{\omega_1}{\omega_0}\right)^2 = \frac{Q_1}{Q_2} \quad (5.2a)$$

where

$$Q_1 = \frac{1}{4}\left\{\left[\bar{\xi}_1 hT_4 + S\bar{\xi}_2 T_9 + \delta\left\{ \mathrm{h}\bar{\xi}_1\left(T_5 + t_0 q_0 \cos Qt_0\right) + \bar{\xi}_2 S\left(T_{10} + \tilde{t}_0 q_3 \cos R\tilde{t}_0\right) \right\}\right\}\right\} \quad (5.2b)$$

$$Q_2 = \left[\left(\bar{\xi}_1 hR_{208} + S\bar{\xi}_2 R_{900}\right) + \left\{ k_1\left(Q\bar{\xi}_1 h\right)^2 R_{209} + k_2\left(Q\bar{\xi}_2 S\right)^2 R_{209} + S\bar{\xi}_1\bar{\xi}_2 R_{901}\right\}\right.$$
$$\left. + \delta\left\{ \bar{\xi}_1 hT_6 + k_1\left(Q\bar{\xi}_1 h\right)^2 T_7 + k_2\left(Q\bar{\xi}_2 S\right)^2 T_8 + S^2\bar{\xi}_2 T_{11} + S\bar{\xi}_1\bar{\xi}_2 T_{12} \right\}\right] \quad (5.2c)$$

6.0 Analysis of result and conclusion.

Since the terms Q_1 and Q_2 are independent of the load parameter λ_D,the result (5.2a) is an algebraic equation that determines λ_D directly and is valid for small values of δ and $\in$. All along, we have tacitly implied that all results, if not explicitly expressed, are uniformly and asymptotically valid. As a consequence, none of the terms in any of the denominators is deemed to vanish. The terms in (5.2a-c) are indicative of the contributions to buckling of the various terms in the governing equations (2.3) - (2.5). For example, the terms multiplying $k_1\left(Q\bar{\xi}_1 h\right)^2, k_2\left(Q\bar{\xi}_2 S\right)^2$ *and* $S\bar{\xi}_1\bar{\xi}_2$ in (5.2a-c) clearly show the contributions to buckling, of the terms $k_1\xi_1^2,\ k_2\xi_2^2$ *and the term* $\xi_1\xi_2$ respectively as they appear in the governing equations. Similarly the terms multiplying $\bar{\xi}_1 h$ and $\bar{\xi}_2 S$ in (5.2a-c) are also the contributions of the terms $\xi_1\xi_0$ *and* $\xi_2\xi_0$ as they appear in equations (2.4) and (2.5). Thus if we assume the presence of only the axisymmetric imperfection, then $\bar{\xi}_2 = 0, \bar{\xi}_1 \neq 0$ and from (5.2) we get

$$\lambda_D\left(\frac{\omega_1}{\omega_2}\right)^2 = \frac{\frac{1}{4}\left\{T_4 + \delta\left(T_5 + t_0 q_0 \cos Qt_0\right)\right\}}{\left[R_{208} + k_1 Q^2\bar{\xi}_1 h R_{209} + \delta\left\{T_6 + k_1 Q^2\bar{\xi}_1 h T_7\right\}\right]} \tag{6.1a}$$

From (6.1a), we note following:

(a) The effect of the coupling term $\xi_1\xi_2$ is absent.

(b) The quadratic term $k_2\xi_2^2$, has no effect .

(c) The effect of the coupling term $\xi_2\xi_0$ is also absent while that of $\xi_1\xi_0$ is the only coupling effect felt.

(d) The only nonlinear term that dominates buckling phenomenon at this stage is $k_1\xi_1^2$.

We can readily obtain the static buckling load λ_s from the governing equations by setting $\bar{\xi}_2 = 0$ and getting

$$(1-\lambda_S)^2 = 4k_1\lambda_S\bar{\xi}_1 \tag{6.1b}$$

We can eliminate the parameter $k_1\bar{\xi}_1$ from (64a), using (64b), to get

$$\left(\frac{\lambda_D}{\lambda_S}\right)\left(\frac{\omega_1}{\omega_0}\right)^2 = \left[\frac{T_4 + \delta\left\{T_5 + q_0\cos Qt_0\right\}}{4\lambda_S R_{208} + h(1-\lambda_S)^2 Q^2 R_{209} + \delta\left\{4\lambda_S T_6 + (1-\lambda_s)^2 \mathrm{h}\, Q^2 T_7\right\}}\right] \tag{6.1c}$$

On the other hand, if we assume the presence of only the non-axisymmetric imperfection $\bar{\xi}_2$ then, we set $\bar{\xi}_1 = 0$ (Danielson's assumption) and get the following result

$$\lambda_D\left(\frac{\omega_1}{\omega_2}\right)^2 = \frac{\frac{1}{4}\left(T_{10} + \tilde{t}_0 q_3 \cos R\tilde{t}_0\right)}{\left[R_{900} + Q^2 k_2\bar{\xi}_2 S R_{210} + \delta\ S\left\{k_2 Q^2\bar{\xi}_2 T_8 + T_{11}\right\}\right]} \tag{6.2a}$$

From (6.2a), we note the following;

(a) The effect of the coupling term $\xi_1\xi_2$ is once again absent.

(b) The quadratic term $k_1\xi_1^2$ has no effect.

(c) The nonlinear term that dominates buckling is the quadratic term $k_2\xi_2^2$.

(d) The effect of the coupling term $\xi_1\xi_0$ is also not felt while that of $\xi_2\xi_0$ is felt.

The static load λ_S in this case of $\bar{\xi}_1 = 0$ is obtained from $(1-\lambda_S)^2 = \frac{3}{2}\sqrt{3k_2}\lambda_S\bar{\xi}_2$ (6.2b)

Using (65b), we readily eliminate the imperfection parameter $\bar{\xi}_2$ from (65a) to get

$$\left(\frac{\lambda_D}{\lambda_S}\right)\left(\frac{\omega_1}{\omega_2}\right)^2 = \frac{\frac{1}{4}\left(T_{10} + \tilde{t}_0 q_3 \cos R\tilde{t}_0\right)}{\left[R_{900}\lambda_S + \frac{2Q^2 R_{210} S(1-\lambda_S)^2\sqrt{k_2}}{3\sqrt{3}} + S\delta\left\{\frac{2Q^2 T_8(1-\lambda_S)^2\sqrt{k_2}}{3\sqrt{3}} + \lambda_S T_{11}\right\}\right]} \tag{6.2c}$$

We thus see from (6.1c) and (6.2c) that in both cases, we are able to eliminate the respective imperfection parameters $\bar{\xi}_1$ and $\bar{\xi}_2$ (as the case may be) so that in each case, we can actually determine the dynamic buckling loaf λ_D from a knowledge of the associated static equivalent λ_S. We now summarize the result as follows:

(a) Danielson's assumption of neglecting both $k_1\xi_1^2$ and $\bar{\xi}_1$ is seen to be superfluous because, by neglecting only $\bar{\xi}_1$, we automatically neglect the effects of $k_1\xi_1^2$; the converse is not true.

(b) Similarly, by neglecting $\bar{\xi}_2$, we automatically neglect the effect of $k_2\xi_2^2$; the converse is not true.

(c) Once an imperfection is neglected, we automatically neglect the effect of the nonlinearity that is in the shape of the neglected imperfection.

(d) The only condition under which the effect of the coupling term $\xi_1\xi_2$ is felt is if none of the imperfections in the shapes of the modes coupling is neglected. The is infact also true of the coupling terms $\xi_1\xi_0$ and $\xi_2\xi_0$ because by neglecting $\bar{\xi}_1$ first and next $\bar{\xi}_2$, we see that effects of $\xi_1\xi_0$ and $\xi_2\xi_0$ are respectively nullified. Thus, with the exception of the effects of the coupling term $\xi_2\xi_0$, Danielson's result could not have accounted for any other coupling effects involved in the formulation since he neglected $\bar{\xi}_1$.

(e) Lastly, Danielson neglected the quadratic term $k_1\xi_1^2$ on the assumption that the coupling term $\xi_1\xi_2$ has a dominant effect ,for condition at initial buckling ,compared to the effect of $k_1\xi_1^2$. We have however seen that by having neglected $\bar{\xi}_1$,the effect of the coupling term $\xi_1\xi_2$ is automatically nullified. Thus ,it is only the quadratic term $k_2\xi_2^2$ and the coupling term $\xi_2\xi_0$ that dominate buckling at this instance and not the coupling term $\xi_1\xi_2$ as envisioned by Danielson. All these deductions (and even more) could not have been arrived at or accounted for if we had adhered strictly to Danielson's assumptions. Thus, we lastly observe, among other things that, while some of Danielson's assumption naturally give a leeway to solving a rather seemingly nonlinear coupled problem, some of his other assumptions , seem, in our judgement, to have unwittingly over- simplified the solution. By setting $\delta = 0$ in (5.2a-c), (6.1a,c) and (6.2c) (that is , no damping), we realize the exact results obtained in [10] .

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