Journal of the Nigerian Association of Mathematical Physics
Volume **11 (November 2007), 323 - 332**

On the static buckling of an externally pressurized finite circular cylindrical shell

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Abstract

The static buckling behaviour of an imperfect finite cylindrical shell, stressed by either a lateral or hydrostatic pressure, is here investigated by assuming that the imperfection can be regarded as the first term in the Fourier sine series expansion .The buckling modes are assumed to be in the shape of the imperfection which is in turn given in the shape of the classical buckling mode .Regular perturbation technique in asymptotic expansions of the relevant parameters is used and a simple expression for determining the static buckling load of the structure is determined. It is observed that , this procedure, perhaps more than other ones ,can be used to analyze relatively more complicated problems particularly where more demands and restrictions are placed on the imperfection parameter. The result is strictly asymptotic.

1.0 Introduction

Several studies have been made, over the years, on the buckling of imperfect cylindrical shells subjected to various kinds of loading conditions. Many of these investigations have severally addressed infinitely long cylindrical shells under various geometrical and structural limitations of the imperfections. Such earlier studies include those by Koiter [1], Amazigo and Fraser [2] and Budiansky and Amazigo [3], among others. Virtually all the mentioned investigations addressed the buckling of infinitely long imperfect cylindrical shells subjected to static loads. On the other hand, Lockhart and Amazigo [4] studied the dynamic buckling of externally pressurized imperfect finite cylindrical shell subjected to a step load. As a special report in their investigation, they obtained the static buckling load of the imperfect circular cylindrical shell investigated. The investigation undertaken here takes a detour from the special report in [4] and reformulates a general procedure that is suitable to a larger, and perhaps, more intricate details of the imperfection of finite circular cylindrical shells.

2.0 Karman-Donnell Equations for finite cylindrical shells

The relevant dimensional Karman-Donnell equations [1-4], in respect of the normal displacement $W(X,Y)$ and Airy stress function $F(X,Y)$ of a finite imperfect circular cylindrical shell of length L, radius R, thickness h, bending stiffness $D=\dfrac{Eh^3}{12\left(1-\upsilon^2\right)}$, where E is the Young's modulus and υ is the Poisson's ratio with P as the external pressure, are respectively given in respect of the compatibility equation and the equilibrium equation as

$$\frac{1}{Eh}\nabla^4 F-\frac{1}{R}W,_{XX}=-S\left(W,\frac{1}{2}W+\overline{W}\right) \qquad (2.1)$$

$$D\nabla^4 W + \frac{1}{R}F,_{XX} = -S\left(W + \overline{W}, F\right) - P \qquad (2.2)$$

where X and Y are the axial and circumferential coordinates respectively and $\overline{W}(X,Y)$ is the stress-free initial normal displacement , otherwise called, the imperfection .Similarly, S is the bilinear operator defined by

$$S(P,Q) = P,_{XX}\, Q,_{YY} + P,_{YY}\, Q,_{XX} - 2P,_{XY}\, Q,_{XY} \qquad (2.3a)$$

while ∇^4 is the usual biharmonic operator, namely

$$\nabla^4 \equiv \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)^2 \qquad (2.3b)$$

and a subscript following a comma indicates partial differentiation. We introduce the following non-dimensional quantities

$$x = \frac{\pi X}{L},\ \mathrm{y} = \frac{\mathrm{Y}}{\mathrm{R}},\ \in \overline{\mathrm{w}} = \frac{\overline{\mathrm{W}}}{\mathrm{h}},\ \mathrm{w} = \frac{W}{h},\ \overline{\lambda} = \frac{\mathrm{L}^2 RP}{\pi^2 D},\ \mathrm{A} = \frac{\mathrm{L}^2\sqrt{12\left(1-\upsilon^2\right)}}{\pi^2 Rh} \qquad (2.4a)$$

$$\xi = \frac{L^2}{(\pi R)^2},\ \mathrm{K}(\xi) = -\left(\frac{A}{1+\xi}\right)^2,\ \mathrm{H} = \frac{\mathrm{h}}{\mathrm{R}} \qquad (2.4b)$$

where $\in$ satisfies the inequality $0 < \in < 1$, and is a small parameter which is a measure of the amplitude of the imperfection while $\overline{\lambda}$ is a non-dimensional load amplitude. As in [2-4], we assume the following:

$$F = -\frac{1}{2}PR\left(X^2 + \frac{1}{2}\alpha Y^2\right) + \frac{Eh^2 L^2 f}{\pi^2 R(1+\xi)^2} \qquad (2.5a)$$

$$W = \frac{PR^2\left(1 - \frac{\alpha\,\upsilon}{2}\right)}{Eh} + h\,\mathrm{w} \qquad (2.5b)$$

where the first terms on the right sides of (2.5a,b) are the pre-buckling approximations of the Airy stress function and the normal displacement respectively. The parameter α takes the value α =1 if the pressure contributes to axial stress through end plates, where as α = 0, if pressure acts laterally. On substituting (2.4a,b) and (2.5a,b) into (2.1) and (2.2) and simplifying, we obtain the following:

$$\overline{\nabla}^4 f - (1+\xi)^2 w,_{\mathrm{xx}} = -(1+\xi)^2 H\overline{S}\left(w, \frac{1}{2}w + \in \overline{w}\right) \qquad (2.6)$$

$$\overline{\nabla}^4 w - K(\xi) f,_{\mathrm{xx}} + \lambda\lambda_C\left[\frac{\alpha}{2}(w + \in \overline{w}),_{\mathrm{xx}} + \xi(w + \in \overline{w}),_{\mathrm{yy}}\right] = -HK(\xi)\overline{S}(w + \in \overline{w}, \mathrm{f}) \qquad (2.7)$$

where

$$\overline{\nabla}^4 \equiv \left(\frac{\partial^2}{\partial x^2} + \xi\frac{\partial^2}{\partial y^2}\right)^2;\ \overline{\lambda} = \lambda\lambda_C,\ \lambda = \frac{\overline{\lambda}}{\lambda_C}; \qquad (2.8a)$$

$$\overline{S}(P,Q) = P,_{\mathrm{xx}}\, Q,_{\mathrm{yy}} + P,_{\mathrm{yy}}\, Q,_{\mathrm{xx}} - 2P,_{\mathrm{xy}}\, Q,_{\mathrm{xy}}$$

and λ_C is the classical buckling load . We assume simply-connected end conditions on the axial coordinate characterized by

$$w = w,_{xx} = f = f,_{xx} = 0 \ \text{ at } \ \mathrm{x} = 0, \pi \qquad (2.8b)$$

3.0 Classical Theory.

The classical buckling load λ_C is defined as the value of the load parameter λ for there to exist a nontrivial solution to the corresponding linear problem of the associated perfect cylindrical shell. Such a result was obtained in [4] as

$$\lambda_C = \left[\frac{\left(1 + n^2 \xi\right)^2 - \dfrac{(1+\xi)^2 K(\xi)}{\left(1 + n^2\xi\right)^2}}{\left(\dfrac{\alpha}{2} + n^2 \xi\right)} \right] \tag{3.1}$$

4.0 Perturbation solution.

The technique adopted here is similar to those by Elishakoff [5] and in the references there cited. Our intention is to asymptotically determine the static buckling load, namely λ_S, (using regular perturbation method) which is defined as the maximum load that the structure can support prior to buckling. Based on the result in [4], we assume the imperfection $\overline{w}(x, y)$ in the form

$$\overline{w}(x, y) = b_1 \sin ny \sin x, \; b_1 \neq 0 \qquad 4.1)$$

We also assume the normal displacement $w(x,y)$ and the Airy stress function $f(x,y)$ as in the following asymptotic series

$$\begin{pmatrix} f(x,y) \\ w(x,y) \end{pmatrix} = \sum_{i=1}^{\infty} \begin{pmatrix} f^{(i)}(x,y) \\ w^{(i)}(x,y) \end{pmatrix} \in^{i} \tag{4.2}$$

and now substitute (4.2) and (2.5a,b) into (2.6) and (2.7) and equate the coefficients of $\in^{i}$, $i = 1,2,3,\cdots$ to obtain the following equations:

$$L^{(1)}\left(f^{(1)}, w^{(1)}\right) \equiv \overline{\nabla}^4 f^{(1)} - (1+\xi)^2 w^{(1)}{}_{,xx} = 0 \tag{4.3}$$

$$L^{(2)}\left(f^{(1)}, w^{(1)}\right) \equiv \overline{\nabla}^4 w^{(1)} - K(\xi) f^{(1)}_{,xx} + \lambda\lambda_C\left[\frac{\alpha}{2}\left(w^{(1)} + \overline{w}\right)_{,xx} + \xi\left(w^{(1)} + \overline{w}\right)_{,yy}\right] = 0 \tag{4.4}$$

$$L^{(1)}\left(f^{(2)}, w^{(2)}\right) = -(1+\xi)^2 H\left[\frac{1}{2}\overline{S}\left(w^{(1)}, w^{(1)}\right) + \overline{S}\left(w^{(1)}, \overline{w}\right)\right] \tag{4.5}$$

$$L^{(2)}\left(f^{(2)}, w^{(2)}\right) = -HK\left[\overline{S}\left(w^{(1)}, f^{(1)}\right) + \overline{S}\left(\overline{w}, f^{(1)}\right)\right] \tag{4.6}$$

$$L^{(1)}\left(f^{(3)}, w^{(3)}\right) = -(1+\xi)^2 H\left[\overline{S}\left(w^{(1)}, w^{(2)}\right) + \overline{S}\left(w^{(2)}, \overline{w}\right)\right] \tag{4.7}$$

$$L^{(3)}\left(f^{(2)}, w^{(2)}\right) = -HK\left[\overline{S}\left(w^{(1)}, f^{(2)}\right) + \overline{S}\left(w^{(2)}, f^{(1)}\right) + \overline{S}\left(\overline{w}, f^{(2)}\right)\right] \tag{4.8}$$

$$w^{(i)} = w^{(i)}_{,xx} = f^{(i)} = f^{(i)}_{,xx} = 0 \quad \text{at } x = 0, \pi. \tag{4.9}$$

For the solution of equations (4.3) - (4.9), we let

$$\begin{pmatrix} f^{(i)}(x,y) \\ w^{(i)}(x,y) \end{pmatrix} = \sum_{p,q=1}^{\infty} \left\{ \begin{pmatrix} f_1^{(i)} \\ w_1^{(i)} \end{pmatrix} \cos py + \begin{pmatrix} f_2^{(i)} \\ w_2^{(i)} \end{pmatrix} \sin py \right\} \sin qx \tag{4.10}$$

4.1 Solution of first order perturbation equations.

We substitute (4.10) into (4.3), for i =1, multiply successively in turn by cosny sinmx and sinny sinmx and get respectively for $p = n$, $q = m$

$$f_1^{(1)} = -\left(\frac{m(1+\xi)}{m^2+n^2\xi}\right)^2 w_1^{(1)} \quad \text{and} \quad f_2^{(1)} = -\left(\frac{m(1+\xi)}{m^2+n^2\xi}\right)^2 w_2^{(1)} \tag{4.11}$$

We now substitute (4.10) into (4.4) and multiply the resultant equation in turn, first by cosnysinmx and after by sinnysinmx , using (4.1) and (4.11) and obtain respectively, in the first and second instances,

$$w_1^{(1)} = 0 \text{ and } w_2^{(1)} = \beta_2 b_1 , \beta_2 = \frac{\lambda_C \lambda\left(\frac{\alpha}{2}+n^2\xi\right)}{\left(1+n^2\xi\right)^2 - \lambda_C\lambda\left(\frac{\alpha}{2}+n^2\xi\right)+\left(\frac{A}{1+n^2\xi}\right)^2} \tag{4.12a}$$

Thus we have

$$w^{(1)} = w_2^{(1)} \sin ny \sin x \tag{4.12b}$$

4.2 Solution of second order perturbation equations.

We next substitute (4.12a,b) into (4.5) and (4.6), for i =2 and simplify to get

$$L^{(1)}\left(f^{(2)}, w^{(2)}\right) = -(1+\xi)^2 n^2 H\left[\frac{1}{2} w_2^{(1)\,2} + b_1 w_2^{(1)}\right](\cos 2ny + \cos 2x) \tag{4.13}$$

$$L^{(2)}\left(f^{(2)}, w^{(2)}\right) = -\mathrm{HKn}^2\left[\mathrm{b}_1 f_2^{(1)} + w_2^{(1)} f_2^{(1)}\right](\cos 2ny + \cos 2x) \tag{4.14}$$

We assume (4.10), for i =2, and substitute same into (4.13) , multiply the resultant equation by cosnrysin mx and see that , for r = 2 and m odd, we get

$$f_1^{(2)} = \frac{\frac{4Q_1}{m\pi} - m^2(1+\xi)^2 w_1^{(2)}}{\left(m^2+4n^2\xi\right)^2} \tag{4.15a}$$

where

$$Q_1 = -(1+\xi)^2 \mathrm{n}^2 H\left[\frac{1}{2} w_2^{(1)\,2} + b_1 w_2^{(1)}\right] \tag{4.15b}$$

Similarly, if we multiply (22) through by sinnrysinmx and simplify , we get

$$f_2^{(2)} = 0 \tag{4.15c}$$

Thus we have

$$f^{(2)} = \sum_{m=1,3,5,\cdots}^{\infty} f_1^{(2)} \cos 2ny \sin mx \tag{4.15d}$$

By denoting the value of $f_1^{(2)}$ at m=1 by $\tilde{f}_1^{(2)}$, we have

$$\tilde{f}_1^{(2)} = \frac{4Q_1}{\pi\left(1+4n^2\xi\right)^2} - \frac{(1+\xi)^2 \tilde{w}_1^2}{\left(1+4n^2\xi\right)^2} \tag{4.15e}$$

where $\tilde{w}_1^2$ is the value of $w_1^{(2)}$ at m = 1. We next substitute (4.10) into (4.14), multiply through in turn by cosnry sinmx and next by sinnry cosmx and in the first case , obtain ,using (4.15d) and for r = 2 , m odd

$$w_1^{(2)} = \frac{\dfrac{4Q_2}{m\pi} + \dfrac{4mA^2 Q_1}{\pi(1+\xi)^2\left(m^2+4n^2\xi\right)^2}}{\left(m^2+4n^2\xi\right)^2 - \lambda\lambda_C\left(\dfrac{\alpha m^2}{2}+4n^2\xi\right)+\left(\dfrac{m^2 A}{m^2+4n^2\xi}\right)^2} \tag{4.16a}$$

$$Q_2 = -HKn^2\left[b_1 f_2^{(1)} + w_2^{(1)} f_2^{(1)}\right] \tag{4.16b}$$

On further simplifying (25a), using (20) and (24a-c),we obtain

$$w_1^{(2)} = b_1\alpha_1 w_2^{(1)} + \alpha_2 w_2^{(1)\,2} \tag{4.17a}$$

where

$$\alpha_1 = \frac{\dfrac{4HKm^2n^2(1+\xi)^2}{m\pi\left(m^2+n^2\xi\right)^2} - \dfrac{4Hmn^2A^2}{\pi\left(m^2+4n^2\xi\right)^2}}{\left(m^2+4n^2\xi\right)^2 - \lambda\lambda_C\left(\dfrac{\alpha m^2}{2}+4n^2\xi\right)+\left(\dfrac{m^2 A}{m^2+4n^2\xi}\right)^2} \tag{4.17b}$$

$$\alpha_2 = \frac{\dfrac{4HKm^2n^2(1+\xi)^2}{m\pi\left(m^2+n^2\xi\right)^2} - \dfrac{2Hmn^2A^2}{\pi\left(m^2+4n^2\xi\right)^2}}{\left(m^2+4n^2\xi\right)^2 - \lambda\lambda_C\left(\dfrac{\alpha m^2}{2}+4n^2\xi\right)+\left(\dfrac{m^2 A}{m^2+4n^2\xi}\right)^2} \tag{4.17c}$$

Henceforth, any function, say $f_k^{(i)}$,evaluated at m=1, will be denoted as $\tilde{f}_k^{(i)}$.Thus at m=1, we have, from (4.16a) - (4.17c), $w_1^{(2)} = \tilde{w}_1^{(2)}$,where $\tilde{w}_1^{(2)} = b_1\tilde{\alpha}_1 w_2^{(1)} + \tilde{\alpha}_2 w_2^{(1)\,2}$ and where

$$\tilde{\alpha}_1 = \frac{\dfrac{4Hkn^2(1+\xi)^2}{\left(1+n^2\xi\right)^2} - \dfrac{4Hn^2A^2}{\left(1+4n^2\xi\right)^2}}{\pi\left[\left(1+4n^2\xi\right)^2 - \lambda\lambda_C\left(\dfrac{\alpha}{2}+4n^2\xi\right)+\left(\dfrac{A}{1+4n^2\xi}\right)^2\right]}, \tag{4.17d}$$

$$\tilde{\alpha}_2 = \frac{\dfrac{4Hkn^2(1+\xi)^2}{\left(1+n^2\xi\right)^2} - \dfrac{2Hn^2A^2}{\left(1+4n^2\xi\right)^2}}{\pi\left[\left(1+4n^2\xi\right)^2 - \lambda\lambda_C\left(\dfrac{\alpha}{2}+4n^2\xi\right)+\left(\dfrac{A}{1+4n^2\xi}\right)^2\right]} \tag{4.17e}$$

Similarly, the value of $f_1^{(2)}$ at m=1, namely $\tilde{f}_1^{(2)}$,is easily evaluated from (4.15e), using (4.17a-c), as $\tilde{f}_1^{(2)} = b_1\tilde{\alpha}_3 w_2^{(1)} + \tilde{\alpha}_4 w_2^{(1)\,2}$ where

$$\tilde{\alpha}_3 = -\left(\frac{1+\xi}{1+4n^2\xi}\right)^2\left(\tilde{\alpha}_1 + \frac{4n^2H}{\pi}\right),\ \tilde{\alpha}_4 = -\left(\frac{1+\xi}{1+4n^2\xi}\right)^2\left(\tilde{\alpha}_2 + \frac{2n^2H}{\pi}\right) \qquad (4.18)$$

We can now write $$w^{(2)} = \sum_{m=1,3,5,\cdots}^{\infty} w_1^{(2)} \cos 2ny \sin mx \qquad (4.19)$$

4.3 Solution of third order perturbation equations.

We now substitute for terms on the right hand sides of (4.7) and (4.8) and obtain

$$L^{(1)}\left(f^{(3)}, w^{(3)}\right) = -(1+\xi)^2 H \sum_{m=1,3,5,\cdots}^{\infty}\left[\left(w_2^{(1)}w_1^{(2)} + b_1 w_1^{(2)}\right)\right. \qquad (4.20)$$

$$\left.\times\left\{\left(4n^2 + m^2n^2\right)\sin x \sin ny \cos 2ny \sin mx\right\} - 4mn^2 \cos ny \sin 2ny \cos x \cos mx\right]$$

$$L^{(1)}\left(f^{(3)}, w^{(3)}\right) = -(1+\xi)^2 H \sum_{m=1,3,5,\cdots}^{\infty}\left[\left(w_2^{(1)}f_1^{(2)} + b_1 f_1^{(2)} + w_1^{(2)} f_2^{(1)}\right)\right. \qquad (4.21)$$

$$\left.\times\left\{\left(4n^2 + m^2n^2\right)\sin x \sin ny \cos 2ny \sin mx\right\} - 4mn^2 \cos ny \sin 2ny \cos x \cos mx\right]$$

Next, we substitute (4.10), for $i = 3$ into (4.20),multiply the resultant equation through by $\sin nry \sin \beta mx$, where r and β are to be determined, and so obtain, for r=1, β =1

$$f_{21}^{(3)} = 2H\left\{\frac{1+\xi}{\pi\left(m^2+n^2\xi\right)}\right\}^2\left[\sum_{m=1,3,5\cdots}^{\infty}\left\{\omega_m\left(4n^2+m^2n^2\right)+4mn^2\theta_m\right\}\left(\mathrm{w}_2^{(1)}w_1^{(2)}+b_1w_1^{(2)}\right)\right]$$

$$-\frac{(1+\xi)^2 m^2 w_{21}^{(3)}}{\left(m^2+n^2\xi\right)^2} \qquad (4.22a)$$

where $$\omega_m = \frac{\pi}{4}\left[2-\frac{1}{1-2m}-\frac{1}{1+2m}\right], \theta_m = \frac{\pi}{4}\left[\frac{1}{2m+1}+\frac{1}{2m-1}\right] \qquad (4.22b)$$

where $f_{21}^{(3)}$ is the value of $f_2^{(3)}$ when r=1, β =1 and $w_{21}^{(3)}$ is the corresponding value of $w_2^{(3)}$ Similarly, when r=3 with β =1 and r=1 with β =3, we have respectively

$$f_{23}^{(3)} = -\frac{\frac{2H(1+\xi)^2}{\pi^2}\left[\sum_{m=1,3,5,\cdots}^{\infty}\left\{\omega_m\left(4\mathrm{n}^2+m^2n^2\right)-4mn^2\theta_m\right\}\left(w_2^{(1)}w_1^{(2)}+b_1w_1^{(2)}\right)\right]-(1+\xi)^2m^2w_{23}^{(3)}}{\left(m^2+9n^2\xi\right)^2} \qquad (4.23)$$

$$f_{24}^{(3)} = \frac{\frac{2H(1+\xi)^2}{\pi^2}\left[\sum_{m=1,3,5,\cdots}^{\infty}\left\{\Omega_m\left(4\mathrm{n}^2+m^2n^2\right)+4\varphi_m mn^2\right\}\left(w_2^{(1)}w_1^{(2)}+b_1w_1^{(2)}\right)\right]-9m^2(1+\xi)^2w_{24}^{(3)}}{\left(9m^2+n^2\xi\right)^2} \qquad (4.24a)$$

where

$$\Omega_m = \frac{\pi}{4}\left[\frac{1}{2m+1}-\frac{1}{1-2m}+\frac{1}{1-4m}-\frac{1}{1+4m}\right], \varphi_{\mathrm{m}} = \left[\frac{1}{5m+1}+\frac{1}{5m-1}+\frac{1}{2m+1}+\frac{1}{2m-1}\right], \qquad (4.24b)$$

where m is odd, and $f_{23}^{(3)}$ and $f_{24}^{(3)}$ are the values of $f_2^{(3)}$ when r=3 with β=1 and r=1 with β=3 respectively . Similarly $w_{23}^{(3)}$ and $\mathrm{w}_{24}^{(3)}$ are the values of $w_2^{(3)}$ for the two respective combinations of r and β. Of the values in (4.22a) – (4.24b), it is only $f_{21}^{(3)}$, evaluated at m=1,that will yield a buckling mode in the shape of the imperfection. By denoting by $\tilde{f}_2^{(3)}$ the value of $f_{21}^{(3)}$ at m=1, we write

$$\tilde{f}_2^{(3)} = \frac{2\omega_1(1+\xi)^2 H\left(5\tilde{\omega}_1 + 4\tilde{\theta}_1\right)\left(w_2^{(1)}\tilde{w}_1^{(2)} + b_1\tilde{w}_1^{(2)}\right)}{\left(1+n^2\xi\right)^2\pi^2} - \frac{(1+\xi)^2\,\tilde{w}_2^{(3)}}{\left(1+n^2\xi\right)^2} \qquad (4.25)$$

where $\tilde{\omega}_1$ *and* $\tilde{\theta}_1$ are the values of ω_m and θ_{m} respectively at m=1 and $\tilde{w}_2^{(3)}$ is also the value of $w_{21}^{(3)}$ at m=1. By multiplying (4.20) by cosnry sin βmx, we get

$$f_1^{(3)} = 0 \qquad (4.26)$$

On substituting (4.10) into (4.21, for $i=3$,and multiplying through by *cosnry* sin βmx ,we get ,for *r* = 1 and $\beta = 1$, (using(4.22a,b))

$$w_{21}^{(3)} = \frac{\left\{\dfrac{2\left(m^2A\right)^2 H}{\pi^2\left(m^2+n^2\xi\right)^2}\sum_{m=1,3,5,\cdots}^{\infty}\left[\left\{\omega_m\left(4\mathrm{n}^2+m^2n^2\right)+4mn^2\theta_m\right\}\left\{w_2^{(1)}w_1^{(2)}+b_1w_1^{(2)}\right\}\right] - \dfrac{HA^2}{(1+\xi)^2}\sum_{m=1,3,5,\cdots}^{\infty}\left[w_2^{(1)}f_1^{(2)}+w_1^{(2)}f_2^{(1)}+b_1f_1^{(2)}\right]\left\{\omega_m\left(4\mathrm{n}^2+m^2n^2\right)+4mn^2\theta_m\right\}\right\}}{\left(m^2+n^2\xi\right)^2 - \lambda\lambda_C\left(\dfrac{\alpha m}{2}+n^2\xi\right)+\left(\dfrac{m^2A}{m^2+n^2\xi}\right)^2}$$

(4.27)

The value of $w_{21}^{(3)}$ when m=1,namely $\tilde{w}_2^{(3)}$, now becomes

$$\tilde{w}_2^{(3)} = \frac{A^2n^2H\left(5\tilde{\omega}_1+4\tilde{\theta}_1\right)}{\beta_1}\left[b_1 w_2^{(1)\,2}\tilde{\alpha}_5 + \tilde{\alpha}_6 w_2^{(1)\,3} + \mathrm{b}_1^2\tilde{\alpha}_7 w_2^{(1)}\right] \qquad (4.28a)$$

where

$$\beta_1 = \left[\left(1+n^2\xi\right)^2 - \lambda\lambda_C\left(\frac{\alpha}{2}+n^2\xi\right)+\left(\frac{A}{1+n^2\xi}\right)^2\right] \qquad (4.28b)$$

$$\tilde{\alpha}_5 = \left[\frac{2(\tilde{\alpha}_1+\tilde{\alpha}_2)}{\pi\left(1+n^2\xi\right)^2} - \frac{\tilde{\alpha}_3}{(1+\xi)^2} - \frac{(1+\xi)^2}{\left(1+n^2\xi\right)^2} + \tilde{\alpha}_4\right], \tilde{\alpha}_6 = \left[\frac{2\tilde{\alpha}_2}{\pi\left(1+n^2\xi\right)^2} - \frac{\tilde{\alpha}_4}{(1+\xi)^2} - \frac{(1+\xi)^2\,\tilde{\alpha}_2}{\left(1+n^2\xi\right)^2}\right] \qquad (4.28c)$$

$$\tilde{\alpha}_7 = \left[\frac{\tilde{\alpha}_1}{\pi\left(1+n^2\xi\right)^2} - \frac{\tilde{\alpha}_3}{(1+\xi)^2} \right] \tag{4.28d}$$

Thus we write $$w^{(3)} = w_2^{(3)} \sin ny \sin x \tag{4.29}$$

So far, we write the normal displacement w(x,y) as

$$w(x,y) = \in w^{(1)} + \in^2 w^{(2)} + \in^3 w^{(3)} + \cdots$$
$$= \in w_2^{(1)} \sin ny \sin x + \in^2 \sum_{m=1,3,5,\cdots}^{\infty} w_1^{(2)} \cos 2ny \sin mx + \in^3 w_2^{(3)} \sin ny \sin x + \cdots \tag{4.30}$$

In reality, we have merely written down the terms $\in^2 \sum_{m=1,3,5,\cdots}^{\infty} w_1^{(2)} \cos 2ny \sin mx$ In (4.30) mainly for completeness because ,as terms of order $\in^2$,they are not in the shape of the imperfection in (4.1). They are henceforth omitted. Despite this omission ,we must however still note that the very terms $\sum_{m=1,3,5,\cdots}^{\infty} w_1^{(2)} \cos 2ny \sin mx$, have been used in determining the term $w_{21}^{(3)}$ $\left(\text{or } \tilde{w}_2^{(3)}\right)$ which is of order $\in^3$ .Thus, we now recast (39) simply as

$$w = \in C_1 + \in^3 C_3 + O\left(\in^4\right) \tag{4.31}$$

where

$$C_1 = w_2^{(1)} \sin ny \sin x = \beta_2 b_1 \sin ny \sin x\,, \tag{4.32a}$$

$$C_3 = w_2^{(3)} \sin ny \sin x = \frac{n^2 A^2 w_2^{(1)3} H\left(5\tilde{\omega}_1 + 4\tilde{\theta}_1\right)\tilde{\alpha}_6 Q \sin ny \sin x}{\beta_1} \tag{4.32b}$$

$$Q = \left[1 + \frac{1}{\tilde{\alpha}_6}\left\{ \tilde{\alpha}_5 b_1 w_2^{(1)\,-1} + b_1^2 \tilde{\alpha}_7 w_2^{(1)\,-2} \right\}\right] \tag{4.32c}$$

where β_2 is as in (4.12). The static buckling load λ_S , is normally evaluated [1-4,6,7] using the condition

$$\frac{d\lambda}{dw} = 0 \tag{4.33}$$

for w(x,y) as given in (4.31). The usual procedure [6,7] is to reverse the series (4.31) in the form

$$\in = d_1 w + d_3 w^3 + \cdots \tag{4.34a}$$

By substituting into (4.34a) for w(x,y) from (4.31) and equating the coefficients of $\in$ and $\in^3$,we have

$$d_1 = \frac{1}{C_1}\ ,\ d_3 = -\frac{C_3}{C_1^4} \tag{4.34b}$$

where ,it must be stressed, C_1 and C_3 depend on λ .The full invocation of (4.33) eventually yields

$$\in = \frac{2}{3}\sqrt{\frac{C_1}{3C_3}} \tag{4.35}$$

which is evaluated at $\lambda = \lambda_S$ .On substituting into (4.35) for C_1 and C_3 from (4.32a,b), we have

$$\left[\left(1+n^2\xi\right)^2 - \lambda_S\lambda_C\left(\frac{\alpha}{2}+n^2\xi\right)+\left(\frac{A}{1+n^2\xi}\right)^2\right]^{\frac{3}{2}} = \frac{3\sqrt{3}}{2}\left|b_1 \in\right| A\lambda_S\lambda_C n\left(\frac{\alpha}{2}+n^2\xi\right)\sqrt{\left(5\tilde{\omega}_1+4\tilde{\theta}_1\right)H\tilde{\alpha}_6 Q} \tag{4.36}$$

where (4.36) is evaluated at $\lambda = \lambda_S$.

5.0 Analysis of result

The result (4.36) is an asymptotic formula and is implicit in the load parameter λ_S .It is valid for small absolute values of $b_1 \in$ as well as for the condition $\left(5\tilde{\omega}_1 + 4\tilde{\theta}_1\right)H\,\tilde{\alpha}_6\,Q > 0$ .The result ,which is not asymptotically exact, clearly shows the dependence of the buckling load λ_S on the various parameters characterizing the nonlinear problem .Guided by Koiter's observation [1,4], the applicability of the result is limited to imperfection whose amplitude is less than one half of the shell thickness $\left(i.e. \in < \frac{1}{2}\right)$. An approximate value of (4.36) can be obtained by setting $Q \approx 1$,to get

$$\left[\left(1+n^2\xi\right)^2 - \lambda_S\lambda_C\left(\frac{\alpha}{2}+n^2\xi\right)+\left(\frac{A}{1+n^2\xi}\right)^2\right]^{\frac{3}{2}} \approx \frac{3\sqrt{3}}{2}\left|b_1 \in\right| A\lambda_S\lambda_C\, n\left(\frac{\alpha}{2}+n^2\xi\right)\left(-\bar{b}\right)^{\frac{1}{2}} \tag{5.1a}$$

where

$$\bar{b} = \left(5\tilde{\omega}_1 + 4\tilde{\theta}_1\right)H\left\{\frac{\tilde{\alpha}_4}{(1+\xi)^2} + \frac{(1+\xi)^2\tilde{\alpha}_2}{\left(1+n^2\xi\right)^2} - \frac{2\tilde{\alpha}_2}{\pi\left(1+n^2\xi\right)^2}\right\} \tag{5.1b}$$

The approximate results (5.1a,b) are valid provided $\bar{b} < 0$. We can regard $\bar{b}$ as the imperfection-sensitivity parameter of the structure. In this case, the structure would be said to be imperfection -sensitive if $\bar{b} < 0$, and imperfection-insensitive if $\bar{b} > 0$ .The case $Q \approx 1$ implies that we have neglected terms of orders $w_2^{(1)\,2}$ and $w_2^{(1)}$ in (4.28a) so that the resultant imperfection-sensitivity parameter $\bar{b}$ does not depend on the load parameter λ_S. If we substitute for λ_C from (3.1), into (4.36) and (5.1a), we get respectively

$$\left(1-\lambda_S\right)^{\frac{3}{2}} = \frac{3\sqrt{3}}{2}\left|b_1 \in\right| A\lambda_S\left\{\left(1+n^2\xi\right)^2 + \left(\frac{A}{1+n^2\xi}\right)^2\right\}^{\frac{1}{3}} n\sqrt{\left(5\tilde{\omega}_1+4\tilde{\theta}_1\right)H\tilde{\alpha}_6 Q} \tag{5.2a}$$

$$(1-\lambda_S)^{\frac{3}{2}} \approx \frac{3\sqrt{3}}{2}|b_1 \in| A\lambda_S \left\{ \left(1+n^2\xi\right)^2 + \left(\frac{A}{1+n^2\xi}\right)^2 \right\}^{\frac{1}{3}} n\ (-\bar{b})^{\frac{1}{2}} \qquad (5.2b)$$

We readily observe from (5.2a,b) that the loss in the buckling strength of the structure is of order $\in^{\frac{2}{3}}$. By using an alternative procedure ,a similar result was obtained in [4] as

$$(1-\lambda_S)^{\frac{3}{2}} = 3\left(-\frac{3\hat{b}}{4}\right)^{\frac{1}{2}}|b_1 \in| \lambda_S \ , \hat{b} < 0 \qquad (5.3)$$

where $\hat{b}$ is the imperfection-sensitivity parameter as defined in [4].The apparent lengthy nature of (5.1a,b) and (5.2a,b) notwithstanding, the striking similarities between (5.2a,b) ,on one hand, and (5.3) on the other hand ,must be appreciated. However the result (5.3) is asymptotically exact. We observe that unlike the results (5.1a,b), the dependence of the static buckling load λ_S on the parameters characterizing the problem is primarily concentrated on the imperfection sensitivity parameter $\hat{b}$ in (5.3).We must stress that while the treatment in [3] focused on highlighting the range of imperfection sensitivity of the structure and did not incorporate imperfection in its formulation, the present investigation actually determines the effects of imperfection on the static buckling load of the structure.

6.0 Conclusion

This investigation has concentrated solely on the use of asymptotic analysis in a regular perturbation appraisal of the problem. We opine that this method is suitable for analysis of more complicated problems including cases where more demands and restrictions are specified on the imperfection.

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