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Newton's equation of motion in the gravitational field of an oblate earth

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Abstract

In this paper, we derived Newton's equation of motion for a satellite in the gravitational scalar field of a uniformly rotating, oblate spheriodal Earth using spheriodal coordinates. The resulting equation is solved for the corresponding precession and the result compared with similar ones.

Keywords: Oblate spheriodal, Planetary equation, Anomalous precession , Successive approximation.

1.0 Introduction

The Newtonian theory of universal gravitation treats the motion planetary bodies around the sun with the assumption that the sun is a perfect sphere. And it is well known how to formulate and solve Newton's equation of motion for a particle of non- zero rest mass in the gravitational fields of massive spherical bodies. These equations are given in the spherical polar coordinates (r, θ, ϕ) with origin at the center of the body by

$$\ddot{r} + r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta = \frac{k}{-r^2} \tag{1.1}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta = 0 \tag{1.2}$$

$$r\ddot{\theta}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\phi\cos\theta = 0 \tag{1.3}$$

Where k = GM_o and M_o is the rest mass of the body and G is universal gravitational constant. Equations (1.1), (1.2) and (1.3) can be solve to give the orbit equation (Newtonian planetary equation) which is given by

$$\frac{d^2u}{d\phi^2} + u = \frac{k}{\ell^2} \tag{1.4}$$

where $u(\phi) = \dfrac{1}{r(\phi)}$

ℓ = angular momentum per unit rest mass
k = GM_o

The solution of the Newton's planetary equation (1.4) is given by

$$r(\phi) = \frac{\frac{l^2}{k}}{1 + e\cos(\phi + \alpha)} \tag{1.5}$$

where e and α are constants of motion. Consequently the orbit of a planet or satellite is a fixed conic section of eccentricity e and epoch α contrary to the fact that the orbits of planet and satellite preccess every one complete revolution.

The Einstein's theory of general relativity equally treats the motion of planetary objects around the sun under the assumption that the sun is a perfect sphere (Schwarzschild space – time). The Einstein's planetary equation as derived from the Schwarzschild's line element and is given by

$$\frac{d^2u}{d\phi^2} + u = \frac{k}{\ell^2} + \frac{3ku^2}{c^2} \tag{1.6}$$

(Anderson 1967) [1], whose solution is $u(\phi) = \frac{k}{\ell^2}\left(1 + e\cos(\phi - \delta\phi)\right)$ (1.7)

corresponding to a conic sectional orbit which precesses in the plane in such a way that it precession angle, Δ is

$$\Delta = \frac{6\pi GM}{c^2\left(1 - e^2\right)a} \tag{1.8}$$

per revolution, where a is the semi-major axis and e is the eccentricity of the orbit. Although the term, $3ku^2$ appearing on the right hand side of c^2 equation (1.6) constitute a perturbation which is responsible for the precession of planetary bodies around the sun and indeed the satellite around their planets, the spherical assumption for the sun and the planets underscore the real fact of nature that all the planets, stars and galaxies in the universe are rotating and are spheroidal in shape. Therefore, treating them as a perfect sphere is at best an approximation for the sake of mathematical convenience.

It is in view of this, that Vinti in 1960 [7] concluded that Gravitational scalar potential of the imperfect spherical earth is governed by the second harmonics (pole of order 3) and the forth harmonics (pole of order 5). Before Vinti's position paper on the need to take care of spheroidal nature of planets, Garfinkel (1959) [2] and Stern (1957) [6] have suggested a general mathematical form for the gravitational scalar potential of the spheriodal earth and showed how to estimate some parameters for use in the study of satellite orbits. Since then, there has remained the need to extend the theory of motion of planets and satellite from the fields of bodies with perfect spherical geometry to those of more general spheroidal geometry. Therefore in this paper we derive an alternative equation for the orbital precession of a satellite in the gravitational scalar potential of an oblate spheriodal and rotating earth for comparison with well known similar theories.

2.0 Theory

2.1 The gravitational field intensity of an oblate spheroidal earth rotating uniformly on its Axis

According to Rikitate et al (1987a) [4] spheroidal coordinates are correlated to rectangular coordinates by

$$X = \left(u^2 + \xi^2\right)^{\frac{1}{2}} \sin\theta\cos\phi$$

$$Y = \left(u^2 + \xi^2\right)^{\frac{1}{2}} \sin\theta\sin\phi$$

$$Z = u\cos\theta \tag{2.1}$$

where u is the radial distance from the centre of the spheroidal body and $\xi = ae$ where a is

the semi-major radius and e is the eccentricity defined as $e = \frac{\left(a^2+b^2\right)^{\frac{1}{2}}}{a}$ in which b is the semi-major axis. Using these coordinates, Rikitake et al (1987b) [5] showed that the gravitational scalar potential of an oblate spheroidal earth rotating uniformly or its axis is

$$\Phi_g(u,\theta,\phi) = \frac{GM}{\xi}\tan^{-1}\frac{\xi}{u} - \frac{\omega^2 a^2 q(u)}{3q(b)}p_2(\cos\theta) + \frac{1}{2}\omega^2\left(u^2+\xi^2\right)\sin^2\theta \tag{2.2}$$

The first two terms on the right handside of this equation represent the potential due to attraction of a spheroidal earth and the last term on the right is the potential due to centrifugal force of a rotating earth

and $$q(u) = -Q_2\left(\frac{iu}{\xi}\right) = \frac{1}{2}\left\{\left(1-\frac{3u^2}{\xi^2}\right)\tan^{-1}\frac{\xi}{u} - \frac{3u}{\xi}\right\} \tag{2.3}$$

$$\tan^{-1}\frac{\xi}{u} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{\xi}{u}\right)^{2n+1} \tag{2.4}$$

following the definition of gravitational field intensity

$$g(u,\theta,\phi) = -\nabla\phi_g = g_u\hat{u} + g_\theta\hat{\theta} + g_\phi\hat{\phi} \tag{2.5}$$

where

$$g_u = -\frac{\partial\Phi_g}{h_u\partial u} = \left(\frac{u^2+\xi^2}{u^2+\xi^2\cos^2\theta}\right)^{\frac{1}{2}}\left\{\frac{GM}{u^2+\xi^2} - \frac{\omega^2 a^2 p'2(\cos\theta)q'(u)}{3q(b)} + \omega^2 u\sin^2\theta\right\} \tag{2.6}$$

$$g_\theta = -\frac{\partial\Phi_g}{h_\theta\partial\theta} = \left(\frac{1}{u^2+\xi^2\cos^2\theta}\right)^{\frac{1}{2}}\left\{\frac{\omega^2 a^2 q(u)p'2(\cos\theta)}{3q(b)} - \omega^2 u\sin^2\theta\right\} \tag{2.7}$$

and $$g_\phi = 0 \tag{2.8}$$

2.2 Instantaneous acceleration of a satellite in oblate spheriodal coordinates

Using equation (2.1) together with the theory of orthogonal curvilinear coordinates, the expression for the instantaneous acceleration of a satellite is given by

$$a = a_u\hat{u} + a_\theta\hat{\theta} + a_\phi\hat{\phi} \tag{2.9}$$

$$a_u(u,\theta,\phi) = \left(\frac{u^2+\xi^2}{u^2+\xi^2\cos^2\theta}\right)^{-\frac{1}{2}}\left\{\ddot{u} + \frac{u\dot{u}^2\xi^2\sin^2\theta}{\left(u^2+\xi^2\cos^2\theta\right)\left(u^2+\xi^2\right)} + u\dot{\theta}\left(\frac{u^2+\xi^2}{u^2+\xi^2\cos^2\theta}\right)\right.$$
$$\left. - u\dot{\phi}\left(\frac{u^2+\xi^2}{u^2+\xi^2\cos^2\theta}\right)\sin^2\theta - \left(\frac{2\dot{u}\dot{\theta}\sin\theta\cos\theta}{u^2+\xi^2\cos^2\theta}\right)\right\} \tag{2.10}$$

$$a_\theta(u,\theta,\phi)=\left(\frac{1}{u^2+\xi^2\cos^2\theta}\right)^{-\frac{1}{2}}\left\{\ddot\theta+\frac{\dot u^2\xi^2}{\left(u^2+\xi^2\right)\left(u^2+\xi^2\cos^2\theta\right)}+\frac{u\dot u\dot\theta}{u^2+\xi^2\cos^2\theta}\right.$$

$$\left.-\frac{\dot\theta^2\xi^2\sin\theta\cos\theta}{u^2+\xi^2\cos^2\theta}\sin^2\theta-\frac{\phi^2\left(u^2+\xi^2\right)\sin\theta\cos\theta}{u^2+\xi^2\cos^2\theta}\right\} \tag{2.11}$$

$$a_\phi(u,\theta,\phi)=\left(u^2+\xi^2\right)^{-\frac{1}{2}}\left\{\left(u^2+\xi^2\right)\ddot\phi\sin\theta+2\left(u^2+\xi^2\right)\dot\theta\dot\phi\cos\theta+2u\dot u\dot\phi\sin\theta\right\} \tag{2.12}$$

2.3 The Equations of Motion

Newton's law of motion in the gravitation field is given by

$$\underline{a}=\underline{g} \tag{2.13}$$

It follows that the components of the equation of motion of a satellite in the gravitational field of a rotating oblate spheroidal earth are given by equating (2.10), (2.11) and (2.12) to equations (2.6), (2.7) and (2.8) respectively, thus obtaining

$$\left(\frac{u^2+\xi^2}{u^2+\xi^2\cos^2\theta}\right)^{-\frac{1}{2}}\left\{\ddot u+\frac{u\dot u^2\xi^2\sin^2\theta}{\left(u^2+\xi^2\cos^2\theta\right)\left(u^2+\xi^2\right)}+u\dot\theta\left(\frac{u^2+\xi^2}{u^2+\xi^2\cos^2\theta}\right)\right.$$

$$\left.-u\dot\phi\left(\frac{u^2+\xi^2}{u^2+\xi^2\cos^2\theta}\right)\sin^2\theta-\left(\frac{2\dot u\dot\theta\sin\theta\cos\theta}{u^2+\xi^2\cos^2\theta}\right)\right\}$$

$$=\left(\frac{u^2+\xi^2}{u^2+\xi^2\cos^2\theta}\right)^{\frac{1}{2}}\left\{\frac{GM}{a^2+\xi^2}-\frac{\omega^2a^2p_2'(\cos\theta)q'(u)}{3q(b)}+\frac{1}{2}\omega^2u\sin^2\theta\right\} \tag{2.14}$$

At the equatorial plane $\theta=\frac{\pi}{2}$ and equation (2.14) becomes

$$\ddot u+\frac{\dot u^2\xi^2}{u\left(u^2+\xi^2\right)}=\frac{GM}{u^2}-\frac{\omega^2a^2q'(u)}{6q(b)}\left(\frac{u^2+\xi^2}{u^2}\right)-\left(\omega^2-\dot\phi^2\right)\left(\frac{u^2+\xi^2}{u}\right) \tag{2.15}$$

In $\theta-$direction we have the equation of motion as

$$\left(\frac{1}{u^2+\xi^2\cos^2\theta}\right)^{-\frac{1}{2}}\left\{\ddot{\theta}+\frac{\dot{u}^2\xi^2}{\left(u^2+\xi^2\right)\left(u^2+\xi^2\cos^2\theta\right)}+\frac{u\dot{u}\dot{\theta}}{u^2+\xi^2\cos^2\theta}\right.$$

$$-\frac{\dot{\theta}^2\xi^2\sin\theta\cos\theta}{u^2+\xi^2\cos^2\theta}\sin^2\theta-\frac{\phi^2\left(u^2+\xi^2\right)\sin\theta\cos\theta}{u^2+\xi^2\cos^2\theta}=-\frac{\partial\Phi_g}{h_\theta\partial\theta} \tag{2.16}$$

$$=\left(\frac{1}{u^2+\xi^2\cos^2\theta}\right)^{\frac{1}{2}}\left\{\frac{\omega^2a^2q(u)p_2'(\cos\theta)}{3q(b)}-\omega^2\left(u^2+\xi^2\right)\sin\theta\cos\theta\right\}$$

In $\phi-$direction we have the equation of motion as

$$\left(u^2+\xi^2\right)^{-\frac{1}{2}}\left\{\ddot{\phi}\sin\theta+2\dot{\theta}\dot{\phi}\cos\theta+\frac{2u\dot{u}\dot{\phi}\sin\theta}{u^2+\xi^2}\right\}=0 \tag{2.17}$$

Equations (2.15, 2.16, and 2.17) are the required equations of motion which can be solved as follows: equation 2.17 integrates uniformly to give

$$\dot{\phi}=\frac{l}{u^2+\xi^2} \tag{2.18}$$

where l is a constant. Equation (2.18) is the azimuthal solution. Rewriting the radial equation (2.15)

$$\ddot{u}+\frac{\dot{u}^2\xi^2}{u\left(u^2+\xi^2\right)}=\frac{-GM}{u^2}-\frac{\omega^2a^2q'(u)}{6q(b)}\left(\frac{u^2+\xi^2}{u^2}\right)-\left(\omega^2-\phi^2\right)\left(\frac{u^2+\xi^2}{u}\right) \tag{2.19}$$

Now by the transformations $\dot{u}(u)=w(u)$ and $w^2(u)=z(u)$ (2.20)

equation (2.15) becomes

$$\frac{dz}{du}+P(u)z=Q(u) \tag{2.21}$$

where

$$P(u)=\frac{2\xi^2}{u\left(u^2+\xi^2\right)} \tag{2.22}$$

$$Q(u)=-\frac{2GM}{u^2}-\frac{\omega^2a^2q'(u)}{3q(b)}\left(\frac{u^2+\xi^2}{u^2}\right)-2\left(\omega^2-\dot{\phi}\right)\left(\frac{u^2+\xi^2}{u}\right) \tag{2.23}$$

Equation (2.21) is a linear differential equation of the first order and the solution is given by

$$Z(u)=e^{-h}\left(\int Q(u)e^{h}du+A\right) \tag{2.24}$$

where $h=\int P(u)du$ and A is the constant of integration. Since $e^{-h}=\dfrac{u^2+\xi^2}{u^2}$

and

$$e^{h}=\frac{u^2}{u^2+\xi^2} \tag{2.25}$$

$$Z(u) = \frac{2GM}{\xi}\tan^{-1}\frac{\xi}{u}\left(\frac{u^2+\xi^2}{u^2}\right) - \frac{\omega^2 a^2 q(u)}{3q(b)}\left(\frac{u^2+\xi^2}{u^2}\right) - \left(\omega^2 - \dot{\phi}^2\right)\left(u^2+\xi^2\right) + A\left(\frac{u^2+\xi^2}{u^2}\right) \tag{2.26}$$

At the apsides $u = u_l$ and $\dot{u} = 0$. Therefore, the constant A above is given by

$$A = \frac{2GM}{\xi}\tan^{-1}\frac{\xi}{u} - \frac{\omega^2 a^2 q(u)}{3q(b)} - \left(\omega^2 - \dot{\phi}^2\right)u_l \tag{2.27}$$

Now from equation (2.19)

$$\ddot{u} = \frac{GM}{u^2} - \frac{\omega^2 a^2 q'(u)}{6q(b)}\left(\frac{u^2+\xi^2}{u^2}\right)\left\{\frac{2GM}{\xi}\tan^{-1}\frac{\xi}{u}\left(\frac{u^2+\xi^2}{u}\right) - \frac{\omega^2 a^2 q(u)}{3q(b)}\left(\frac{u^2+\xi^2}{u}\right) - \left(\omega^2 - \dot{\phi}^2\right)\left(u^2+\xi^2\right) + A\left(\frac{u^2+\xi^2}{u}\right)\right\} \tag{2.28}$$

or

$$\ddot{u} = \frac{GM}{u^2} - \frac{\omega^2 a^2 q'(u)}{6q(b)}\left(\frac{u^2+\xi^2}{u^2}\right) - \frac{2GM}{\xi}\tan^{-1}\frac{\xi}{u} + \frac{\omega^2 a^2 \xi^2 q(u)}{3q(b)} - \frac{\xi^2\left(\omega^2 - \dot{\phi}^2\right)}{u} + \frac{A\xi^2}{u^3} \tag{2.29}$$

using the transformation $u(\phi) = \dfrac{1}{v(\phi)}$ then it follows from equation (2.18) that $\dot{u}(\phi) = \dfrac{l}{1+\xi^2 v^2}\dfrac{dv}{d\phi}$ and equation (2.28) becomes

$$-\frac{l^2 v^2}{1+\xi^2 v^2}\left\{\frac{1}{1+\xi^2 v^2}\frac{d^2 v}{d\phi^2} - \frac{2\xi^2}{\left(1+\xi^2 v^2\right)^2}\left(\frac{dv}{d\phi}\right)^2\right\} = -GMv^2 - \frac{\omega^2 a^2 q'\left(\frac{1}{v}\right)\left(1+\xi v^2\right)}{6q(b)}$$
$$-\left(\omega^2 - \dot{\phi}^2\right)\left(\frac{1+\xi^2 v^2}{v}\right) - 2\xi GMv^3\tan^{-1}(\xi v) + \frac{\omega^2 a^2 \xi^2 q'\left(\frac{1}{v}\right)v^3}{3q(b)} + \xi^2\left(\omega^2 - \dot{\phi}^2\right)v - A\xi^2 v^3 \tag{2.30}$$

or

$$\frac{d^2 v}{d\phi^2} - \frac{2\xi^2 v^2}{\left(1+\xi^2 v^2\right)^2}\left(\frac{dv}{d\phi}\right)^2 = \frac{GM}{l}\left(1+\xi^2 v^2\right)^2 + \frac{\omega^2 a^2 q'\left(\frac{1}{v}\right)\left(1+\xi^2 v^2\right)^3}{6q(b)l^2 v^2}$$
$$+ 2\xi\frac{GM\tan^{-1}\xi v\left(1+\xi^2 v^2\right)^2}{l^2} - \frac{\omega^2 a^2 q'\left(\frac{1}{v}\right)\left(1+\xi^2 v^2\right)^2 v}{6q(b)l^2}$$

$$-\xi^2 \frac{\left(\omega^2-\dot{\phi}^2\right)\left(1+\xi^2 v^2\right)^2}{l^2 v} - \frac{A\xi^2\left(1+\xi^2 v\right)^2}{l^2} \tag{2.31}$$

From equation (1.3) and (1.4)

$$q\left(\frac{1}{v}\right) = \frac{1}{2}\tan^{-1}\xi v + \frac{3}{2\xi^2 v^2}\tan^{-1}\xi v - \frac{3}{2\xi v}$$

$$q'\left(\frac{1}{v}\right) = \frac{\xi v^{-2}}{2\left(\xi^2 v^2\right)} - \frac{3}{2\xi\left(1+\xi^2 v^{-2}\right)}\frac{3\tan^{-1}\xi v}{\xi^2 v} - \frac{3}{2\xi} \tag{2.32}$$

Using equation (2.32) in equation (2.31) it becomes

$$\frac{d^2 v}{d\phi^2} - \frac{2\xi^2 v^2}{1+\xi^2 v^2}\left(\frac{dv}{d\phi}\right)^2 = \frac{GM}{l^2} + \frac{4\xi^2 GMv^2}{l^2} - \frac{\omega^2 a^2 \xi^3 v^2}{6q(b)l^2} - \frac{A\xi^2 v}{l^2} + o\left(v^3\right) \tag{2.33}$$

or
$$\frac{d^2 v}{d\phi^2} + v\left(1 - \frac{A\xi^2}{l^2}\right) = \frac{GM}{l^2} + \left(4\xi^2 GM - \frac{\omega^2 a^2 \xi^2}{6q(b)}\right)\frac{v^2}{l^2} + o\left(v^3\right) \tag{2.34}$$

This equation is hereafter called the *complete Newton's equation* of motion for a satellite moving in the equatorial plane of a uniformly rotating oblate spheroidal body in terms of the angular coordinate ϕ. This equation is in agreement with previous result (Howusu and Musongon 2004) [3].

If $\frac{A\xi^2}{l^2} < 1$ and $\frac{\omega^2 a^{\omega} \xi^3}{6q(b)} < \frac{A\xi^2 GM}{l^2}$, equation (2.34) reduces approximately to

$$\frac{d^2 v}{d\phi^2} + v = \frac{GM}{l^2} + \frac{4\xi^2 GMv^2}{l^2} \tag{2.35}$$

The equation (2.35) compares favourably with Einstein's planetary equation of motion. Solving equation (2.35) using the standard method of successive approximation for the corresponding precession, Δ we obtain

$$\Delta = \frac{8\pi\xi^2 (GM)^2}{l^4} \tag{2.36}$$

radian per revolution. This is the Newtonian perihelion precession of satellite due to the earth's uniform rotation on its axis and the equatorial bulge.

3.0 Summary and conclusion

In this paper we derive the "complete Newton's equation of motion" for a satellite moving in the gravitational field of an oblate spheroidal earth which is rotating uniformly on its axis. This equation which apart from recognizing the fact that, Earth is oblate spheroidal also takes into account the fact that, earth is not stationary (as given by Howusu and Musongong (2005) [3] in their paper) but rotates uniformly on its axis. We differ from Howusu and Musongong in that, we added the rotational term in our formulation in order to take care of the Earth periodic rotation every twenty four hours. Therefore, our result is available for comparison with their result and other similar equations. Equation (2.35) is solved for the corresponding precession using method of successive approximation to be $\Delta = \frac{8\pi\xi^2 (GM)^2}{l^2}$ radian per revolution. Our theoretical values for the precession compare adequately with that of Einstein's

{Anderson, (1967) [1]}, and therefore stand on equal footing as others. Consequently, the way is now open for further physical interpretation and hence experimental investigation of this

The most profound fit of this paper is that we derived the exact and complete expression of the 'radial' equation of motion for closed orbits in the equatorial plane in term of the azimuthal angular coordinate as (2.34). This is hence forth available for comparison with the well known corresponding equation in the field a spherical body. For the sake of comparison it may be recalled that the radial equation of motion in equatorial plane of a stationary homogenous spherical body in terms of the azimuthal angular coordinate is given by

$$\frac{d^2 v}{d\phi^2} = \frac{k}{l^2} - v \tag{2.37}$$

Thus,the major difference between (2.34) and (2.37) is that, the right hand side of the former contains terms of all orders of v. And the second major difference is that, the left-hand side of (2.33) contains the term in $\left(\frac{d\upsilon}{d\phi}\right)^2$ which is not present in other similar equations.

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