Journal of the Nigerian Association of Mathematical Physics
Volume **11 (November 2007), 311 - 322**

Probabilistic dynamic stability of a damped spherical shell pressurized by a random load

A. M. Ette
Department of Mathematics and Computer Science
Federal University of Technology, Owerri, Nigeria.
e- mail: tonimonsette@ yahoo.com

Abstract

This investigation examines the dynamic stability of a damped imperfect spherical shell within the precinct of a random dynamic load applied just after the initial time. The statistical characterizations of the random load ,such as the mean and the autocorrelation , are assumed given and non-vanishing .In particular, the autocorrelation of the random dynamic load is a stationary noise that is correlated as an exponentially decaying harmonic function of time . Such stochastic and random characterizations of the dynamic load function confer some element of randomness on the normal displacement whose statistical mean we shall first seek for the determination of the dynamic buckling load . Lastly, the dynamic buckling load is determined via a suitable maximization and certain useful deductions are made . Assuming that the variance of the random load is R_0 and using the mean normal displacement as a relevant statistical characterization of the response, it is observed that the dynamic buckling load is of order R_0^{-1}, that is $O\left(\frac{1}{R_0}\right)$, of the load variance R_0

1.0 Introduction

Over the years, investigations into the dynamic buckling of most commonly used elastic structures have concentrated primarily on those loading histories that have been observed to be purely deterministic in nature. Such loading histories include step loading, rectangular loading, impulsive loading and even periodic loading , among a few others. These loadings express the time variable either implicitly or explicitly and thereby resulting either in autonomous cases or non-autonomous problems as the case may be. However, while random dynamic loadings (as opposed to deterministic loadings) have been severally investigated in most nonlinear dynamical systems, they seem not to be frequently discussed in the context of dynamic buckling .Common daily experiences dictate that practical dynamic loads need not always be strictly deterministic but could have some embodiments of randomness which may be either stationary or non-stationary . Amongst such commonly known random dynamic loads are Earthquake ground motion, wind gusts, launch phase of missile flight and pyrotechnic firing , all which are treated as non-stationary processes. We stress that although the possibility of randomness of dynamic loadings in nonlinear dynamical systems has long been recognized , very few previous investigations (this author is however unaware of such previous works) have actually addressed the subject matter in the context of dynamic buckling .

2.0 Formulation

The relevant differential equations for the case of deterministic loading in respect of the undamped structure investigated here, were sufficiently derived by Danielson [1] when he considered the normal displacement $W(x,y,\hat{t})$ at a point on the shell surface to be given by

$$W(x,y,\hat{t}) = \xi_0(\hat{t})\mathrm{W}_0(\mathrm{x},\mathrm{y}) + \xi_1(\hat{t})\mathrm{W}_1(\mathrm{x},\mathrm{y}) + \xi_2(\hat{t})\mathrm{W}_2(\mathrm{x},\mathrm{y}) \quad (2.1)$$

where $\mathrm{W}_0(\mathrm{x},\mathrm{y})$, $\mathrm{W}_1(\mathrm{x},\mathrm{y})$ and $\mathrm{W}_2(\mathrm{x},\mathrm{y})$ are the pre-buckling symmetric mode, the axisymmetric buckling mode and the non-axisymmetric buckling mode respectively while $\xi_0(t), \xi_1(t)$ and $\xi_2(t)$ are their respective time dependent amplitudes .Danielson took the imperfection $\overline{W}(x,y)$ to be in the shape of the buckling modes , namely

$$\overline{W}(x,y) = \overline{\xi}_1\ \mathrm{W}_1(\mathrm{x},\mathrm{y}) + \overline{\xi}_2\ \mathrm{W}_2(x,y) \quad (2.2)$$

where $\overline{\xi}_1$ and $\overline{\xi}_2$ are their respective deterministic amplitudes satisfying the inequalities $0<\overline{\xi}_1<1$ and $0<\overline{\xi}_2<1$. By substituting equations (2.1) and (2.2) into the relevant compatibility equation as well as the dynamic equilibrium equation characterizing an imperfect spherical shell, Danielson derived the following equations ,as here modified to include viscous damping on only the buckling modes:

$$\frac{1}{\omega_0^2}\frac{d^2\xi_0}{d\hat{t}^2} + \xi_0 = \lambda F(\hat{t}), \hat{t} > 0 \quad (2.3)$$

$$\frac{1}{\omega_1^2}\frac{d^2\xi_1}{d\hat{t}^2} + \frac{c_o d\xi_1}{d\hat{t}} + \xi_1(1-\xi_0) - k_1\xi_1^2 + k_2\xi_2^2 = \overline{\xi}_1\xi_0\ , \hat{\mathrm{t}} > 0 \quad (2.4)$$

$$\frac{1}{\omega_2^2}\frac{d^2\xi_2}{d\hat{t}^2} + \frac{c_o d\xi_2}{d\hat{t}} + \xi_2(1-\xi_0) + \xi_1\xi_2 = \overline{\xi}_2\xi_0\ ,\ \hat{t} > 0 \quad (2.5)$$

$$\xi_i(0) = \frac{d\xi_i(0)}{d\hat{t}} = 0\ , \mathrm{i} = 0,1,2. \quad (2.6)$$

Here, λ Is a nondimensional load amplitude , satisfying the inequality $0<\lambda<1$ and nondimensionalized with respect to the classical buckling load λ_C while $F(\hat{t})$

Is, in our case, a randomly stochastic time dependent univariate load function with known statistical characterizations such as mean and autocorrelation $R_f(\hat{t})$, both which are assumed non-vanishing. The parameter c_o satisfies the inequality $0<c_o<1$ and indicates a viscous damping constant which we assume to be the same in the two buckling modes $\xi_1(\hat{t})$ and $\xi_2(\hat{t})$.We again stress that for ease of analysis, we have assumed damping only on the damping modes. In Danielson's investigation, the load function was a step load whereby $F(\hat{t})$=1 for $\hat{t}>0$. The parameters ω_0, ω_1 *and* ω_2 are the circular frequencies of the associated modes while $k_1>0$, $\mathrm{k}_2>0$ are constants considered known .Our pre-occupation is to determine a certain value of λ, called the dynamic buckling load , denoted by λ_D, and satisfying the inequality $0<\lambda_D<\lambda_S<\lambda_C<1$, where λ_S is the static buckling load . We define the dynamic buckling load λ_D as the largest load parameter for which the solution of the problem remains bounded for all time $\hat{t}>0$.For solution, we adopt regular perturbation procedure and first determine uniformly valid asymptotic values of $\xi_0(t), \xi_1(t)$ and $\xi_2(t)$.The random load function $F(\hat{t})$confers some element of randomness on the normal displacement $W(x,y,\hat{t})$, whose mean $\nabla(x,y,\hat{t})$ we shall next determine in terms of the statistical characterizations of the random load .Lastly , we determine the dynamic buckling load λ_D by using the maximization [2,3,4] given by

$$\frac{d\lambda}{d\,\nabla}=0 \tag{2.7}$$

Actually, the mean normal displacement ∇, at this level of analysis , is functionally dependent on only the time variable because, we have earlier on eliminated the spatial dependence by the simplification that yielded the equations (2.3) - (2.5).

For the averaging process to be initiated, we define the Mathematical expectation $\langle\cdots\rangle$ by

$$\langle\cdots\rangle=\lim T\to\infty\ \left\{\frac{1}{\mathrm{T}}\int_0^T(\cdots)\,d\hat{t}\right\} \tag{2.8}$$

3.0 Solution of the problem

For ease of the analysis, we shall ignore the inertia of the pre-buckling mode and thus approximate the pre-buckling mode by

$$\xi_0(\hat{t})=\lambda F(\hat{t}) \tag{3.1a}$$

We let $t=\omega_1\hat{t}$ and get

$$\frac{d\xi_n}{d\hat{t}}=\omega_1\frac{d\xi_n}{dt},\ \mathrm{n}=1,2 \tag{3.1b}$$

We substitute (3.1a,b) into equations (2.4) - (2.6) and simplify to get

$$\frac{d^2\xi_1}{dt^2}+2\delta\frac{d\xi_1}{dt}+\xi_1(1-\in Qf(t))-k_1\xi_1^2+k_2\xi_2^2=\bar{\xi}_1\in Qf(t) \tag{3.2a}$$

$$\frac{d^2\xi_2}{dt^2}+2\delta P\frac{d\xi_2}{dt}+\xi_2(P-\in f(t))+P\xi_1\xi_2=\bar{\xi}_2\in f(t) \tag{3.2b}$$

$$\xi_i(0)=\frac{d\xi_i(0)}{dt}=0,\ \mathrm{i}=1,2;\ 0<\mathrm{t}<\infty, \tag{3.2c}$$

where $\mathrm{Q}=\left(\frac{\omega_1}{\omega_2}\right)^2,\ \mathrm{P}=\left(\frac{\omega_2}{\omega_1}\right)^2=\frac{1}{Q},\in=\lambda\left(\frac{\omega_2}{\omega_1}\right)^2,\ \omega_2<\omega_1,\ \ 2\delta=c_o,\ f(t)=F\left(\frac{t}{\omega_1}\right)$ (3.2d)

We consider $0<\in<1$ and now let

$$\xi_1(t)=\sum_{i=1}^{\infty}\eta_i(t)\in^i,\ \xi_2(t)=\sum_{i=1}^{\infty}\zeta_i(t)\in^i \tag{3.3}$$

We substitute (3.3) into equations (3.2a-c) , using (3.3) and equate equations of the same orders of $\in$ to get

$$M\eta_1\equiv\frac{d^2\eta_1}{dt^2}+2\delta\frac{d\eta_1}{dt}+\eta_1=Q\bar{\xi}_1f(t) \tag{3.4a}$$

$$M\eta_2=Qf(t)\eta_1+k_1\eta_1^2-k_2\zeta_1^2 \tag{3.4b}$$

$$N\zeta_1\equiv\frac{d^2\zeta_1}{dt^2}+2\delta P\frac{d\zeta_1}{dt}+P\zeta_1=\bar{\xi}_2f(t) \tag{3.5a}$$

$$N\zeta_2=-P\eta_1\zeta_1+\zeta_1f(t) \tag{3.5b}$$

$$\eta_i(0)=\frac{d\eta_i(0)}{dt}=\zeta_i(0)=\frac{d\zeta_i(0)}{dt}=0,\ \mathrm{i}=1,2,3,\cdots \tag{3.6}$$

The solution of (3.4a) is

$$\eta_1(t)=e^{-\delta t}\{a_1\cos\psi\,\mathrm{t}+\mathrm{b}_1\sin\psi\,\mathrm{t}\}+\frac{Q\bar{\xi}_1}{\psi}\int_0^{t}f(\tau)e^{-\delta(t-\tau)}\sin\psi(t-\tau) \tag{3.7a}$$

where

$$\psi = \sqrt{\left(1-\delta^2\right)} \ , 0<\delta<1 \tag{3.7b}$$

On using the appropriate initial conditions for $\eta_1(t)$ as in (3.6), we have $a_1 = b_1 = 0$ so that we have

$$\eta_1(t) = Q\bar{\xi}_1 \int_0^t h(\tau) f(t-\tau)\,\mathrm{d}\tau, \ \mathrm{h}(\tau) = \frac{e^{-\delta\tau}\sin\psi\tau}{\psi} \tag{3.7c}$$

Similarly, the solution of (3.5a) is

$$\zeta_1(t) = \bar{\xi}_2 \int_0^t q(\tau) f(t-\tau)\,\mathrm{d}\tau, \mathrm{q}(\tau) = \frac{e^{-(\delta \mathrm{P})\tau}\sin\varphi\tau}{\varphi}; \varphi = \sqrt{\mathrm{P} - \delta^2 P^2}\ ; 0<\delta\,\mathrm{P}<p \tag{3.8}$$

On solving (3.4b), we have

$$\eta_2(t) = \int_0^t h(\tau)\left\{ Qf(t-\tau)\eta_1(t-\tau) + k_1\eta_1^2(t-\tau) - k_2\zeta_1^2(t-\tau) \right\}\mathrm{d}\tau \tag{3.9}$$

The solution of (3.5b) is

$$\zeta_2(t) = -P\int_0^t q(\tau)\eta_1(t-\tau)\zeta_1(t-\tau)\,\mathrm{d}\tau + \int_0^t q(\tau) f(t-\tau)\,\zeta_1(t-\tau)\,\mathrm{d}\tau \tag{3.10}$$

Thus we have, after expanding the terms in (3.9) ,

$$\begin{aligned}
\xi_1(t) &= \in \eta_1(t) + \in^2 \eta_2(t) + \cdots \\
&= \in \int_0^t h(\tau) f(t-\tau) d\tau + \in^2 \Bigg[\bar{\xi}_1 Q \int_0^t \int_0^{t-\tau_1} h(\tau_1) h(\tau_2) f(t-\tau_1) f(t-\tau_1-\tau_2) d\tau_1 d\tau_2 \\
&+ \left(\mathrm{Q}\bar{\xi}_1\right)^2 k_1 \int_0^t \int_0^{t-\tau_1} \int_0^{t-\tau_1} h(\tau_1) h(\tau_2) h(\tau_3) f(t-\tau_1-\tau_2) f(t-\tau_1-\tau_3) d\tau_1 d\tau_2 d\tau_3 \\
&- \bar{\xi}_2^2 k_2 \int_0^t \int_0^{t-\tau_1} \int_0^{t-\tau_1} h(\tau_1) q(\tau_2) q(\tau_3) f(t-\tau_1-\tau_2) f(t-\tau_1-\tau_3) d\tau_1 d\tau_2 d\tau_3 \Bigg] + \cdots
\end{aligned} \tag{3.11}$$

Similarly, on simplifying (3.10), we have

$$\begin{aligned}
\xi_2(t) &= \in \int_0^t q(\tau) f(t-\tau)\,\mathrm{d}\tau + \in^2 \Bigg[-PQ\bar{\xi}_1\bar{\xi}_2 \int_0^t \int_0^{t-\tau_1} \int_0^{t-\tau_1} q(\tau_1) h(\tau_2) q(\tau_3) f(t-\tau_1-\tau_2) \times \\
& f(t-\tau_1-\tau_3)\, d\tau_1 d\tau_2 d\tau_3 + \bar{\xi}_2 \int_0^t \int_0^{t-\tau_1} q(\tau_1) q(\tau_2) f(t-\tau_1) f(t-\tau_1-\tau_2)\, d\tau_1 d\tau_2 \Bigg] + \cdots
\end{aligned} \tag{3.12}$$

We shall now evaluate the mean normal displacement $\nabla(t)$ which is achieved by assuming the averaging process in (2.8) and noting that the autocorrelation $R_f(t)$ of the random load function $f(t)$ is given by

$$R_f(\tau) = \left\langle f(t) f(t+\tau) \right\rangle = R_0 e^{-\alpha|\tau|}\cos\Omega\,\tau, 0<\alpha,\ 0<\Omega \tag{3.13}$$

Thus, when $\tau = 0$, we have $R_f(0) = R_0$ which is the variance of the load over the interval of definition of the problem. The autocorrelation in (3.13) was used by Fraser [5] , Amazigo [6] and Cederbaum et al [7] , among others . The analysis presented here is related to those by Spanos et al [8] and Lin and Cai [9]. We note that the normal displacement *W*(*t*) now takes the form

$$W(t) = \in \{\eta_1(t) + \zeta_1(t)\} + \in^2 \{\eta_2(t) + \zeta_2(t)\} + \cdots$$

$$= \in \left[Q\bar{\xi}_1 \int_0^t h(\tau) f(t-\tau)\, \mathrm{d}\tau + \bar{\xi}_2 \int_0^t q(\tau) f(t-\tau)\, \mathrm{d}\tau \right]$$

$$+ \in^2 \left[Q^2 \bar{\xi}_1 \int_0^t \int_0^{t-\tau_1} h(\tau_1) h(\tau_2) f(t-\tau_1) f(t-\tau_1-\tau_2)\, \mathrm{d}\tau_1 \mathrm{d}\tau_2 \right.$$

$$+ (\mathrm{Q}\bar{\xi}_1)^2 k_1 \int_0^t \int_0^{t-\tau_1} \int_0^{t-\tau_1} h(\tau_1) h(\tau_2) h(\tau_3) f(t-\tau_1-\tau_2) f(t-\tau_1-\tau_3) d\tau_1 d\tau_2 d\tau_3$$

$$- \bar{\xi}_2^2 k_2 \int_0^t \int_0^{t-\tau_1} \int_0^{t-\tau_1} h(\tau_1) q(\tau_2) q(\tau_3) f(t-\tau_1-\tau_2) f(t-\tau_1-\tau_3) d\tau_1 d\tau_2 d\tau_3$$

$$- PQ\bar{\xi}_1 \bar{\xi}_2 \int_0^t \int_0^{t-\tau_1} \int_0^{t-\tau_1} q(\tau_1) h(\tau_2) q(\tau_3) f(t-\tau_1-\tau_2) f(t-\tau_1-\tau_3) d\tau_1 d\tau_2 d\tau_3$$

$$\left. + \bar{\xi}_2 \int_0^t \int_0^{t-\tau_1} q(\tau_1) q(\tau_2) f(t-\tau_1) f(t-\tau_1-\tau_2)\, d\tau_1 d\tau_2 \right] + \cdots \tag{3.14}$$

Thus, the mean normal displacement ∇, becomes

$$\nabla = \langle W(t) \rangle = \in \left[Q\bar{\xi}_1 \int_0^\infty h(\tau) \langle f(t-\tau) \rangle \mathrm{d}\tau + \bar{\xi}_2 \int_0^\infty q(\tau) \langle f(t-\tau) \rangle \mathrm{d}\tau \right]$$

$$+ \in^2 \left[Q^2 \bar{\xi}_1 \int_0^\infty \int_0^\infty h(\tau_1) h(\tau_2) \langle f(t-\tau_1) f(t-\tau_1-\tau_2) \rangle \mathrm{d}\tau_1 \mathrm{d}\tau_2 \right.$$

$$+ (\mathrm{Q}\bar{\xi}_1)^2 k_1 \int_0^\infty \int_0^\infty \int_0^\infty h(\tau_1) h(\tau_2) h(\tau_3) \langle f(t-\tau_1-\tau_2) f(t-\tau_1-\tau_3) \rangle d\tau_1 d\tau_2 d\tau_3$$

$$- \bar{\xi}_2^2 k_2 \int_0^\infty \int_0^\infty \int_0^\infty h(\tau_1) q(\tau_2) q(\tau_3) \langle f(t-\tau_1-\tau_2) f(t-\tau_1-\tau_3) \rangle d\tau_1 d\tau_2 d\tau_3$$

$$- PQ\bar{\xi}_1 \bar{\xi}_2 \int_0^\infty \int_0^\infty \int_0^\infty q(\tau_1) h(\tau_2) q(\tau_3) \langle f(t-\tau_1-\tau_2) f(t-\tau_1-\tau_3) \rangle d\tau_1 d\tau_2 d\tau_3$$

$$\left. + \bar{\xi}_2 \int_0^\infty \int_0^\infty q(\tau_1) q(\tau_2) \langle f(t-\tau_1) f(t-\tau_1-\tau_2) \rangle d\tau_1 d\tau_2 \right] + \cdots \tag{3.15}$$

We note that based on the averaging process in (2.8), the mean normal displacement ∇ is evaluated at infinite time .We shall now evaluate each of the integrals in (3.15).The first integral in (3.15) is evaluated to give

$$Q\bar{\xi}_1 \int_0^\infty h(\tau) \langle f(t-\tau) \rangle \mathrm{d}\tau = \frac{Q\bar{\xi}_1 r_1}{\psi^2 + \delta^2} \;;\; \mathrm{r}_1 = \langle f(t-\tau) \rangle \tag{3.16a}$$

where we have, for convenience and for ease of further analysis, taken the mean $\langle f(t-\tau) \rangle = r_1$ of the random load f(t) to be constant over the interval of averaging. This need not be the case in all situations. Similarly, the second integral in (3.15) gives

$$\bar{\xi}_2 \int_0^\infty q(\tau) \langle f(t-\tau) \rangle \mathrm{d}\tau = \frac{\bar{\xi}_2 r_1}{\varphi^2 + \delta^2 P^2} \tag{3.16b}$$

The third integral in (3.15) gives

$$Q^2\bar{\xi}_1 \int_0^\infty\int_0^\infty h(\tau_1)h(\tau_2)\langle f(t-\tau_1)f(t-\tau_1-\tau_2)\rangle \mathrm{d}\tau_1 d\tau_{2_1} = Q^2\bar{\xi}_1 \int_0^\infty\int_0^\infty h(\tau_1)h(\tau_2)\mathrm{R}_f(\tau)\mathrm{d}\tau_1 d\tau_2$$

$$= Q^2\bar{\xi}_1 R_0 \int_0^\infty\int_0^\infty h(\tau_1)h(\tau_2) e^{-\alpha\|\tau\|}\cos\Omega\tau \;\; \mathrm{d}\tau_1 d\tau_2 = \frac{Q^2\bar{\xi}_1 R_0 F_0}{2\psi\left(\psi^2+\delta^2\right)} \tag{3.17a}$$

where $$F_0 = \left[\frac{\psi+\Omega}{(\psi+\Omega)^2+(\delta+\alpha)^2} + \frac{\psi-\Omega}{(\psi-\Omega)^2+(\delta+\alpha)^2}\right] \tag{3.17b}$$

The fourth integral in (3.15) is evaluated as follows :

$$\left(\mathrm{Q}\bar{\xi}_1\right)^2 k_1 \int_0^\infty\int_0^\infty\int_0^\infty h(\tau_1)h(\tau_2)h(\tau_3)\langle f(t-\tau_1-\tau_2)f(t-\tau_1-\tau_3)\rangle d\tau_1 d\tau_2 d\tau_3$$

$$= \left(\mathrm{Q}\bar{\xi}_1\right)^2 k_1 \left\{\int_0^\infty h(\tau_1)d\tau_1\right\}\left\{\int_0^\infty\int_0^\infty h(\tau_2)h(\tau_3)R_f(\tau_2-\tau_3)d\tau_2 d\tau_3\right\} \tag{3.18a}$$

$$= \left(\mathrm{Q}\bar{\xi}_1\right)^2 k_1 R_0 \left\{\int_0^\infty h(\tau_1)d\tau_1\right\}\int_0^\infty\int_0^\infty h(\tau_2)h(\tau_3)\, e^{-\alpha(\tau_2-\tau_3)} \cos\Omega(\tau_2-\tau_3)d\tau_2 d\tau_3$$

$$= \frac{\left(\mathrm{Q}\bar{\xi}_1\right)^2 k_1 R_0}{4\left(\psi^2+\delta^2\right)} \int_0^\infty\int_0^\infty e^{-(\delta+\alpha)\tau_2} e^{-(\delta-\alpha)\tau_3}[\cos(\psi+\Omega)\tau_2\cos(\psi+\Omega)\tau_3 + \sin(\psi+\Omega)\tau_2\sin(\psi+\Omega)\tau_3$$

$$\begin{aligned} &+\cos(\psi-\Omega)\tau_2\cos(\psi-\Omega)\tau_3 + \sin(\psi-\Omega)\tau_2\sin(\psi-\Omega)\tau_3 \\ &-\cos(\psi+\Omega)\tau_2\cos(\psi-\Omega)\tau_3 + \sin(\psi+\Omega)\tau_2\sin(\psi-\Omega)\tau_3 \\ &-\cos(\psi-\Omega)\tau_2\cos(\psi+\Omega)\tau_3 + \sin(\psi-\Omega)\tau_2\sin(\psi+\Omega)\tau_3]\mathrm{d}\tau_2\mathrm{d}\tau_3 \end{aligned} \tag{3.18b}$$

$$= \frac{\left(\mathrm{Q}\bar{\xi}_1\right)^2 k_1 R_0 F_1}{4\left(\psi^2+\delta^2\right)} \tag{3.19a}$$

where

$$\begin{aligned} F_1 = \Bigg[&\left[\left\{\frac{(\delta+\alpha)}{(\delta+\alpha)^2+(\psi+\Omega)^2}\right\}\left\{\frac{(\delta-\alpha)}{(\delta-\alpha)^2+(\psi+\Omega)^2}\right\} + \left\{\frac{(\psi+\Omega)}{(\delta+\alpha)^2+(\psi+\Omega)^2}\right\}\times\right. \\ &\left\{\frac{(\psi+\Omega)}{(\delta-\alpha)^2+(\psi+\Omega)^2}\right\} + \left\{\frac{(\delta+\alpha)}{(\delta+\alpha)^2+(\psi-\Omega)^2}\right\}\left\{\frac{(\delta-\alpha)}{(\delta-\alpha)^2+(\psi-\Omega)^2}\right\} \\ &+\left\{\frac{(\psi-\Omega)}{(\delta+\alpha)^2+(\psi-\Omega)^2}\right\}\left\{\frac{(\psi-\Omega)}{(\delta-\alpha)^2+(\psi-\Omega)^2}\right\} \\ &-\left\{\frac{(\delta+\alpha)}{(\delta+\alpha)^2+(\psi+\Omega)^2}\right\}\left\{\frac{(\delta-\alpha)}{(\delta-\alpha)^2+(\psi-\Omega)^2}\right\} + \left\{\frac{(\psi+\Omega)}{(\delta+\alpha)^2+(\psi+\Omega)^2}\right\}\times \\ &\left\{\frac{(\psi-\Omega)}{(\delta-\alpha)^2+(\psi-\Omega)^2}\right\} - \left\{\frac{(\delta+\alpha)}{(\delta+\alpha)^2+(\psi-\Omega)^2}\right\}\left\{\frac{(\delta-\alpha)}{(\delta-\alpha)^2+(\psi+\Omega)^2}\right\} \\ &+\left\{\frac{(\psi-\Omega)}{(\delta+\alpha)^2+(\psi-\Omega)^2}\right\}\left\{\frac{(\psi+\Omega)}{(\delta-\alpha)^2+(\psi+\Omega)^2}\right\}\Bigg], \delta>\alpha \end{aligned} \tag{3.19b}$$

The fifth integral in (3.15) is evaluated as follows:

$$-\bar{\xi}_2^2 k_2 \int_0^\infty\int_0^\infty\int_0^\infty h(\tau_1)q(\tau_2)q(\tau_3)\langle f(t-\tau_1-\tau_2)f(t-\tau_1-\tau_3)\rangle d\tau_1 d\tau_2 d\tau_3$$

$$= -\bar{\xi}_2^2 k_2 \left\{\int_0^\infty h(\tau_1)d\tau_1\right\}\left\{\int_0^\infty\int_0^\infty q(\tau_2)q(\tau_3)R_f(\tau_2-\tau_3)d\tau_2 d\tau_3\right\} \quad (3.20a)$$

$$= -\bar{\xi}_2^2 k_2 R_0 \left\{\int_0^\infty h(\tau_1)d\tau_1\right\}\left\{\int_0^\infty\int_0^\infty q(\tau_2)q(\tau_3)e^{-\alpha(\tau_2-\tau_3)}\cos\Omega(\tau_2-\tau_3)d\tau_2 d\tau_3\right\}$$

$$= -\frac{\bar{\xi}_2^2 k_2 R_0}{(\psi^2+\delta^2)}\int_0^\infty\int_0^\infty e^{-(\delta P+\alpha)\tau_2}e^{-(\delta P-\alpha)\tau_3}\sin\varphi\tau_2\sin\varphi\tau_3\cos\Omega(\tau_2-\tau_3)d\tau_2 d\tau_3$$

$$= -\frac{\bar{\xi}_2^2 k_2 R_0}{4(\psi^2+\delta^2)}\int_0^\infty\int_0^\infty e^{-(\delta P+\alpha)\tau_2}e^{-(\delta P-\alpha)\tau_3} \times [\cos(\varphi+\Omega)\tau_2\cos(\varphi+\Omega)\tau_3$$

$$+\sin(\varphi+\Omega)\tau_2\sin(\varphi+\Omega)\tau_3+\cos(\varphi-\Omega)\tau_2\cos(\varphi-\Omega)\tau_3+\sin(\varphi-\Omega)\tau_2\sin(\varphi-\Omega)\tau_3$$

$$-\cos(\varphi+\Omega)\tau_2\cos(\varphi-\Omega)\tau_3+\sin(\varphi+\Omega)\tau_2\sin(\varphi-\Omega)\tau_3$$

$$-\cos(\varphi-\Omega)\tau_2\cos(\varphi+\Omega)\tau_3+\sin(\varphi-\Omega)\tau_2\sin(\varphi+\Omega)\tau_3]d\tau_2 d\tau_3, \ \delta P>\alpha \quad (3.20b)$$

$$= -\frac{\bar{\xi}_2^2 k_2 R_0 F_2}{4(\psi^2+\delta^2)} \quad (3.21a)$$

where
$$F_2 = \left[\left\{\frac{(\delta P+\alpha)}{(\delta P+\alpha)^2+(\varphi+\Omega)^2}\right\}\left\{\frac{(\delta P-\alpha)}{(\delta P-\alpha)^2+(\varphi+\Omega)^2}\right\}+\left\{\frac{(\varphi+\Omega)}{(\delta P+\alpha)^2+(\varphi+\Omega)^2}\right\}\times\right.$$

$$\left\{\frac{(\varphi+\Omega)}{(\delta P-\alpha)^2+(\varphi+\Omega)^2}\right\}+\left\{\frac{(\delta P+\alpha)}{(\delta P+\alpha)^2+(\varphi-\Omega)^2}\right\}\left\{\frac{(\delta P-\alpha)}{(\delta P-\alpha)^2+(\varphi-\Omega)^2}\right\}$$

$$+\left\{\frac{(\varphi-\Omega)}{(\delta P+\alpha)^2+(\varphi-\Omega)^2}\right\}\left\{\frac{(\varphi-\Omega)}{(\delta P-\alpha)^2+(\varphi-\Omega)^2}\right\}$$

$$-\left\{\frac{(\delta P+\alpha)}{(\delta P+\alpha)^2+(\varphi+\Omega)^2}\right\}\left\{\frac{(\delta P-\alpha)}{(\delta P-\alpha)^2+(\varphi-\Omega)^2}\right\}+\left\{\frac{(\varphi+\Omega)}{(\delta P+\alpha)^2+(\varphi+\Omega)^2}\right\}\times$$

$$\left\{\frac{(\varphi-\Omega)}{(\delta P-\alpha)^2+(\varphi-\Omega)^2}\right\}-\left\{\frac{(\delta P+\alpha)}{(\delta P+\alpha)^2+(\varphi-\Omega)^2}\right\}\left\{\frac{(\delta P-\alpha)}{(\delta P-\alpha)^2+(\varphi+\Omega)^2}\right\}$$

$$\left.+\left\{\frac{(\varphi-\Omega)}{(\delta P+\alpha)^2+(\varphi-\Omega)^2}\right\}\left\{\frac{(\varphi+\Omega)}{(\delta P-\alpha)^2+(\varphi+\Omega)^2}\right\}\right], \ \delta P>\alpha \quad (3.21b)$$

We next evaluate the sixth integral in (3.15) as follows:

$$-PQ\bar{\xi}_1\bar{\xi}_2\int_0^\infty\int_0^\infty\int_0^\infty q(\tau_1)h(\tau_2)q(\tau_3)\langle f(t-\tau_1-\tau_2)f(t-\tau_1-\tau_3)\rangle d\tau_1 d\tau_2 d\tau_3$$

$$= -PQ\bar{\xi}_1\bar{\xi}_2\left\{\int_0^\infty q(\tau_1)d\tau_1\right\}\left\{\int_0^\infty\int_0^\infty h(\tau_2)q(\tau_3)R_f(\tau_3-\tau_2)d\tau_2 d\tau_3\right\} \quad (3.22a)$$

$$= -\frac{PQ\bar{\bar{\xi}}_1\bar{\bar{\xi}}_2 R_0}{\left(\varphi^2+\delta^2 P^2\right)}\int_0^\infty\int_0^\infty e^{-(\delta+\alpha)\tau_2} e^{-(\delta P-\alpha)\tau_3}\sin\psi\tau_2\sin\varphi\tau_3\cos\Omega(\tau_2-\tau_3)\; d\tau_2 d\tau_3$$

$$= -\frac{PQ\bar{\bar{\xi}}_1\bar{\bar{\xi}}_2 R_0}{4\left(\varphi^2+\delta^2 P^2\right)}\int_0^\infty\int_0^\infty e^{-(\delta+\alpha)\tau_2} e^{-(\delta P-\alpha)\tau_3}[\cos(\psi+\Omega)\tau_2\cos(\varphi+\Omega)\tau_3 \tag{3.22b}$$

$$+\sin(\psi+\Omega)\tau_2\sin(\varphi+\Omega)\tau_3+\cos(\psi-\Omega)\tau_2\cos(\varphi-\Omega)\tau_3$$
$$+\sin(\psi-\Omega)\tau_2\sin(\varphi-\Omega)\tau_3-\cos(\psi+\Omega)\tau_2\cos(\varphi-\Omega)\tau_3$$
$$+\sin(\psi+\Omega)\tau_2\sin(\varphi-\Omega)\tau_3-\cos(\psi-\Omega)\tau_2\cos(\varphi+\Omega)\tau_3$$
$$+\sin(\psi-\Omega)\tau_2\sin(\varphi+\Omega)\tau_3]d\tau_2 d\tau_3$$

$$= \frac{PQ\;\bar{\xi}_1\bar{\xi}_2 R_0 F_3}{4\left(\varphi^2+\delta^2 P^2\right)} \tag{3.23a}$$

where $F_3 = \left[\left\{\frac{(\delta+\alpha)}{(\delta+\alpha)^2+(\psi+\Omega)^2}\right\}\left\{\frac{(\delta P-\alpha)}{(\delta P-\alpha)^2+(\varphi+\Omega)^2}\right\}+\left\{\frac{(\psi+\Omega)}{(\delta+\alpha)^2+(+\Omega)^2}\right\}\times\right.$

$$\left\{\frac{(\varphi+\Omega)}{(\delta P-\alpha)^2+(\varphi+\Omega)^2}\right\}+\left\{\frac{(\delta+\alpha)}{(\delta+\alpha)^2+(\psi-\Omega)^2}\right\}\left\{\frac{(\delta P-\alpha)}{(\delta P-\alpha)^2+(\varphi-\Omega)^2}\right\}$$

$$+\left\{\frac{(\psi-\Omega)}{(\delta+\alpha)^2+(\psi-\Omega)^2}\right\}\left\{\frac{(\varphi-\Omega)}{(\delta P-\alpha)^2+(\varphi-\Omega)^2}\right\}$$

$$-\left\{\frac{(\delta+\alpha)}{(\delta+\alpha)^2+(\psi+\Omega)^2}\right\}\left\{\frac{(\delta P-\alpha)}{(\delta P-\alpha)^2+(\varphi-\Omega)^2}\right\}+\left\{\frac{(\varphi+\Omega)}{(\delta+\alpha)^2+(\varphi+\Omega)^2}\right\}\times$$

$$\left\{\frac{(\varphi-\Omega)}{(\delta P-\alpha)^2+(\varphi-\Omega)^2}\right\}-\left\{\frac{(\delta+\alpha)}{(\delta+\alpha)^2+(\psi-\Omega)^2}\right\}\left\{\frac{(\delta P-\alpha)}{(\delta P-\alpha)^2+(\varphi+\Omega)^2}\right\}$$

$$\left.+\left\{\frac{(\psi-\Omega)}{(\delta+\alpha)^2+(\psi-\Omega)^2}\right\}\left\{\frac{(\varphi+\Omega)}{(\delta P-\alpha)^2+(\varphi+\Omega)^2}\right\}\right],\ \delta P>\alpha \tag{3.23b}$$

The last integral in (3.15a) is now evaluated thus:

$$\bar{\bar{\xi}}_2\int_0^\infty\int_0^\infty q(\tau_1)q(\tau_2)\langle f(t-\tau_1)f(t-\tau_1-\tau_2)\rangle d\tau_1 d\tau_2$$

$$= \bar{\bar{\xi}}_2\left\{\int_0^\infty q(\tau_1)d\tau_1\right\}\left\{\int_0^\infty q(\tau_2)R_f(\tau_2)d\tau_2\right\} \tag{3.24a}$$

$$= \frac{\bar{\bar{\xi}}_2 R_0}{2\left\{\varphi^2+(\delta P)^2\right\}}\int_0^\infty e^{-(\delta P+\alpha)}\{\sin(\varphi+\Omega)\tau_2+\sin(\varphi-\Omega)\tau_3\}d\tau_2 = \frac{\bar{\bar{\xi}}_2 R_0 F_4}{2\left\{\varphi^2+(\delta P)^2\right\}}$$

(3.24b)

$$F_4 = \left[\frac{\psi+\Omega}{(\psi+\Omega)^2+(\delta P+\alpha)^2}+\frac{\varphi-\Omega}{(\varphi-\Omega)^2+(\delta P+\alpha)^2}\right] \tag{3.25}$$

Thus the mean normal displacement ∇ follows from the full simplification of (3.15) to give

$$\nabla = \in C_1 + \in^2 C_2^2 + \cdots \tag{3.26a}$$

where

$$C_1 = \left[\frac{Q\bar{\xi}_1 r_1}{\psi^2 + \delta^2} + \frac{\bar{\xi}_2 r_1}{\varphi^2 + \delta^2 P^2} \right] ; \mathrm{C}_2 = \left[\frac{Q^2 \bar{\xi}_1 R_0 F_0}{2\psi\left(\psi^2 + \delta^2\right)} + \frac{\left(\mathrm{Q}\bar{\xi}_1\right)^2 k_1 R_0 F_1}{4\left(\psi^2 + \delta^2\right)} - \frac{\bar{\xi}_2^2 k_2 R_0 F_2}{4\left(\psi^2 + \delta^2\right)} \right.$$

$$\left. - \frac{PQ\bar{\xi}_1\bar{\xi}_2 R_0 F_3}{4\left(\varphi^2 + \delta^2 P^2\right)} + \frac{\bar{\xi}_2 R_0 F_4}{2\left\{\varphi^2 + (\delta \mathrm{P})^2\right\}} \right] \tag{3.26b}$$

4.0 Dynamic buckling load λ_D

We observe that the mean normal displacement ∇ in (3.26a,b) depends on the load parameter λ through the small parameter $\in$ .To determine the dynamic buckling load λ_D**,** we [2,3,4] first have to reverse the series (3.26a,b) so that

$$\in = d_1\nabla + d_2\nabla^2 + \cdots \tag{4.1a}$$

By substituting into (4.1a) for ∇ from (3.26a), and equating the coefficients of $\in$ and $\in^2$ we have

$$d_1 = \frac{1}{C_1}, \mathrm{C}_2 = -\frac{C_2}{C_1^3} \tag{4.1b}$$

The maximization in (2.7) is now easily accomplished through (4.1a) to give

$$\nabla_D = \nabla(\lambda_D) = -\frac{d_1}{2d_2} = \frac{C_1^2}{2C_2} \tag{4.1c}$$

where ∇_D is the value of ∇ at λ_D **.**If we evaluate (4.1a) at $\lambda = \lambda_D$**,** we have

$$\in_D = \frac{C_1}{4C_2} \tag{4.1d}$$

where $\in_D$ is the value of $\in$ at $\lambda = \lambda_D$ **.**On substituting into (4.1d) for all the terms , we have

$$\lambda_D\left(\frac{\omega_2}{\omega_1}\right)^2 = \frac{C_1}{4C_2} \tag{4.2}$$

which gives a formula for evaluating the dynamic load λ_D **.**

5.0 Analysis of result and conclusion

For clarity, we expand (4.2) in full in the following way

$$\lambda_D\left(\frac{\omega_2}{\omega_0}\right)^2 = \frac{\frac{r_1}{4R_0}\left(\frac{\omega_1}{\omega_0}\right)^2\left[\frac{Q\bar{\xi}_1}{\psi^2+\delta^2} + \frac{\bar{\xi}_2}{\varphi^2+\delta^2 P^2}\right]}{\left[\frac{Q^2\bar{\xi}_1 F_0}{2\psi\left(\psi^2+\delta^2\right)} + \frac{\left(\mathrm{Q}\bar{\xi}_1\right)^2 k_1 F_1}{4\left(\psi^2+\delta^2\right)} - \frac{\bar{\xi}_2^2 k_2 F_2}{4\left(\psi^2+\delta^2\right)} - \frac{PQ\bar{\xi}_1\bar{\xi}_2 F_3}{4\left(\varphi^2+\delta^2 P^2\right)} + \frac{\bar{\xi}_2 F_4}{2\left\{\varphi^2+(\delta \mathrm{P})^2\right\}}\right]} \tag{5.1}$$

We readily observe that the dynamic buckling load λ_D is of order R_0^{-1}, that is $O\left(\frac{1}{R_0}\right)$ **,**of the variance of the random load and where we have re-introduced the circular frequency ω_1 through the relation $\left(\frac{\omega_2}{\omega_0}\right) = \left(\frac{\omega_2}{\omega_1}\right)\left(\frac{\omega_1}{\omega_0}\right)$. Because the load parameter λ_D appears only on the left hand side of (5.1), it is therefore clear that equation (5.1) is an algebraic equation that determines λ_D directly and needs no further simplification. The result gives the dynamic buckling load of the structure in the presence of both axisymmetric and non-axisymmetric imperfections .The terms multiplying k_1, k_2 and $\bar{\xi}_1\bar{\xi}_2$ in (5.1) are the obvious contributions to dynamic buckling of the quadratic terms $k_1\xi_1^2, k_2\xi_2^2$ and the coupling term $\xi_1\xi_2$ respectively in the governing equations .Similarly, the terms multiplying F_0 *and* F_4 in (5.1) are also the contributions of the coupling terms $\xi_1\xi_0$ and $\xi_2\xi_0$ respectively .Thus , if we assume the presence of only the axisymmetric imperfection, then $\bar{\xi}_1 \neq 0, \bar{\xi}_2 = 0$ and from (5.1) we have the following result

$$\lambda_D\left(\frac{\omega_2}{\omega_0}\right)^2 = \frac{r_1\left(\frac{\omega_1}{\omega_0}\right)^2}{R_0Q\left\{\frac{2F_0}{\psi} + F_1k_1\bar{\xi}_1Q\right\}} \tag{5.2a}$$

From (5.2a) we observe the following:

(a) The effects of the coupling terms $\xi_1\xi_2$ and $\xi_2\xi_0$ are absent . (b) The effect of the quadratic term $k_2\xi_2^2$ is also absent. (c)The effects of the quadratic term $k_1\xi_1^2$ and the coupling term $\xi_1\xi_0$ are however not absent. The corresponding static buckling load λ_S can be obtained from the governing equations (2.4) and (2.5) by dropping the inertia terms, the viscous damping terms as well as , setting $f(\bar{t})=1$, $\bar{\xi}_2=0$ to get

$$(1-\lambda_S)^2 = 4k_1\lambda_S\bar{\xi}_1 \tag{5.2b}$$

If we eliminate the imperfection parameter from (5.2a) using (5.2b),we get

$$\left(\frac{\lambda_D}{\lambda_S}\right)\left(\frac{\omega_2}{\omega_0}\right)^2 = \frac{4r_1\left(\frac{\omega_1}{\omega_0}\right)^2}{R_0Q\left\{\frac{8\lambda_S F_0}{\psi} + (1-\lambda_S)^2F_1\bar{\xi}_1Q\right\}} \tag{5.2c}$$

If however we assume the presence of only the non-axisymmetric imperfections, then $\bar{\xi}_1 = 0, \bar{\xi}_2 \neq 0$ and the result (5.1) becomes

$$\lambda_D\left(\frac{\omega_2}{\omega_0}\right)^2 = \frac{r_1\left(\frac{\omega_1}{\omega_0}\right)^2}{R_0\left\{2F_0 - \bar{\xi}_2k_2QF_2\left(\frac{\varphi^2+\delta^2P^2}{\psi^2+\delta^2}\right)\right\}} \tag{5.3a}$$

We note from (5.3a) that (a) the effects of the coupling terms $\xi_1\xi_0$ and $\xi_1\xi_2$ are absent (b) The effect of the quadratic term $k_1\xi_1^2$ is also absent .(c) The effects of the quadratic term $k_2\xi_2^2$ and the coupling term $\xi_2\xi_0$ are however not absent.

The corresponding static buckling load λ_S for this case is

$$(1-\lambda_S)^2 = \frac{3\lambda_S\bar{\xi}_2}{2}\sqrt{3k_2} \tag{5.3b}$$

On eliminating the imperfection parameter from (5.3a) using (5.3b), we have

$$\left(\frac{\lambda_D}{\lambda_S}\right)\left(\frac{\omega_2}{\omega_0}\right)^2 = \frac{r_1\left(\frac{\omega_1}{\omega_0}\right)^2}{2R_0\left\{F_0 - \frac{QF_2}{3}\sqrt{\frac{k_2}{3}}\left(\frac{\varphi^2+\delta^2P^2}{\psi^2+\delta^2}\right)\right\}} \tag{5.3c}$$

From both (5.2c) and (5.3c), we observe that it is possible to obtain the dynamic buckling load λ_D in terms of the associated static buckling load λ_S.Besides, from (5.2c) and (5.3c), we are able to determine the values of λ_D by by-passing the imperfection parameters. Lastly we conclude from (5.1), (5.2) and (5.3) that the only condition for the coupling effects to be felt is that the imperfections in the shapes of the modes coupling be not neglected.

References

[1] Danielson, D. A.; Dynamic buckling loads of imperfection-sensitive structures from perturbation procedures AIAA J. 7,1506-1510 (1969).

[2] Amazigo, J. C.; Buckling of stochastically imperfect structures. In, Buckling of Structures, IUTAM symposium Cambridge, USA, 172-182, 1974.

[3] Amazigo, J. C.; Dynamic Buckling of structures with random imperfections. Stochastic problem in mechanics. Ed. H. Leipolz, University of waterloo press, 243-254, 1974.

[4] Ette, A. M.; On the dynamic buckling of stochastically imperfect finite cylindrical shells under step loading. J. Nig. Asso. Math. Physics, 8,35-46,(2004).

[5] Fraser, W. B.; Buckling of structures with random imperfections. Ph.D. thesis, Harvard University, 1968.

[6] Amazigo, J. C.; Buckling of stochastically imperfect columns on Nonlinear elastic foundations Quart. Appl. Math.29,403-409 (1971).

[7] Cederbaum, G., Librescu, L. and Elishakoff, I.; Response of laminated plates to non-stationary random excitation, Structural safety, 6.99-113 (1989).

[8] Spanos, P. D., Tein, W. Y. and Ghanem, R. C.; Heuristic spectral estimation of Bivarate non-stationary process. Meccanica, 31,207-218, 1996.

[9] Lin, Y. K. and Cai. G. O.; Probabilistic Structural Dynamics, MacGraw-Hill, New-York, 1995.