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Stability of triangular points in the generalised photogravitational restricted three-body problem with variable mass

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Abstract

The stability of triangular points under the influence of radiation pressure of the bigger primary, oblateness of the smaller primary and variation in mass of the third infinitesimal body has been investigated. It is found that these points are stable for $0 \leq \mu \leq \mu_c$, and unstable for $\mu_c \leq \mu \leq \frac{1}{2}$. It is also seen that the range of stability decreases due to radiation pressure, oblateness and variation in mass of the respective body.

***Keywords*:** stability, triangular points, generalised photogravitational RTBP, variable mass.

1.0 Introduction

The classical restricted three-body problem is unable to discuss the motion of the third infinitesimal body when one of the participating bodies is a source of radiation or an oblate spheroid or of variable mass. In recent times, many perturbing forces, i.e. radiation, oblateness, variation of the mass etc. have been included in the study of the restricted three-body problem. One such variation when the bigger primary is a source of radiation, the smaller one is an oblate spheroid and the mass of the third infinitesimal body varies with time, is of considerable interest in the study of the solar system.

Radzievskii (1950) [8] formulated the photogravitational restricted three body problem and discussed it for three specific bodies: the sun, a planet and a dust particle. Chernikov (1970) [3] extended his work by including aberational deceleration (the Poynting Robertson effect). He demonstrated the instability of the solutions. Sharma (1982) [9] studied the linear stability of triangular libration point of the restricted problem when the more massive primary is a source of radiation and an oblate spheroid as well. Simmons et al. (1985) [10] gave a classical treatment of the more general problem with radiation emanating from both primaries. Dankowicz (1997) [4] described the motion of grains in orbit around asteroids under the influence of radiation pressure originating in the flux of solar photons. His recent paper (2002) accounts for gravitational interactions with the asteroid and the sun and the radiation pressure from the sun.

Vidyakin (1974) [17] studied the effect of oblateness of both primaries on the existence of five stationary solutions. Subbarao and Sharma (1975) [16] investigated the restricted problem with one of the primaries as an oblate spheroid and proved that there was an increase in the coriolis force and the centrifugal force due to oblateness.

Singh and Ishwar (1984, 1985) [11, 12] studied the effect of small perturbations in the coriolis and the centrifugal forces on the location and stability of equilibrium points in the restricted problem with variable mass under the assumption that the mass of the third body varies with time.

They investigated the above problem with the help of Jeans' law (1928) [6] and space-time transformation comparing it to the transformation of Meshcherskii (1949) [7]. The first order normalization in the above problem, which has applications in the study of the nonlinear stability, was also performed by Singh (2006) [15]. In the further paper (1999), they examined the stability of triangular points when both primaries are sources of radiation and oblate spheroids as well. The stability of collinear equilibrium points in the above problem was studied by Singh (2005) [14]. By taking small perturbations in the coriolis and the centrifugal forces in the latter two problems, AbdulRaheem and Singh (2004, 2006) [1, 2] discussed the combined effects of perturbations, radiation and oblateness on the location, and on the stability, of equilibrium points.

In this paper, we wish to study the "stability of triangular points in the generalised photogravitational restricted three-body problem with variable mass". The problem is generalized in the sense that the smaller primary is taken as an oblate spheroid. It is photogravitational because of the bigger primary being a source of radiation.

Singh (2005) [14] investigated the stability of collinear equilibrium points under the effects of both oblateness and radiation, of both primaries; while the present paper considers the radiation and the oblateness of the bigger and smaller primaries respectively under the assumption that the mass of the third body varies with time.

2.0 Equations of motion

The equations of motion of the third body of infinitesimal variable mass when the bigger primary is a source of radiation and smaller one is an oblate spheroid, are

$$\xi'' - 2\omega\eta' = \frac{\partial\Omega}{\partial\xi} ,$$

and

$$\eta'' + 2\omega\xi' = \frac{\partial\Omega}{\partial\eta} , \tag{2.1}$$

with

$$\Omega = \frac{1}{2}\left(\omega^2 + \frac{\beta^2}{4}\right)\left(\xi^2 + \eta^2\right) + \gamma^{\frac{3}{2}}\frac{(1-\mu)}{r_1}q_1 + \gamma^{\frac{3}{2}}\frac{\mu}{r_2} + \gamma^{\frac{5}{2}}A_2\frac{\mu}{2r_2^3}, \tag{2.2}$$

$$r_1^2 = \left(\xi + \mu\gamma^{\frac{1}{2}}\right)^2 + \eta^2,$$
$$r_2^2 = \left[\xi - (1-\mu)\gamma^{\frac{1}{2}}\right]^2 + \eta^2, \gamma = \frac{m}{m_0}, \tag{2.3}$$

Here, the mass m of the third body varies with time t such that $m = m_0$ at $t = 0$. The parameter μ is the ratio of the mass of the smaller primary to the total mass of the primaries and $0 < \mu < ½$. Dashes indicate differentiation with respect to Γ where $dt = \gamma^{-k} d\Gamma$, k is the constant of proportionality. β is a constant due to the variation in mass governed by Jeans' law (1928) [6], ω is the mean motion of the primaries given by

$$\omega^2 = 1 + \frac{3}{2}A_2, \quad 0 < A_2 << 1, \tag{2.4}$$

where A_2 is the oblateness coefficient of the smaller primary q_1 is the mass reduction factor constant of the bigger primary given by the relation

$$q_1 = 1 - \frac{F_p}{F_g}, \quad 0 < 1 - q_1 << 1, \tag{2.5}$$

in which F_g and F_p being gravitational attraction force and the solar radiation pressure force respectively.

3.0 Locations of triangular points

The locations of triangular points are the solutions of

$$\frac{\partial\Omega}{\partial\xi}=0,$$
$$\frac{\partial\Omega}{\partial\eta}=0,\ \eta\neq 0.$$

These yield us

$$\frac{q_1}{r_1^3}-\frac{1}{r_2^3}-\frac{3}{2}A_2\frac{\gamma}{r_2^5}=0,$$
$$r_1=\gamma^{\frac{1}{2}}\left[q_1\Big/\left(\omega^2+\frac{\beta^2}{4}\right)\right]^{\frac{1}{3}},$$
$$r_2=\gamma^{\frac{1}{2}}\left[1+\left\{\left(\frac{\beta^2}{4}+1\right)^{\frac{5}{3}}-1\right\}\left(\frac{\beta^2}{4}+1\right)^{-1}\frac{A_2}{2}\right]\left(\frac{\beta^2}{4}+1\right)^{\frac{1}{3}}, \qquad \text{-(3.1)}$$

In order to find the coordinates of triangular points we write $q_1=1-\beta_1,\quad 0<\beta_1<<1,$ and make use of equations (2.3), (2.4), (3.1) and, then neglect second and higher order terms in β_1, A_2.

The coordinates of triangular points L_4 and L_5 are

$$\xi=\gamma^{\frac{1}{2}}\left[\frac{1}{2}-\mu-\frac{\beta_1}{3}\left(\frac{\beta^2}{4}+1\right)^{-2/3}-\frac{A_2}{2}\right],$$

$$\eta=\pm\frac{1}{2}\gamma^{\frac{1}{2}}\left(\frac{\beta^2}{4}+1\right)^{-1/3}\left[4-\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}}-\frac{4}{3}\beta_1+\left\{2\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}}-4\left(\frac{\beta^2}{4}+1\right)^{-1}A_2\right\}\right]^{\frac{1}{2}}, \qquad (3.2)$$

where the positive sign corresponds to L_4 and the negative to L_5. These points form simple triangles with the primaries different from the case of the classical problem where these points make equilateral triangles. It is evident that the positions of these points are affected by those factors which appear due to radiation pressure, oblateness and variation in mass of the respective body. If these factors are omitted i.e. when $\beta_1=0, A_2=0, \gamma=1, \beta=0,$ we get the same results as the classical case.

4.0 Stability of triangular points

We consider now what happens to the infinitesimal body if it is displaced a little from one of the triangular points. We assume that the body has given a very small displacement and small velocity. If its motion is rapid departure from the vicinity of the point, we can call such a position of equilibrium an unstable one. If however, the body merely oscillates about the point, it is said to be a stable position. In order to examine the stability of a solution we apply this small displacement method.

Let the position of any triangular point be (ξ_0,η_0) and let the infinitesimal body be displaced to the point $\xi_0+u,\eta_0+v,$ where u,v are small displacements.

Then substituting these quantities into the equations of motion (2.1) and expanding in a Taylor's series, we obtain the linear variational equations as

$$u''-2\omega v' = u\left(\Omega^0_{\xi\xi}\right)+v\left(\Omega^0_{\xi\eta}\right),$$
$$v''+2\omega u' = u\left(\Omega^0_{\eta\xi}\right)+v\left(\Omega^0_{\eta\eta}\right), \qquad (4.1)$$

Here only linear terms in u and v have been taken. The second partial derivatives of Ω are denoted by subscripts. The superscript 0 indicates that the derivative is to be evaluated at the triangular points $\left(\xi_0,\eta_0\right)$.

The determinantal equation obtained by inserting a trial solution of the form

$$u = Ae^{\lambda\Gamma},\ \ v = Be^{\lambda\Gamma},$$

into equations of (4.1) is

$$\begin{vmatrix} \lambda^2-\Omega^0_{\xi\xi} & -2\omega\lambda-\Omega^0_{\xi\eta} \\ 2\omega\lambda-\Omega^0_{\xi\eta} & \lambda^2-\Omega^0_{\eta\eta} \end{vmatrix} = 0$$

The fourth-order characteristic equation for λ is

$$\lambda^4-\left(\Omega^0_{\xi\xi}+\Omega^0_{\eta\eta}-4\omega^2\right)\lambda^2+\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}-\left(\Omega^0_{\xi\eta}\right)^2 = 0 \qquad (4.2)$$

If all the λ_i obtained from equation (4.2) are pure imaginary numbers, then u and v are periodic and thus give stable periodic solutions in the vicinity of $\left(\xi_0,\eta_0\right)$. if, however, any of the λ_i are real or complex numbers, then u and v increase with time so that the solution is unstable. It can happen, however, that the solution contains constant terms in the place of exponentials. The solution is then stable if the remaining exponentials are purely imaginary. In the case of triangular solutions, we have

$$\Omega^0_{\xi\xi} = \frac{3}{4}\left(\frac{\beta^2}{4}+1\right)^{\frac{5}{3}}\left[1+a_1\beta_1+a_2A_2+\mu\left(b_1\beta_1+b_2A_2\right)\right],$$

$$\Omega^0_{\eta\eta} = \frac{3}{4}\left(\frac{\beta^2}{4}+1\right)\left[4-\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}}+a_3\beta_1+a_4A_2+\mu b_3\beta_1\right],$$

$$\Omega^0_{\xi\eta} = \pm\frac{3}{4}\left(\frac{\beta^2}{4}+1\right)^{\frac{4}{3}}\left[1-2\mu+a_5\beta_1+a_6A_2-\mu\left(b_4\beta_1+b_5A_2\right)\right]\times\left[4-\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}}+a_7\beta_1+b_6A_2\right]^{\frac{1}{2}}$$ so that

$$\Omega^0_{\xi\xi}+\Omega^0_{\eta\eta}-4\omega^2 = 3\left(\frac{\beta^2}{4}+1\right)-4-\frac{3}{2}A_2+3\left(\frac{\beta^2}{4}+1\right)A_2\mu,$$

$$\Omega^0_{\xi\xi}\cdot\Omega^0_{\eta\eta} = \frac{9}{16}\left(\frac{\beta^2}{4}+1\right)^{\frac{8}{3}}\left[4-\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}}+a_8\beta_1+a_9A_2+\mu\left(b_7\beta_1+b_8A_2\right),\right]$$

$$\Omega^0_{\xi\xi}\cdot\Omega^0_{\eta\eta}-\left(\Omega^0_{\xi\eta}\right)^2 = \frac{9}{4}\left(\frac{\beta^2}{4}+1\right)^{\frac{8}{3}}\left[4-\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}}+\left\{\frac{4}{3}-\frac{2}{3}\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}}\right\}\beta_1+\right.$$

$$\left\{2\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}}+16\left(\frac{\beta^2}{4}+1\right)^{-1}-5\left(\frac{\beta^2}{4}+1\right)^{-1/3}\right\}A_2\Bigg]\mu(1-\mu), \quad (4.3)$$

with

$$a_1=\frac{2}{3}\left[1-2\left(\frac{\beta^2}{4}+1\right)^{-2/3}\right],\ b_1=\frac{8}{3}\left(\frac{\beta^2}{4}+1\right)^{-2/3}-\frac{2}{3}$$

$$a_2=\frac{5}{2}\left(\frac{\beta^2}{4}+1\right)^{-1}-2,\ b_2=4,$$

$$a_3=\frac{4}{3}-\frac{2}{3}\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}},\ b_3=\frac{2}{3}\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}}-\frac{8}{3},$$

$$a_4=2\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}}+6\left(\frac{\beta^2}{4}+1\right)^{-1}-\frac{5}{2}\left(\frac{\beta^2}{4}+1\right)^{-1/3},\ b_4=\frac{2}{3},$$

$$a_5=\frac{2}{3}\left[1-\left(\frac{\beta^2}{4}+1\right)^{-2/3}\right],\ b_5=5\left(\frac{\beta^2}{4}+1\right)^{-1},$$

$$a_6=\frac{5}{2}\left(\frac{\beta^2}{4}+1\right)^{-1}-1,\ b_6=2\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}}-4\left(\frac{\beta^2}{4}+1\right)^{-1},$$

$$a_7=-\frac{4}{3},\ b_7=\frac{4}{3}\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}}+\frac{32}{3}\left(\frac{\beta^2}{4}+1\right)^{-2/3}-8,$$

$$a_8=\frac{16}{3}-\frac{4}{3}\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}}-\frac{16}{3}\left(\frac{\beta^2}{4}+1\right)^{-2/3},\ b_8=16-4\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}},$$

$$a_9=4\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}}+16\left(\frac{\beta^2}{4}+1\right)^{-1}-5\left(\frac{\beta^2}{4}+1\right)^{-1/3}-8, \quad (4.41)$$

Applying equations (4.3) and (4.4) in the equation (4.2), we obtain

$$\lambda^4+b\lambda^2+c=0$$

where

$$b=-\left[3\left(\frac{\beta^2}{4}+1\right)-4-\frac{3}{2}A_2+3\left(\frac{\beta^2}{4}+1\right)A_2\mu\right],$$

$$c=\frac{9}{4}\left(\frac{\beta^2}{4}+1\right)^{\frac{8}{3}}\left[4-\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}}+\left\{\frac{4}{3}-\frac{2}{3}\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}}\right\}\beta_1+\right.$$

$$\left\{2\left(\frac{\beta^2}{4}+1\right)^{\frac{2}{3}}+16\left(\frac{\beta^2}{4}+1\right)^{-1}-5\left(\frac{\beta^2}{4}+1\right)^{-1/3}\right\}A_2\Bigg](1-\mu)\mu,$$

This is a quadratic equation in λ^2. Its roots are

$$\lambda^2 = \frac{1}{2}\left[-b \pm \sqrt{\Delta}\right], \tag{4.5}$$

where Δ is the discriminant given by $\Delta = f\mu^2 + g\mu + h,$ with

$$f = 36\left(\frac{\beta^2}{4}+1\right)^{\frac{8}{3}} - 9\left(\frac{\beta^2}{4}+1\right)^{\frac{10}{3}} + \left\{12\left(\frac{\beta^2}{4}+1\right)^{\frac{8}{3}} - 6\left(\frac{\beta^2}{4}+1\right)^{\frac{10}{3}}\right\}\beta_1 +$$

$$+\left\{18\left(\frac{\beta^2}{4}+1\right)^{\frac{10}{3}} + 144\left(\frac{\beta^2}{4}+1\right)^{\frac{5}{3}} - 45\left(\frac{\beta^2}{4}+1\right)^{\frac{7}{3}}\right\}A_2 > 0,$$

$$g = -\left[f + \left\{24\left(\frac{\beta^2}{4}+1\right) - 18\left(\frac{\beta^2}{4}+1\right)^2\right\}A_2\right] < 0,$$

$$h = 16 + 9\left(\frac{\beta^2}{4}+1\right)^2 - 24\left(\frac{\beta^2}{4}+1\right) - \left\{9\left(\frac{\beta^2}{4}+1\right) - 12\right\}A_2 > 0,$$

Now,

$$\frac{d\Delta}{d\mu} = 2f\mu + g < 0 \qquad \text{for} \qquad 0 < \mu < \frac{1}{2}, \ (\Delta)_{\mu=0} = h \approx 1,$$

$$(\Delta)_{\mu=\frac{1}{2}} = -\frac{f}{4} - \left\{12\left(\frac{\beta^2}{4}+1\right) - 9\left(\frac{\beta^2}{4}+1\right)^2\right\}A_2 + h \approx -\frac{23}{4}, \ [As\ \beta_1 << 1, A_2 << 1.]$$

Therefore Δ is a strictly decreasing function of μ in the closed interval [0, ½] and has values of opposite signs at the end points μ=0 and μ = ½ . Consequently, there is one and only one value of μ say μ_c in the open interval (0, ½) for which Δ vanishes.

There are three possible regions of the values of μ:

(*i*) When $0 \le \mu \le \mu_c$, Δ is positive, the values of λ^2 given by (4.5) are negative and all the four roots of the characteristic equation are distinct pure imaginary. This shows that the triangular point in question is stable.

(*ii*) When $\mu = \mu_c$, Δ is zero. Both the values of λ^2 given by equation (4.5) are same. So the solutions of the variational equations contain secular terms and consequently the triangular point is unstable.

(*iii*) When $\mu_c < \mu \le$ ½ , Δ is negative. This indicates that the real parts of two of the characteristic roots are positive and so the triangular point is unstable.

Hence for $0 \le \mu < \mu_c$ we have stability and for $\mu_c \le \mu \le$ ½ we have instability.

5.0 Critical mass and concluding remarks

The critical value μ_c of the mass parameter is a root of the equation $\Delta = 0$. Restricting only linear terms in β^2 in the expansion of the type $\left(1+\frac{\beta^2}{4}\right)^n$ and neglecting the second and higher order terms in $\beta^2, \beta_1 and A_2$, we have

$$\mu_c = \mu_0 + P, \tag{5.1}$$

where μ_0 is the critical value of μ in the classical case given as

$$\mu_0 = \frac{1}{2}\left(1-\sqrt{\frac{23}{27}}\right) = 0.03852\cdots \tag{5.2}$$

$$P = -\frac{19}{9\sqrt{621}}\beta^2 - \frac{2}{27\sqrt{69}}\beta_1 + \frac{1}{9}\left(1-\frac{13}{\sqrt{69}}\right)A_2 \tag{5.3}$$

The range of stability increases or decreases or remains unchanged according as $P >\leq 0$. It is here noticed that μ_c depends upon the factors β_1, A_2, β^2 appearing due to radiation pressure, oblateness and variation of the mass of the respective body. This is contrary to the classical case where the critical mass is a constant quantity. If these factors are omitted i.e., when $\beta_1 = 0, A_2 = 0, \beta^2 = 0,$ the stability behaviour of triangular points coincides with the classical case.

If $\beta_1 = 0, A_2 = 0,$ then μ_c becomes the same as that of the unperturbed result of Singh and Ishwar (1985) [12]. If $\beta = 0, A_2 = 0, \mu_c$ becomes the result of Sharma (1982) [9] in which $A_1 = 0$ If $\beta=0$, μ_c corresponds to the result obtained by Singh and Ishwar (1999) [11] in which $q_2 = 1, A_1 = 0$.

Hence μ_c is different from others so obtained. It is clear from the equation (5.3) that the range of stability decreases due to radiation pressure, oblateness or variation of the mass of the respective body. It is noticed that, due to the introduction of the variable mass, the range of stability decreases fast. If the third body is of constant mass, i.e. $\beta = 0, \gamma = 1$, then it verifies the unperturbed results of AbdulRaheem & Singh (2006) [2], while the bigger and the smaller primaries be spherical and non luminous respectively. In this case, equation (3.2) provides us the same positions of triangular points as those of AbdulRaheem and Singh (2004) [1]. However, equation (3.2) shows that the locations of the triangular points are affected by radiation, oblateness and varying mass of the respective body and they form simple triangles with the line joining the primaries. It is also observed, in the fourth section, that the triangular points $L_{4,5}$ in the present problem are stable for $0 \leq \mu < \mu_c$ and unstable for $\mu_c \leq \mu \leq ½$, while the collinear points $L_{1,2,3}$ in Singh (2005) [14] remain unstable for any value of the mass ratio μ.

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