Journal of the Nigerian Association of Mathematical Physics
***Volume* 11 (November 2007), 261 - 268**
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Construction of two-step block Simpson type method with large region of absolute stability

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Abstract

In this note, we describe the construction of two-step Block Simpson type method with non-uniform order, in continuous approximation form, with large regions of absolute stability. All the discrete schemes used in the block method come from a single continuous formulation. When used in block form, the suggested approach is self- starting, accurate and efficient. Numerical result is included to further justify our present method.

Keywords: Region of Absolute stability (RAS), Simpson's scheme, Block Simpson Type (BST).

AMS (MOS) subject classifications: 65LO5 CR subject classification G.1.7

1.0 Introduction

Our interest is on the numerical solution of the IVP

$$y' = f(x,y) \quad y(a) = y \in R^m \qquad (1.1)$$
$$x \in [a,b],\ y(x) \in R^m$$

but whose solution we have assumed is required on a given set of mesh

$$\prod = \{\ x_n / x_n = a + nh,\ n = 0,1,.....,N\ \},\ h = \frac{(x - x_0)}{N}$$

Here, *N* is a positive integer, h is a fixed step size, specifically $x_0 = a$ *while* y_n, is an approximation to $y(x_n)$, where $y(x)$ is the theoretical solution to (1.1).

The novel property of the present method we shall discuss is that of simultaneously, producing approximations to the solution of the initial value problem in a block at points $x_{n+1}, x_{n+2}, \cdots, x_{n+N}$. Although this method will be formulated in terms of linear multi-step methods, we shall see that they preserve the traditional Runge-Kutta advantage of being self-starting and of permitting easy change of step length. Their advantage over conventional Runge-Kutta methods lies in the fact that they are less expensive in terms of function evaluation for given order see Chollom and Jackiewicz (2003).

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2.0 The multi-step collocation method

In earlier works, multi-step collocation methods based on a matrix inversion algorithm (Onumanyi et al (1994) [5], Yahaya (2004) [8]) for the solution of ODE's of the form (1.1) has been developed.

Following Onumanyi et al (1994).[5]. Consider the collocation method defined for the step $[x_n x_{n+k}]$ by

$$u(x)=\sum_{f=0}^{t-1}\phi_j(x)y_{n+j}+h\sum_{f=0}^{m-1}\psi_j\ f\left(\bar{x}_j,u(\bar{x}_j)\right) \qquad (2.1)$$

where *t* and *m* denote respectively the numbers of interpolation and collocation points used. We assume that $\phi_j(x),\ j=0,......t-1 \quad and \quad \psi_j(x),\ j=0,......m-1$ can be represented in polynomial form

$$\phi_j(x)=\sum_{j=0}^{t+m-1}\phi_{j,j+1}x^i \quad h\Psi_j(x)=h\sum_{j=0}^{t+m-1}\Psi_{j,j+1}x^i \qquad (2.2)$$

then $u(x)$ satisfies the conditions $\quad u(x_{n+i})=y_{n+i,}i=0,....,t-1 \qquad (2.3)$

$$u'(\bar{x}_j)=f\left(\bar{x}_j,u(\bar{x}_j)\right),\quad j=0,.......m-1 \qquad (2.4)$$

with the following conditions on $\phi_j(x),\ j=0,......,t-1 \quad and \quad \psi_j(x),\ j=0,......m-1$

$$\phi_j(x_{n+i})=\delta_{ij},\ j=0,.....t-1:\quad i=0,...,t-1 \qquad (2.5)$$

$$h\psi_j'(\bar{x}_i)=\delta_{ij},\ j=0,...m-1;\quad i-0,...,m-1 \qquad (2.6)$$

and

$$\phi_j'\ (\bar{x}_i)=0\ ,\quad j=0,...t-1;\quad i=0,...,m-1 \qquad (2.7)$$

$$h\psi_j'\ (\bar{x}_i)=\delta_{ij}\, j=0,...m-1;\quad i=0....,m-1 \qquad (2.8)$$

Next writing the equations (2.5)-(2.8) in a matrix equation form, we have

$$DC=I \qquad (2.9)$$

Where I is the identity matrix of dimension $(m+t)\times(m+t)$ and

$$D=\begin{pmatrix}
1 & ... & x_n & x_n^2 & ... & x_n^{t+m-1}\\
1 & ... & x_{n+1} & x_{n+1}^2 & ... & x_{n+1}^{t+m-1}\\
. & & . & . & . & .\\
. & & . & . & . & .\\
. & & . & . & . & .\\
1 & ... & x_{n+t-1} & x_{n+t-1}^2 & ... & x_{n+t-1}^{t+m-1}\\
0 & ... & 1 & 2\bar{x}_0 & ... & (t+m-1)\bar{x}_0^{t+m-2}\\
. & & . & . & & .\\
. & & . & . & & .\\
. & & . & . & & .\\
0 & ... & 1 & 2\bar{x}_{m-1} & ... & (t+m-1)x_{m-1}^{t+m-2}
\end{pmatrix} \qquad (2.10$$

$$C = \begin{bmatrix} \phi_{01} & \phi_{11} & \dots & \phi_{t-1.1} & h\Psi_{01} & \dots & h\Psi_{m-1.1} \\ \phi_{01} & \phi_{12} & \dots & \phi_{t-1.2} & h\Psi_{02} & \dots & h\Psi_{m-1.2} \\ . & . & & . & . & & . \\ . & . & & . & . & & . \\ . & . & & . & . & & . \\ \phi_{0,t+m} & \phi_{1,t+m} & \dots & \phi_{t-1,t+m} & h\Psi_{0,t+m} & \dots & h\Psi_{m-1,t+m} \end{bmatrix} \tag{2.11}$$

From (2.9), the columns of C, which give the continuous coefficients $\phi_j(x),\ j = 0,......t-1 \quad and \quad \psi_j(x),\ j = 0,......m-1$ can be obtained from the corresponding columns of D^{-1}. Thus we have explicitly (2.1) in the form.

$$u(x) = \left(y_{n,...,} y_{n+t+1,} f_{n,...,} f_{n+m-1}\right) c^{\mathrm{T}} \left(1, x, ..., x^{t+m-1}\right)^{\mathrm{T}} \tag{2.12}$$

where in (2.12), *T* denotes the "transpose of"

3.0 Construction of BST (Block Simpson Type) method with large region of absolute stability

In this section we consider the derivation of the continuous formulation of the two-step Block Simpson's type method. Consider the following form of the collocation method defined for the step $\left[x_{n,} x_{n+2}\right]$ by $u(x) = \sum_{j=0}^{t-1} \Phi_j(x) y_{n+j} + h \sum_{j=0}^{m-1} \varphi_j(x) f\left(\overline{x}_j, u\left(\overline{x}_j\right)\right)$ (3.1)

The case we are interested in is that of $k = t = 2$; where k is the step length of the proposed method. $\overline{x}_j = x_{n+j},\ j = 0,1,2$, then (3.1) becomes

$$u(x) = \Phi_0(x) y_n + h\left[\Psi_0(x) f_n + \Psi_1(x) f_{n+1} + \Psi_1(x) f_{n+1} + \Psi_2(x) f_{n+2}\right] \tag{3.2}$$

Yielding the following form for the matrix D in (2.10)

$$D = \begin{pmatrix} 1 & X_n & X_n^2 & X_n^3 \\ 0 & 1 & 2X_n & 3X_n^2 \\ 0 & 1 & 2X_{n+1} & 3X_{n+1}^2 \\ 0 & 1 & 2X_{n+2} & 3X_{n+2}^2 \end{pmatrix}$$

Then the continuous formula (3.2) takes the following form;

$$u(x) = y_n + \frac{\left[2\left(x - x_{n+1}\right)^3 - 3h\left(x - x_{n+1}\right)^2 + 5h^3\right]}{12h^2} f_n + \left[-\frac{\left(x - x_{n+1}\right)^3 + 3h^2\left(x - x_{n+1}\right) + 2h^3}{3h^2}\right] f_{n+1}$$

$$+ \frac{\left[2\left(x - x_{n+1}\right)^3 + 3h\left(x - x_{n+1}\right) - h^3\right]}{12h^2} f_{n+2} \tag{3.3}$$

Next by evaluating equation (3.3) at $x = x_{n+2}$ we obtain;

$$y_{n+2} - y_n = \frac{h}{3}\left\{f_n + 4f_{n+1} + f_{n+2}\right\} \tag{3.4}$$

which is of order P = 4, error constant $E_5 = -{}^{1}/_{90}$ which is the popular discrete Simpson scheme with empty region of Absolute stability (see Lambert 1973) [4]. Similarly at $x = x_{n+1}$, we have

$$y_{n+1} - y_n = \frac{h}{12}\{5f_n + 8f_{n+1} - f_{n+2}\} \tag{3.5}$$

which is of order P = 3, error constant $E_5 = -{}^1/_{24}$ and whose RHS can be labeled negative Adams Moulton Method (AMM), whereas the left hand is on $\left[x_{n,}x_{n+1}\right]$, unlike AMM which is on $\left[x_{n+1,}x_{n+2}\right]$. The two discrete schemes, that is (3.4) and (3.5) constitute the proposed block method

$$\left.\begin{aligned} y_{n+1} - y_n &= \frac{h}{12}\{5f_n + 8f_{n+1} - f_{n+2}\} \\ y_{n+2} - y_n &= \frac{h}{3}\{f_n + 4f_{n+1} + f_{n+2}\} \end{aligned}\right\} \tag{3.6}$$

4.0 Analysis of the proposed method

The convergence analysis of the newly constructed two step block Simpson's type (BST) method (integrators) is determined using the approach by Fatunla (1992) [2], where each block integrator is represented as single step block r-point multi-step method of the form.

$$A^{(o)}Y_m = \sum_{i=1}^{k} A^{(i)}Y_{m-i} + h\sum_{i=0}^{k} B^{(i)}F_{m-i} \tag{4.1}$$

where $A^{(i)}, B^{(i)}$, $i = 0(1)k$ are r by r matrices respectively with elements $a_{ij}^{(i)}, b_{ij}^{(i)}$ *for* $i, j = 1(1)r$.

Specifically, $A^{(0)}$ is an $r \times r$ identity matrix, Y_m , Y_{m-i}, F_m *and* F_{m-i} are vectors of numerical estimate describe below. With the *r*-vector Y_m *and*, F_m (for $n = mr$, $m = 0,1\ldots$) specified as

$$Y_m = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ . \\ . \\ y_{n+r} \end{pmatrix}, \quad F_m = \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ . \\ . \\ f_{n+r} \end{pmatrix}, Y_{m-i} = \begin{pmatrix} y_{n-r} \\ . \\ . \\ y_{n-1} \\ y_n \end{pmatrix}, F_{m-i} = \begin{pmatrix} f_{n-r} \\ . \\ .. \\ f_{n-1} \\ f_n \end{pmatrix} \tag{4.2}$$

***Definition* 4.1 (zero stability)**

For $n = mr$, for some integer $m \geq 0$, a block method (4.1)-(4.2) is said to be zero-stable if the roots R_j, $j = 1(1)k$ of the first characteristic polynomial $\rho(R)$ specified as $\rho(R) = \det\left[\sum_{i=0}^{k} A^{(i)}R^{k-i}\right] = 0$ satisfies $\left|R_j\right| \leq 1$ and for those roots with $\left|R_j\right| \leq 1$, the multiplicity must not exceed one.

***Definition* 4.2(A–stability)**

A numerical method is said to be A- stable if its region of absolute stability contain the whole of the left-hand half-plane $\mathrm{Re}(h\lambda) < 0$.

***Stability Analysis of* 3.6**

For easy analysis; thus in matrix equation form, we have

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{A^{(0)}}\begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}}_{A^{(1)}}\begin{pmatrix} y_{n-1} \\ y_n \end{pmatrix} + h\left[\underbrace{\begin{pmatrix} \frac{8}{12} & \frac{-1}{12} \\ \frac{4}{3} & \frac{1}{3} \end{pmatrix}}_{B^{0}}\begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & \frac{5}{12} \\ 0 & \frac{1}{3} \end{pmatrix}}_{B^{(1)}}\begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix}\right] \tag{4.3}$$

The first characteristic polynomial of the Block method (4.3) is

$$\begin{aligned}\rho(\lambda) &= \det\left(\lambda\left(A^{0}\right) - A^{(1)}\right) \\ &= \det\left[\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\right] \\ &= \det\begin{pmatrix}\lambda, -1 \\ 0, \lambda - 1\end{pmatrix} = \lambda(\lambda-1) - 0 \\ &= \lambda(\lambda-1),\ which\ imply\ \lambda = \ 0\ or\ \lambda = 1\end{aligned} \qquad (4.4)$$

From definition (4.1) and equation (4.4), the block method (3.6) is also consistent as its order P > 1. From Henrici (1962), we can safely assert the convergence of the proposed Block method (3.6).

In the same vein we present pictorial representation of known discrete Simpson and the present block formulation.

***Remarks* 4.3**

Using the matlab package, we were able to plot the stability regions of the proposed lock method (3.6). This is done by reformulating the block method as general linear methods to obtain the values of the matrices A, B, U and V which are then substituted into the stability matrix and the stability function. Then the utilization of maple package yields the stability polynomial of the block method. Using a matlab program we plot the absolute stability region of proposed Two-step block Simpson's type (BST) method and the well known Discrete Simpson's scheme with empty region of absolute stability as shown below.

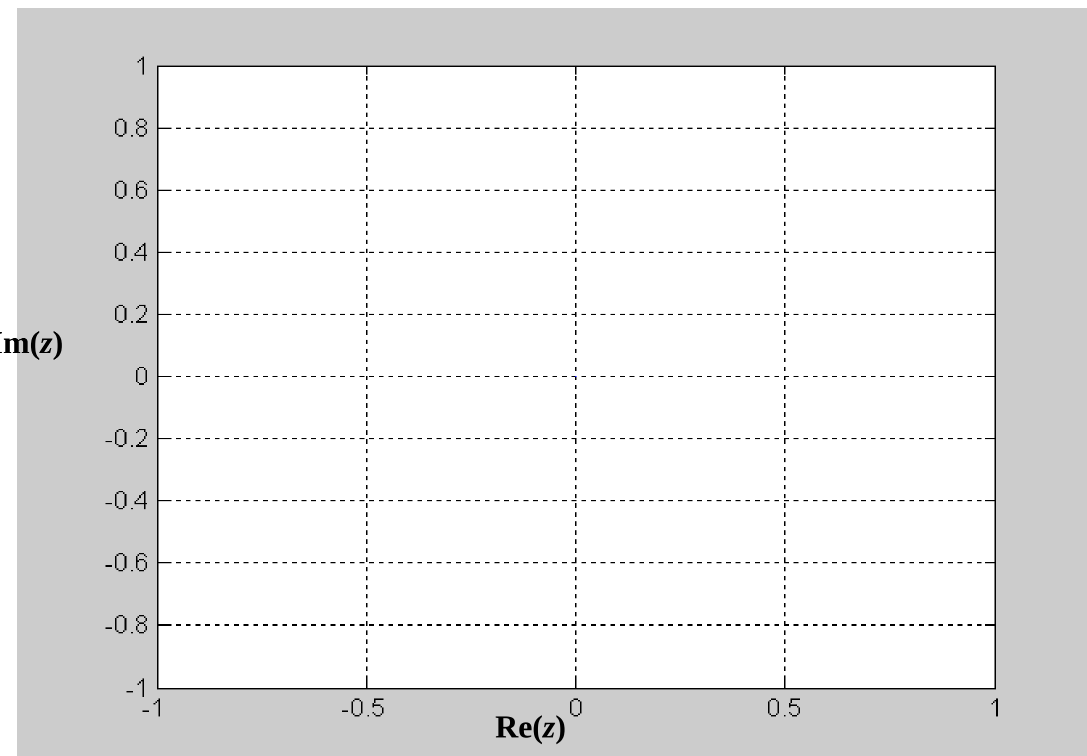

Figure 1: Stability Region of Discrete Simpson Scheme with Empty Region Absolute Stability (RAS)

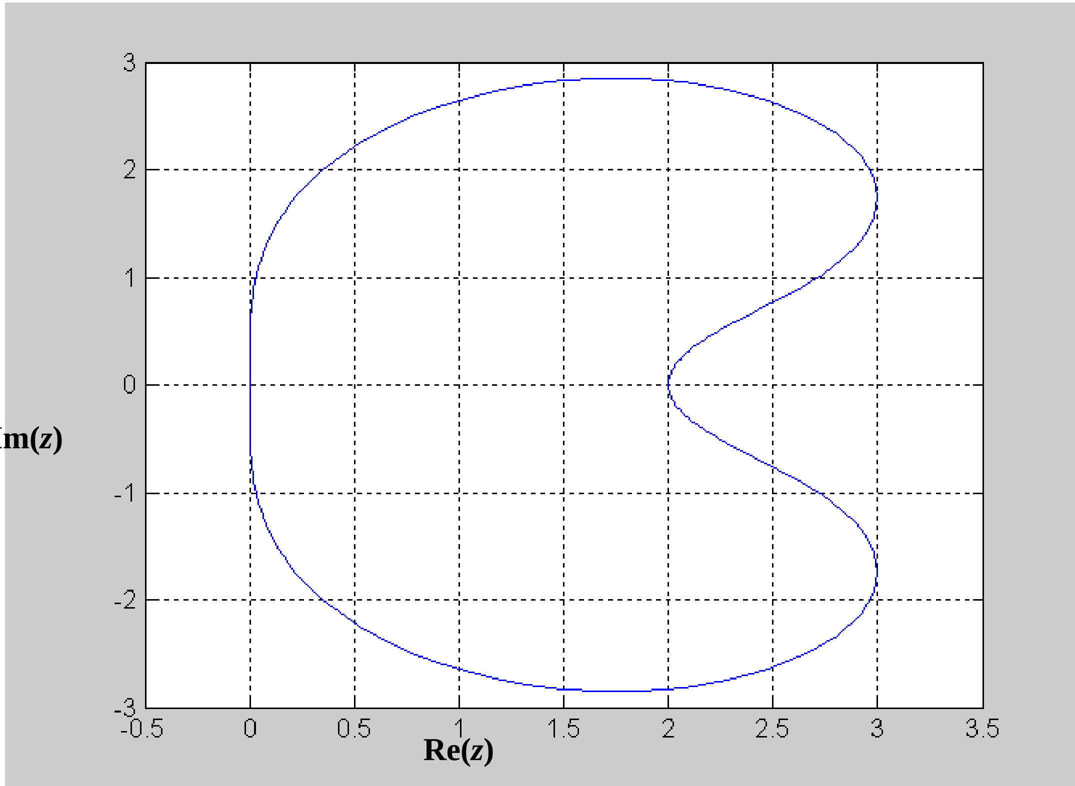

Figure 2: Region of Absolute Stability (RAS) of Proposed Block Simpson Type (BST)

Note: from definition (4.2) and figure 2 above the proposed two step block Simpson type (BST) method (3.6) is A-stable.

5.0 Numerical experiment

To illustrate the potentials of the new formulas constructed in this paper, we will compare their performance on the same problem when Simpson's scheme is used singly (Note that it is not self starting) we shall use classical 4th order Runge-Kutta method to get it started (i.e. to obtain y_1). While the proposed Two-step Block Simpson type method is self- starting on its own.

$$y' = -y, y(0) = 1, y(x) = \ell^{-x}, h = 0.1$$

Table 5.1: The result of Proposed Two-Step Block Simpson's Type (BST) Method, which do not require a starting value

N	*X*	APPROXIMATE VALUE	EXACT VALUE	EXACT ERROR
0	0	1	1	0
1	0.1	0.904833837	0.904837418	3.58144E-06
2	0.2	0.818731118	0.818730753	3.64822E-07
3	0.3	0.740815619	0.740818221	2.60218E-06
4	0.4	0.670320643	0.670320046	5.97364E-07
5	0.5	0.6065288	0.60653066	1.86021E-06
6	0.6	0.54881237	0.548811636	7.33706E-07
7	0.7	0.496584002	0.496585304	1.30159E-06
8	0.8	0.449329765	0.449328964	8.00883E-07

N	*X*	APPROXIMATE VALUE	EXACT VALUE	EXACT ERROR
9	0.9	0.406568775	0.40656966	8.84641E-07
10	1	0.367880261	0.367879441	8.19729E-07
11	1.1	0.332870508	0.332871084	5.75798E-7
12	1.2	0.301165017	0.301194212	8.05388E-7
13	1.3	0.272561443	0.272531793	3.49934E-07
14	1.4	0.246597733	0.246596964	7.69258E-07
15	1.5	0.223129973	0.22313016	1.87148E-07
16	1.6	0.201897238	0.201896518	7.19805E-07
17	1.7	0.182983452	0.182683524	7.18527E-07
18	1.8	0.165299551	0.165398888	6.62978E-07
19	1.9	0.149568627	0.149568619	7.87736E-07
20	2	0.135335886	0.135335283	6.03063E-07

Table 5.2: Simpson's Scheme only, using a starting value (y_1) computed from 4th order R-K Method

N	*x*	Approximate value	Exact value	Exact error
0	0	1	1	0.00000E+00
1	0.1	0.897391667	0.904837418	7.44575-03
2	0.2	0.81961398	0.818730753	9.60645E-03
3	0.3	0.733728798	0.740818221	7.08942E-03
4	0.4	0.672133398	0.670320046	1.81335E-03
5	0.5	0.599664567	0.60653066	6.86609E-03
6	0.6	0.55139388	0.548811636	2.58224E-03
7	0.7	0.489828933	0.496585304	6.57637E-03
8	0.8	0.452616348	0.449328964	3.28738E-03
9	0.9	0.399824957	0.40656966	6.74470E-03
10	1	0.371824976	0.367879441	3.94554E-03
11	1.1	0.326052382	0.332871084	6.81870E-03
12	1.2	0.305764993	0.301194212	4.57078E-03
13	1.3	0.265563197	0.272531793	6.96860E-03
14	1.4	0.251772	0.246596964	5.17504E-03
15	1.5	0.215943377	0.22313016	7.18678E-03
16	1.6	0.207664984	0.201896518	5.76847E-03
17	1.7	0.175216065	0.182683524	7.46746E-03
18	1.8	0.171658719	0.165298888	6.35983E-03
19	1.9	0.141762291	0.149568619	7.80633E-03
20	2	0.142292054	0.135335283	6.95677E-03

6.0 Conclusion

Novelty of the work reported in this paper is that (i.e. the two step Block Simpson's Type (BST) method proposed is self starting, convergent and A-stable as shown by the plotted region of absolute stability (Figure 4.2). Also, it can treat directly both the initial and boundary condition. This method has been tested on simple ODE's and shown to perform satisfactorily, without recourse to any other method to provide a starting value.

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