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Time dependent two phase flows in Magnetohydrodynamics: A Greens function approach

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Abstract

The present article presents a mathematical model to study the time dependent two phase magneto-hydrodynamic (MHD) flow in a parallel plat channel having one phase occupied by electrically conducting fluid and the other phase by non-conducting fluid. Both the phases were incompressible and the flow is assumed to be time dependent. The two regions are coupled by equating the velocity and shear stress at the interface. Using the Green's function approach, expressions for velocity in both phases were obtained for general class of time dependent movement of boundary or sudden change in pressure gradient or both. As a special case, expressions for time dependent velocity fields in both phases were obtained due to sudden change in the pressure gradient.

1.0 Introduction

The study of two phase flow is of great importance because of their applications in several industrial and physical fields. Due to its practical applications, in last thirty years two-phase flow has been widely researched and reported in literature under several idealized assumption [1-5]. For instance, Shail [5] considered the two-phase flow of conducting and non-conducting incompressible fluids in a horizontal channel in the presence of transverse magnetic field. It was shown theoretically that significant increase in the flow rate of conducting fluid can be obtained for suitable choice of ratio of viscosities and the depth of the two phases. Lohrasbi and Sahai [6] studied the thermal characteristics of [5], by accommodating the viscous and Ohmic-dissipation in the energy equation. The two-phase flow was further examined by Malashetty and Leela [7-8] by considering both phases as an electrically conducting and solved the momentum and energy equation analytically for short circuit and open circuit cases respectively.

In all above discussed works momentum and energy equations are assumed to be independent of time. The present mathematical model addresses the transient tow phase flow of viscous, incompressible fluid in a horizontal parallel plate channel having one phase occupied by electrically-conducting fluid and other by non-conducting fluid. A closed form solution is obtained using Green's function approach.

2.0 Governing equation and solutions

The physical situation corresponds to that of the time dependent two phase *mhd* (magneto-hydrodynamic) flow in a horizontal parallel plate channel partially filled with conducting fluid. The physical model shown in Fig. 1 consists of two infinite parallel plates extending *x*- and *z* – directions. The region $0 \le y \le d$ and $d \le y \le h$ are occupied by two different fluids of density ρ_1 and ρ_2 which are electrically non-conducting and conducting respectively. The viscosities of both fluids are also assumed to be different. A constant magnetic field of uniform strength B_o is applied in the *y* – direction. The flow formation inside the channel is due to the application of constant external pressure gradient or (and) by the sudden movement of the boundaries. With these assumptions, the dimensionless governing equations for both phases and corresponding initial, boundary and interface conditions are:

$$\frac{\partial U_1}{\partial t} = -P + \frac{\partial^2 U_1}{\partial Y^2} \tag{2.1}$$

$$\frac{\partial U_2}{\partial t} = -P\alpha + \gamma\alpha \frac{\partial^2 U_2}{\partial Y^2} - \alpha M^2 U_2 \tag{2.2}$$

The initial boundary and interface conditions are:

$$\begin{aligned} &U_1(0,Y) = F_1(Y),\ U_2(0,Y) = F_2(Y), \\ &U_1(t,0) = f_1(t), \\ &U_2(t,H) = f_2(t) \end{aligned} \tag{2.3}$$

$$U_1(t,1) = U_2(t,1),\ \frac{\delta U_1(t,1)}{\partial Y} = Y\frac{\delta U_2(t,1)}{\delta Y} \tag{2.4}$$

The dimensional quantities introduced in equations (2.1 - 2.4) are given in list of symbols. Equation (2.3) represents the general nature of initial and boundary conditions. A specific situation can be handled by proper selection of initial and boundary conditions. It is more convenient to solve the problem with homogeneous boundary conditions. To attain, this let

$$U_i(Y,t) = V_i(Y,t) + \phi_i(Y)\ f_1(t) + \varepsilon_i(Y)\ f_2(t),\ i = 1,\ 2 \tag{2.5}$$

This relation transforms the governing equations (2.1) – (2.4) into the following set of equations:

$$\frac{d^2\phi_1}{dY_1^2} = 0 \tag{2.6}$$

$$\gamma\frac{d^2\phi_2}{dY^2} - M^2\phi_2 = 0 \tag{2.7}$$

$$\left.\begin{aligned} &\phi_1(0) = 1;\ \phi_2(H) = 0, \\ &\phi_1(1) = \phi_2(1),\ \frac{d\phi_1(1)}{dY} = \gamma\frac{d\phi_2(1)}{dY} \end{aligned}\right\} \tag{2.8}$$

$$\frac{d^2\varepsilon_1}{dY_1^2} = 0 \tag{2.9}$$

$$\gamma\frac{d^2\varepsilon_2}{dY^2} - M^2\varepsilon_2 = 0 \tag{2.10}$$

$$\left.\begin{aligned} &\varepsilon_1(0) = 0;\ \varepsilon_2(H) = 1, \\ &\varepsilon_1(1) = \varepsilon_2(1),\ \frac{d\varepsilon_1(1)}{dY} = \gamma\frac{d\varepsilon_2(1)}{dY} \end{aligned}\right\} \tag{2.11}$$

$$\frac{\partial V_1}{\partial t} = -P + \frac{\partial^2 V_1}{\partial Y^2} \qquad (2.12)$$

$$\frac{\partial V_2}{\partial t} = -P\alpha + \gamma\alpha \frac{\partial^2 V_2}{\partial Y^2} - \alpha M^2 V_2 \qquad (2.13a)$$

$$V_1(t,0) = 0,\ V_2(t,H) = 0,\ V_1(t,1) = V_2(t,1) \qquad (2.13b)$$

$$\frac{dV_1(t,1)}{dY} = \mu_r \frac{dV_2(t,1)}{dY} \qquad (2.14)$$

$$V_i(0,Y) = F_i(Y) - \phi_i(Y)\ f_1(0) - \varepsilon_i(Y)\ f_2(0) = F_i^*,\quad i = 1,\ 2 \qquad (2.15)$$

$$P_1 = P + \phi_1 \frac{df_1}{dt} + \varepsilon_1 \frac{df_2}{dt} \qquad (2.16)$$

$$P_2 = P\alpha + \phi_1 \frac{df_1}{dt} + \varepsilon_1 \frac{df_2}{dt} \qquad (2.17)$$

The solutions of equations (2.6 - 2.8) are:

$$\phi_1(Y) = A_1 + A_2 Y, \qquad (2.18)$$

$$\phi_2(Y) = A_3\ Exp\left(\frac{MY}{\sqrt{\gamma}}\right) + A_4\ Exp\left(-\frac{MY}{\sqrt{\gamma}}\right) \qquad (2.19)$$

where A_2 = 1, and A_1, A_3 and A_4 are found from the solution of the following set of equations:

$$\begin{bmatrix} 0 & Exp\left(\frac{MH}{\sqrt{\gamma}}\right) & Exp\left(-\frac{MH}{\sqrt{\gamma}}\right) \\ -1 & Exp\left(\frac{M}{\sqrt{\gamma}}\right) & Exp\left(-\frac{M}{\sqrt{\gamma}}\right) \\ -1 & M\sqrt{\gamma}\, Exp\left(\frac{M}{\sqrt{\gamma}}\right) & -M\sqrt{\gamma}\, Exp\left(-\frac{M}{\sqrt{\gamma}}\right) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Also, the solutions of equations (2.9 -2.11) are:

$$\varepsilon_1(Y) = A_5 Y + A_6, \qquad (2.20)$$

$$\varepsilon_2(Y) = A_7\ Exp\left(\frac{MY}{\sqrt{\gamma}}\right) + A_8\ Exp\left(-\frac{MY}{\sqrt{\gamma}}\right), \qquad (2.21)$$

where A_6 = 0, and A_5, A_7 and A_8 are found from the solution of the following set of equations:

$$\begin{bmatrix} 0 & Exp\left(\frac{MH}{\sqrt{\gamma}}\right) & Exp\left(-\frac{MH}{\sqrt{\gamma}}\right) \\ -1 & Exp\left(\frac{M}{\sqrt{\gamma}}\right) & -Exp\left(-\frac{M}{\sqrt{\gamma}}\right) \\ -1 & M\sqrt{\gamma}\, Exp\left(\frac{M}{\sqrt{\gamma}}\right) & -M\sqrt{\gamma}\, Exp\left(-\frac{M}{\sqrt{\gamma}}\right) \end{bmatrix} \begin{bmatrix} A_5 \\ A_7 \\ A_8 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Using Green's function method, equations (2.12 – 2.15) assume the following form:

$$V_1(t,Y) = \int_0^1 G_{11}(Y,t/Y',t')_{t'=0} F_1^*(Y')\,dY' - \int_{t'=0}^{t} dt' \int_0^1 G_{11}(Y,t/Y',t')(P)\,dY'$$

$$+ \int_1^H G_{12}(Y,t/Y',t')_{t'=0} F_2^*(Y')\,dY' - \int_{t'=0}^{t} dt' \int_1^H G_{12}(Y,t/Y',t')\alpha(P)\,dY' \qquad (2.22)$$

$$V_2(t,Y) = \int_0^1 G_{21}(Y,t/Y',t')_{t'=0} F_1^*(Y')\,dY' - \int_{t''=0}^{t} dt' \int_0^1 G_{21}(Y,t/Y',t')(P_1)\,dY'$$

$$+ \int_1^H G_{22}(Y,t/Y',t')_{t'=0} F_2^*(Y')\,dY' - \int_{t''=0}^{t} dt' \int_1^H G_{22}(Y,t/Y',t')\alpha(P_2)\,dY' \qquad (2.23)$$

Where G_{ij} is the appropriate Green's function found from the homogeneous version of the governing equations (2.12 – 2.15) are:

$$G_{ij}(Y,t/Y',t') = \sum_{n=1}^{\infty} \frac{Exp\left(-\beta_n^2(t-t')\right)}{N_n} \eta_i \psi_{in}(Y)\psi_{jn}(Y'),$$

where $\qquad i = 1,\ 2,\ \&\ j = 1,\ 2,\ \eta_1 = 1,\ \eta_2 = \dfrac{1}{\alpha}. \qquad (2.24)$

$$N_n = \int_0^1 \psi_{1n}^2(Y')\,dY' + \left(\frac{1.}{\alpha}\right)\int_1^H \psi_{2n}^2(Y')\,dY' \qquad (2.25)$$

$$\psi_{1n}(Y) = \sin(\beta_n Y), \qquad (2.26)$$

$$\psi_{2n}(Y) = A_{2n}\,Cos(\lambda_n Y) + \beta_{2n} Sin(\lambda_n Y), \qquad (2.27)$$

where $\lambda_n^2 = \left(\dfrac{\beta_n^2}{\alpha\gamma}\right) - \left(\dfrac{M^2}{\gamma}\right).$

The values of constants A_{2n} and B_{2n} are found from the solution of the following set of equations:

$$\begin{bmatrix} Cos(\lambda_n) & Sin(\lambda_n) \\ -\gamma\lambda_n Sin(\lambda_n) & \gamma\lambda_n Cos(\lambda_n) \end{bmatrix} \begin{bmatrix} A_{2n} \\ B_{2n} \end{bmatrix} = \begin{bmatrix} Sin(\beta_n) \\ \beta_{2n}\,Cos(\beta_n) \end{bmatrix}$$

The eigen-values λ_n and B_n are found as the roots of

$$A_{2n}\,Cos(H\lambda_n) + B_{2n} Sin(H\lambda_n) = 0 \qquad (2.28)$$

3.0 Particular case

The physical situation, in which flow inside the channel is solely caused by uniform pressure gradient, i.e. $F_1(Y) = F_2(Y) = f_1(t) = f_2(t) = 0$, which yields:

$$P_1 = P, P_2 = \alpha P \text{ and } U_i = V_i,\ i = 1,\ 2$$

Using these values in equations (2.22) and (2.23) the dimensional velocity in both phases are:

$$U_1(Y,t) = -\int_{t'=0}^{t} dt \int_0^1 G_{11}(Y,t/Y',t')\psi_{1n}(Y)\psi_{1n}(Y')P\,dY'$$

$$-\int_{t'=0}^{t} dt' \int_{1}^{H} G_{12}(Y,t/Y',t')\psi_{1n}(Y)\psi_{2n}(Y')\,PdY' \tag{3.1}$$

$$U_2(Y,t) = -\int_{t'}^{t} dt' \int_{0}^{1} G_{21}(Y,t/Y',t')\psi_{2n}(Y)\psi_{1n}(Y')P\,dY'$$

$$-\int_{t'}^{t} dt' \int_{1}^{H} G_{22}(Y,t/Y',t')\psi_{2n}(Y)\psi_{2n}(Y')\,PdY' \tag{3.2}$$

$$U_1(Y,t) = -P\sum_{n=1}^{\infty}\frac{\left[1.-Exp\left(-\beta_n^2 t\right)\right]}{\beta_n^3 N_n} Sin(\beta_n Y)[1.-Cos(\beta_n)] - P\sum_{n=1}^{\infty}\frac{\left[1.-Exp\left(-\beta_n^2 t\right)\right]}{\beta_n^2 N_n\lambda_n} Sin(\beta_n Y)$$
$$[A_{2n}\{Sin(\lambda_n H)-Sin(\lambda_n)\}-B_{2n}\{Cos(\lambda_n H)-Cos(\lambda_n)\}] \tag{3.3}$$

$$U_1(Y,t) = -P\sum_{n=1}^{\infty}\frac{\left[1.-Exp\left(\beta_n^2 t\right)\right][A_{2n}Cos(\beta_n Y)+B_{2n}Sin(\beta_n Y)][1.-Cos(\beta_n)]}{\beta^3 N_n}$$

$$-P\sum_{n=1}^{\infty}\frac{\left[1.-Exp\left(-\beta_n^2 t\right)\right][A_{2n}Cos(\lambda_n Y)+B_{2n}Sin(\lambda_n Y)]}{\beta_n^3 N_n\lambda_n} \tag{3.4}$$
$$[A_{2n}\{Sin(\lambda_n H)-Sin(\lambda_n)\}-B_{2n}\{Cos(\lambda_n H)-Cos(\lambda_n)\}]$$

$$N_n = \int_0^1 \psi_{1n}^2(Y')\,dY' + \left(\frac{1.}{\alpha}\right)\int_1^H \psi_{2n}^2(Y')\,dY' = \left[\frac{Y'}{2}-\frac{Sin(2C_nY')}{4}\right]_0^1 + \left(\frac{A_{2n}^2}{\alpha}\right)\left[\frac{Y'}{2}-\frac{Sin(2\lambda_n Y')}{4\lambda_n}\right]_1^H$$
$$+\left(\frac{B_{2n}^2}{\alpha}\right)\left[\frac{Y'}{2}-\frac{Sin(2\lambda_n Y')}{4\lambda_n}\right]_1^H - \left(\frac{A_{2n}B_{2n}}{\alpha\lambda_n}\right)[Cos(2\lambda_n Y')]_1^H$$

4.0 Steady-state solution

$$\frac{\partial^2 U_1}{\partial Y^2} = P \tag{4.1}$$

$$\gamma\frac{\partial^2 U_2}{\partial Y^2} - M^2 U_2 = P, \tag{4.2}$$

$$U_1 = \frac{PY^2}{2} + C_1 Y + C_2, \tag{4.3}$$

$$U_2 = C_3 Cos\ h\left(\frac{MY}{\sqrt{\lambda}}\right) + C_4 Sin\ h\left(\frac{MY}{\sqrt{\lambda}}\right) - \frac{P}{M^2}, \tag{4.4}$$

where $C_2=0$, and C_1, C_3 and C_4 are found from the solution of the following set of equations:

$$\begin{bmatrix} 0 & Cos\ h\left(\frac{MH}{\sqrt{\lambda}}\right) & Sin\ h\left(\frac{MH}{\sqrt{\lambda}}\right) \\ -1 & Cos\ h\left(\frac{M}{\sqrt{\lambda}}\right) & Sin\ h\left(\frac{M}{\sqrt{\lambda}}\right) \\ -1 & M\sqrt{\lambda}\ Sin\left(\frac{M}{\sqrt{\lambda}}\right) & M\sqrt{\lambda}\ Cos\ h\left(\frac{M}{\sqrt{\lambda}}\right) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} \frac{P}{M^2} \\ \frac{P}{2} + \frac{P}{M^2} \\ P \end{bmatrix} \tag{4.5}$$

5.0 Concluding remarks

The novel feature of this work is to present an analytical solution of unsteady two phase flow where one phase is conducting and other phase is non-conducting using Green's function approach.

List of Symbols

d = width of the non-conducting fluid
h = total width of channel
H = dimensionless total width of the channel (h/d)
B_o = magnetic field strength
u_1 = velocity field in non-conducting fluid
u_2 = velocity field in conducting fluid
U_1 = dimensionless velocity in non-conducting fluid ($u_1 d/\mu_1$)
U_2 = dimensionless velocity in conducting fluid ($u_2 d/\mu_1$)
y = dimensionless transverse co-ordinate (h/d)
dp/dx = dimensionless axial pressure gradient
P = dimensionless axial pressure gradient ($-dp/dx(d^3/\rho_1\ v_1^2)$)
M = Hartman number ($B_0 d\sqrt{\sigma/\mu_1}$)
$f_1(t)$ = dimensionless velocity of lower bounding wall
$f_2(t)$ = dimensionless velocity of upper bounding wall
$F_1(t)$ = dimensionless initial velocity in non-conducting fluid
$F_2(t)$ = dimensionless initial velocity in conducting fluid

Greek Symbols

μ = dynamic viscosity
v = kinematic viscosity
ρ = density of fluid
σ = electrical conductivity of conducting fluid
Υ = ratio of dynamic viscosities (μ_2/μ_1)
α = ratio of densities of fluids (ρ_1/ρ_2)

Subscripts

1 = non-conducting fluid domain
2 = conducting fluid domain

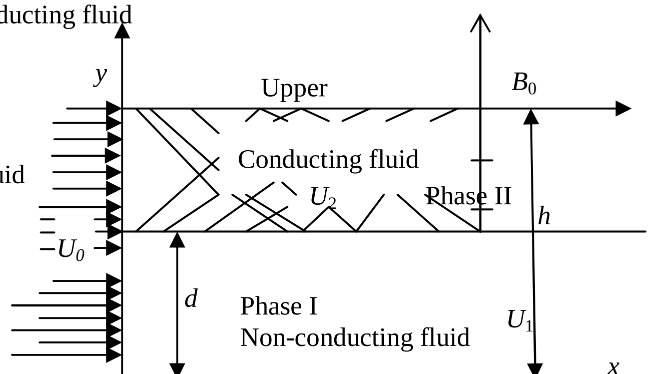

Figure 1: Flow domain

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