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Finite element solution of the Boussinesq wave equation

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Abstract

In this work, we investigate a Boussinesq-type flow model for nonlinear dispersive waves by developing a computational model based on the finite element discretisation technique. Hermite interpolation functions were used to interpolate approximation elements. The system is modeled using a time dependent equation. Solution to the model is obtained, through a combination of two different schemes namely: a time approximation scheme (the Newmark Method) and the eigenvalue finite element method. Using this schemes, discrete solutions of the model at different time steps, were obtained. Graphical illustrations of solutions for the transient displacements at the center and right end of the rod are presented. The results obtained are very accurate and the model efficient.

***Keywords*:** Boussinesq equation, nonlinear dispersive waves, finite element model, Hermite interpolation functions, Newmark Algorithm, Eigenvalue method.

1. Introduction

The Boussinesq equation was first described by a French scientist, Joseph Boussinesq in the 1870s when he modeled equations for the propagation of long waves on the surface of water with small amplitudes. Boussinesq -type equations can be considered as the first model for non-linear, dispersive wave propagation. In this work, a non-linear one-dimensional Boussinesq -type wave equation is solved using two different schemes – the Finite element/Eigenvalue scheme and the Finite element/Newmark algorithm scheme - and the result investigated for its applicability to flows and design of flow control structures (hydraulic structures) such as spillways, weirs, embankments which are usually come across in engineering practice.

One way of describing the global and local flow characteristics of a flow problem is through mathematical modeling of the flow problem, i.e. by developing governing equations based on simplifying assumptions. Numerical modeling of higher order flow equations for flow situations has wide practical application particularly for flow measuring devices such as weir and venturi flumes as well as structures such as railway embankments. Other related works include Makhankov (1978) [4], Liu (1993) [3], and EqWorld (2006) [2], the solutions obtained by these researchers were not very informative about the characterization the behaviour of Boussinesq waves. This study therefore arises from the need to provide very accurate and informative solutions that will correctly describe the behaviour of Boussinesq waves. The features of the finite element method, makes these solutions very accurate and efficient. This is done as follows:

Discretization of the domain using large number of finite elements, provides very accurate solutions, and information about the displacement / and behaviour of the wave at any point in the domain can be easily obtained as compared to other schemes like the Finite difference method. This work is restricted to investigate the solution of nonlinear one-dimensional Boussinesq-type equation, which is widely employed to describe the dispersive and nonlinear characteristics of the wave problems predominant in coastal engineering.

The Bousinesq wave equation is studied in this work because of it's importance in the design and construction of structures and mega structures that subjected to heavy wave impact, as experienced in offshore and deep offshore oil drilling rigs. We note that the energy that waves generate, are quite enormous. They can deliver very high energy impact on structures that are within their wave frontier. Their effects are quite disastrous. Thus, it is important to have mega structures like oil rigs (which are located along the path of waves) designed and manufactured to have the strength, toughness and elasticity to withstand the impact of waves.

The concept of finite element is well treated in many standard texts see (Spyrakos, 1996 [6], Spyrakos and Raftoyiannis, 1997 [7]; Reddy, 1984 [5], EqWorld, 2006) [2].

2.0 Mathematical formulation

The Boussinesq equations can be expressed in one of its forms as:

$$\phi_{tt} - gh'\frac{\partial^2\phi}{\partial x^2} + \alpha h'^2\frac{\partial^2\phi_{tt}}{\partial x^2} - \alpha g h'^3\frac{\partial^4\phi}{\partial x^2} = gF_t \tag{2.1a}$$

where, g = gravitational acceleration,

h = water depth measured from the still water level.

α = constant = $-\frac{1}{3}(1+\beta)$

α_1 = constant = $-\frac{1}{3}\beta$

β = constant whose value determines the dispersive properties of the equation

F_t = source function.

ϕ = water surface elevation ϕ_{tt} is the second derivative with respect to time t, of the water surface elevation ϕ, that is, $\phi_{tt} = \frac{\partial^2\phi}{\partial t^2}$. The above equation can be written as

$$-a\frac{\partial^2\phi}{\partial x^2} + b\frac{\partial^4\phi}{\partial x^4} + b'\frac{\partial^2\phi_{tt}}{\partial x^2} + \phi_{tt} = f,\ \ 0 \le x \le L,\ \ 0 \le t \tag{2.1b}$$

where $a = gh', b' = \alpha h'^2 = -\frac{1}{3}(1+\beta)h'^2, b = -\alpha_1 g h'^3 = \frac{\beta}{3} g h'^3, f = -gF_t$ and L is the length of the domain.

2.1 Finite Element Formulation

In developing the model for solving the Boussinesq equation, we used the finite element method. The finite element formulation of the problem (a time dependent problem) involved two steps:

2.1.1 Spatial approximation

Here the solution ϕ of the equation under consideration was approximated using:

$$\phi(x,t) \approx \phi^e(x,t) = \sum_{j=1}^{n} \phi_j^e(t)\psi_j^e(x) \tag{2.2}$$

and the spatial finite element model of the equation is developed using procedures of static or steady state problems, while retaining all time dependent terms in the formulation. Equation (2.2) represents the spatial approximation of ϕ for any time t. When the solution is separable into functions of time and space only, $\phi(x,t) = T(t)X(x)$, equation 2.2 is clearly justified. Even when the solution is not separable, it can represent a good approximation provided a sufficiently small time step is used.

2.1.2 Temporal approximation

Here the system of ordinary differential equations were further approximated in time. This step enabled conversion of the system of ordinary differential equations into a set of algebraic equations among ϕ_j^e at a time $t_{n-1} = (n+1)\Delta t$ where Δt is the increment and n is an integer.

It should be noted that all time approximation schemes seek to find ϕ_j^e at a time t_{n-1} using known values of ϕ_j^e from previous times. Thus at the end of the two stage approximation, one has a continuous spatial solution at discrete intervals of time.

2.2 Semi-discrete finite element model

We conducted a semi-discrete formulation which involved approximating the independent variable. Applying the steps to equation 2.1b we obtain:

$$0 = \int_{x_A}^{x_B} w\left[-\frac{\partial}{\partial x}\left(a\frac{\partial\phi}{\partial x}\right) + \frac{\partial^2}{\partial x^2}\left(b\frac{\partial^2\phi}{\partial x^2}\right) + b'\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\phi}{\partial t^2}\right) + \frac{\partial^2\phi}{\partial t^2} - f\right]dx$$

$$= \int_{x_A}^{x_B}\left[\frac{\partial w}{\partial x}a\frac{\partial\phi}{\partial x} + \frac{\partial^2 w}{\partial x^2}b\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}b'\frac{\partial^2\phi}{\partial t^2} + \frac{\partial^2\phi}{\partial t^2} - wf\right]dx$$

$$+\left[w\left[\left(-a\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial x}\left(b'\frac{\partial^2\phi}{\partial t^2}\right) + \frac{\partial}{\partial x}\left(b\frac{\partial^2\phi}{\partial x^2}\right)\right] + \frac{\partial w}{\partial x}\left[\left(-b\frac{\partial^2\phi}{\partial x^2}\right) + \left(-b'\frac{\partial^2\phi}{\partial t^2}\right)\right]\right]_{x_A}^{x_B}$$

$$= \int_{x_A}^{x_B}\left[a\frac{\partial w}{\partial x}\frac{\partial\phi}{\partial x} + b\frac{\partial^2 w}{\partial x^2}\frac{\partial^2\phi}{\partial x^2} + b'\frac{\partial^2 w}{\partial x^2}\frac{\partial^2\phi}{\partial t^2} + \frac{\partial^2\phi}{\partial t^2} - wf\right]dx$$

$$-\hat{Q}_1 w(x_A) - \hat{Q}_3 w(x_B) - \hat{Q}_2\left(\frac{\partial w}{\partial x}\right)\Bigg|_{x_A} - \hat{Q}_4\left(\frac{\partial w}{\partial x}\right)\Bigg|_{x_B} \qquad (2.3)$$

where,

$$\hat{Q}_1 = \left[\left(-a\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial x}\left(b\frac{\partial^2\phi}{\partial x^2}\right) + \frac{\partial}{\partial x}\left(b'\frac{\partial^2\phi}{\partial t^2}\right)\right]\Bigg|_{x_A}, \quad \hat{Q}_2 = \left(b\frac{\partial^2\phi}{\partial x^2} + b'\frac{\partial^2\phi}{\partial t^2}\right)\Bigg|_{x_A}$$

$$\hat{Q}_3 = -\left[\left(-a\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial x}\left(b\frac{\partial^2\phi}{\partial x^2}\right) + \frac{\partial}{\partial x}\left(b'\frac{\partial^2\phi}{\partial t^2}\right)\right]\Bigg|_{x_B}, \quad \hat{Q}_4 = -\left(b\frac{\partial^2\phi}{\partial x^2} + b'\frac{\partial^2\phi}{\partial t^2}\right)\Bigg|_{x_B}$$

x_A and x_B are the end points of a linear element of length h.

Next we interpolated ϕ using equation 2.2. Thus at any arbitrary fixed time $t > 0$, the function ϕ can be approximated by a linear combination of the ψ_i with $\phi_j^e(t)$

being the value of ϕ at a time t at the j^{th} node of the element Ω^e. The finite element solution obtained at the end of the analysis is continuous in space but not in time. The solution is of the form:

$$\phi(x,t) = \sum_{j=1}^{n} \phi_j^e(t_s)\psi_j^e(x) = \sum_{j=1}^{n} (\phi_j^s)^e \psi_j^e(x) \qquad s = (1,2\ldots..)$$

Substituting $w=\psi_i$ and (2) into (3), we obtain

$$0 = \int_{x_A}^{x_B} \left[\begin{array}{l} a\frac{d\psi_i}{dx}\left(\sum_{j=1}^{n}\phi_j\frac{d\psi_j}{dx}\right) + b\frac{d^2\psi_i}{dx^2}\left(\sum_{j=1}^{n}\phi_j\frac{d^2\psi}{dx^2}\right) \\ +b'\frac{d^2\psi_i}{dx^2}\left(\sum_{j=1}^{n}\frac{d^2\phi_j}{dt^2}\psi_j\right) + \psi_i\left(\sum_{j=1}^{n}\frac{d^2\phi_j}{dt^2}\psi_j\right) - \psi_i f \end{array} \right] dx$$

$$-\hat{Q}_1\psi_i(x_A) - \hat{Q}_3\psi_i(x_B) - \hat{Q}_2\left(-\frac{d\psi_i}{dx}\right)\Bigg|_{x_A} - \hat{Q}_4\left(-\frac{d\psi_i}{dx}\right)\Bigg|_{x_B}$$

$$= \sum_{j=1}^{n}\left[\left[K^1{}_{ij} + K^2{}_{ij}\right]\phi_j + \left[M_{ij}^1 + M_{ij}^2\right]\frac{d^2\phi_j}{dx^2}\right] + F_i \qquad (2.4)$$

where $K^1{}_{ij} = \int_{x_A}^{x_B} a\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx$, $K^2{}_{ij} = \int_{x_A}^{x_B} b\frac{d^2\psi_i}{dx^2}\frac{d^2\psi_j}{dx^2}dx$,

$$M_{ij}^1 = \int_{x_A}^{x_B} b'\frac{\partial^2\psi_i}{\partial x^2}\psi_j dx, \quad M_{ij}^2 = \int_{x_A}^{x_B}\psi_i\psi_j dx, \quad F_i = \int_{x_A}^{x_B}\psi_i f dx + \hat{Q}_i$$

Putting (2.4) in matrix form, we have $[K]\{\phi\} + [M]\{\ddot{\phi}\} = F$ (2.5)

where $[K]=\left[K_{ij}^1\right]+\left[K_{ij}^2\right]$, $[M] = \left[M_{ij}^1\right] + \left[M_{ij}^2\right]$

2.2.1 Interpolation Functions

A four-parameter polynomial was selected for ϕ to match the four conditions in the element (two per node). This gives:

$$\phi(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3 \qquad (2.6)$$

Expressing c_i in terms of the primary nodal variables, putting in matrix form and inverting using Cramer's rule, we obtained the Hermite interpolation functions which when expressed in terms of the local coordinate $\overline{x}$ gives

$$\psi_1^e = 1 - 3\left(\frac{\overline{x}}{h_e}\right)^2 + 2\left(\frac{\overline{x}}{h_e}\right)^3, \psi_2^e = -\overline{x}\left(1-\frac{\overline{x}}{h_e}\right)^2, \psi_3^e = 3\left(\frac{\overline{x}}{h_e}\right)^2 - 2\left(\frac{\overline{x}}{h_e}\right)^3$$

$$\psi_4^e = -\overline{x}\left[\left(\frac{\overline{x}}{h_e}\right)^2 - \frac{\overline{x}}{h_e}\right] \qquad (2.7)$$

The matrices $\left[K_{ij}^{1}\right]$, $\left[K_{ij}^{2}\right]$,$\left[\mathrm{M}_{\mathrm{ij}}^{1}\right]$, $\left[M_{ij}^{2}\right]$ and $\left[F\right]$ are computed from equation 2.7 to yield:

$$\left[M_{ij}^{1}\right]=\frac{b'}{30h}\begin{bmatrix}-36 & 3h & 36 & 3h\\ 33h & -4h^2 & -3h & h^2\\ 36 & -3h & -36 & -3h\\ 3h & h^2 & -33h & -4h^2\end{bmatrix},\ \left[M_{ij}^{2}\right]=\frac{h}{420}\begin{bmatrix}156 & -22h & 54 & 13h\\ -22h & 4h^2 & -13h & -3h^2\\ 54 & -13h & 156 & 22h\\ 13h & -3h^2 & 22h & 4h^2\end{bmatrix}$$

$$\left[K_{ij}^{1}\right]=\frac{a}{30h}\begin{bmatrix}36 & -3h & -36 & -3h\\ -3h & 4h^2 & 3h & -h^2\\ -36 & 3h & 36 & 3h\\ -3h & -h^2 & 3h & 4h^2\end{bmatrix},\ \left[K_{ij}^{2}\right]=\frac{2b}{h^3}\begin{bmatrix}6 & -3h & -6 & -3h\\ -3h & 2h^2 & 3h & h^2\\ -6 & 3h & 6 & 3h\\ -3h & h^2 & 3h & 2h^2\end{bmatrix}$$

$$\left[F_i\right]=\frac{fh}{12}\begin{bmatrix}6\\ -h\\ 6\\ h\end{bmatrix}+\begin{bmatrix}\hat{Q}_1\\ \hat{Q}_2\\ \hat{Q}_3\\ \hat{Q}_4\end{bmatrix}$$

$$\left[M\right]=\left[M_{ij}^{1}\right]+\left[M_{ij}^{2}\right]=$$

$$\left(\frac{B}{30\cdot h}\right)\cdot\begin{bmatrix}-36 & 3h & 36 & 3h\\ 33h & -4h^2 & -3h & h^2\\ 36 & -3h & -36 & -3h\\ 3h & h^2 & -33h & -4h^2\end{bmatrix}+\left(\frac{h}{420}\right)\cdot\begin{bmatrix}156 & -22h & 54 & 13h\\ -22h & 4h^2 & -13h & -3h^2\\ 54 & -13h & 156 & 22h\\ 13h & -3h^2 & 22h & 4h^2\end{bmatrix}\rightarrow\begin{bmatrix}\frac{-6}{5}\cdot\frac{B}{h}+\frac{13}{35}\cdot h & \frac{1}{10}\cdot\frac{B}{h}-\frac{17}{210}\cdot h & \frac{6}{5}\cdot\frac{B}{h}+\frac{9}{70}\cdot h & \frac{1}{10}\cdot\frac{B}{h}+\frac{19}{420}\cdot h\\ \frac{17}{10}\cdot\frac{B}{h}-\frac{17}{210}\cdot h & \frac{-8}{15}\cdot\frac{B}{h}+\frac{4}{105}\cdot h & \frac{-1}{10}\cdot\frac{B}{h}-\frac{19}{420}\cdot h & \frac{1}{30}\cdot h\cdot B-\frac{3}{140}\cdot h\\ \frac{6}{5}\cdot\frac{B}{h}+\frac{9}{70}\cdot h & \frac{-1}{10}\cdot\frac{B}{h}-\frac{19}{420}\cdot h & \frac{-6}{5}\cdot\frac{B}{h}+\frac{13}{35}\cdot h & \frac{-1}{10}\cdot\frac{B}{h}+\frac{17}{210}\cdot h\\ \frac{1}{10}\cdot\frac{B}{h}+\frac{19}{420}\cdot h & \frac{1}{30}\cdot h\cdot B-\frac{3}{140}\cdot h & \frac{-17}{10}\cdot\frac{B}{h}+\frac{17}{210}\cdot h & \frac{-8}{15}\cdot\frac{B}{h}+\frac{4}{105}\cdot h\end{bmatrix}$$

where $B = b'$

$$\left[K\right]=\left[K_{ij}^{1}\right]+\left[K_{ij}^{2}\right]=$$

$$\left[\left(\frac{a}{30\cdot h}\right)\cdot\begin{bmatrix}36 & -3h & -36 & -3h\\ -3h & 4h^2 & 3h & -h^2\\ -36 & 3h & 36 & 3h\\ -3h & -h^2 & 3h & 4h^2\end{bmatrix}\right]+\left(\frac{2\cdot b}{h^3}\right)\cdot\begin{bmatrix}6 & -3h & -6 & -3h\\ -3h & 2h^2 & 3h & h^2\\ -6 & 3h & 6 & 3h\\ -3h & h^2 & 3h & 2h^2\end{bmatrix}\rightarrow\begin{bmatrix}\frac{6}{5}\cdot\frac{a}{h}+12\cdot\frac{b}{h^3} & \frac{-1}{10}\cdot\frac{a}{h}-6\cdot\frac{b}{h^3} & \frac{-6}{5}\cdot\frac{a}{h}-12\cdot\frac{b}{h^3} & \frac{-1}{10}\cdot\frac{a}{h}-6\cdot\frac{b}{h^3}\\ \frac{-1}{10}\cdot\frac{a}{h}-6\cdot\frac{b}{h^3} & \frac{8}{15}\cdot\frac{a}{h}+8\cdot\frac{b}{h^3} & \frac{1}{10}\cdot\frac{a}{h}+6\cdot\frac{b}{h^3} & \frac{-1}{30}\cdot a\cdot h+2\cdot\frac{b}{h}\\ \frac{-6}{5}\cdot\frac{a}{h}-12\cdot\frac{b}{h^3} & \frac{1}{10}\cdot\frac{a}{h}+6\cdot\frac{b}{h^3} & \frac{6}{5}\cdot\frac{a}{h}+12\cdot\frac{b}{h^3} & \frac{1}{10}\cdot\frac{a}{h}+6\cdot\frac{b}{h^3}\\ \frac{-1}{10}\cdot\frac{a}{h}-6\cdot\frac{b}{h^3} & \frac{-1}{30}\cdot a\cdot h+2\cdot\frac{b}{h} & \frac{1}{10}\cdot\frac{a}{h}+6\cdot\frac{b}{h^3} & \frac{8}{15}\cdot\frac{a}{h}+8\cdot\frac{b}{h^3}\end{bmatrix}$$

Equation 2.5 is a hyperbolic equation and can be solved using the Newmark Method. For the interval $0 < \tau < \Delta t$, the interval corresponding to the time step, the acceleration is expressed as:

$$\ddot{\phi}_{t+\tau}=\ddot{\phi}_t\left(1-\frac{\tau}{\Delta t}\right)+\ddot{\phi}_{t+\Delta t}\left(\frac{\tau}{\Delta t}\right) \qquad (2.8)$$

Integrating (2.8) yields

$$\dot{\phi}_{t+\tau} = \dot{\phi}_t + \ddot{\phi}_t\left(\tau - \frac{\tau^2}{2\Delta t}\right) + \ddot{\phi}_{t+\Delta t}\left(\frac{\tau^2}{2\Delta t}\right) \qquad (2.9)$$

At τ = Δt, equation 2.9 yields

$$\dot{\phi}_{t+\Delta t} = \dot{\phi}_t + \left(\frac{\ddot{\phi}_t + \phi_{t+\Delta t}}{2}\right)\Delta t = \dot{\phi}_t + \Delta t\ddot{\phi}_{AVG} \qquad (2.10)$$

that is the increment in the velocity is based on the approximate average acceleration on the interval (0, *Δt*). Integrating equation 2.9 yields:

$$\phi_{t+\tau} = \phi_t + \tau\dot{\phi}_t + \ddot{\phi}_t\left(\frac{\tau^2}{2} + \frac{\tau^3}{6\Delta t}\right) + \ddot{\phi}_{t+\Delta t}\left(\frac{\tau^3}{6\Delta t}\right) \qquad (2.11)$$

Also with τ = *Δt*, equation 2.11 yields:

$$\phi_{t+\Delta t} = \phi_t + \Delta t\phi_t + \left(\frac{2\ddot{\phi}_t + \ddot{\phi}_{t+\Delta t}}{6}\right)\Delta t^2 \qquad (2.12)$$

These expressions are employed with the differential equation 2.5 to yield the conditionally stable average acceleration algorithm. The Newmark's generalized equations from equations 2.10 and 2.12 are:

$$\dot{\phi}_{t+\Delta t} = \dot{\phi}_t + \left((1-\delta)\ddot{\phi}_t + \delta\ddot{\phi}_{t+\Delta t}\right)\Delta t^2 \qquad (2.13)$$

$$\phi_{t+\Delta t} = \phi_t + \Delta t\dot{\phi}_t + \left(\left(\frac{1}{2} - \alpha\right)\ddot{\phi}_t + \alpha\ddot{\phi}_{t+\Delta t}\right)\Delta t^2 \qquad (2.14)$$

where δ and α are parameters to be chosen for accuracy and stability. The method is unconditionally stable as long as the parameters δ and α are chosen to satisfy, δ = 0.5 and $\alpha \geq 0.25(\delta + 0.5)^2$. Equation 2.14 was solved for $\phi_{t+\Delta t}$ and substituted into equation 2.13 to yield: $\dot{\phi}_{t+\Delta t} = \dot{\phi}_t + \delta\left(\frac{\phi_{t+\Delta t} - \phi_t - \Delta t\dot{\phi}_t}{\alpha\Delta t}\right) + d_2\Delta t\ddot{\phi}_t$ where $d_2 = 1 - \frac{\delta}{2\alpha}$ and then into the differential equation evaluated at *t* + *Δt* to yield

$$([M] + \alpha\Delta t^2[K])\phi_{t+\Delta t} = [M](\phi_t + \Delta t\dot{\phi}_t + d_1\Delta t^2\ddot{\phi}_t) + \alpha\Delta t^2\{F\} \qquad (2.15)$$

where $d_1 = \frac{1}{2} - \alpha$. Equation 2.15, together with the two equations for velocity and acceleration at *t* + Δ*t*, namely

$$\dot{\phi}_{t+\Delta t} = \dot{\phi}_t + \delta\left(\frac{\phi_{t+\Delta t} - \phi_t - \Delta t\dot{\phi}_t}{\alpha\Delta t}\right) + d_2\Delta t\ddot{\phi}_t \qquad (2.16)$$

and

$$\ddot{\phi}_{t+\Delta t} = \left(\frac{\phi_{t+\Delta t} - \phi_t - \dot{\phi}_t}{\alpha\Delta t^2}\right) - \left(\frac{d_1\phi_t}{\alpha}\right) \qquad (2.17)$$

respectively were used to step ahead in time to determine the solution. In order to start the process the acceleration at time $t = 0$ was needed. This was determined using equation (2.5) evaluated at $t = 0$ *i.e.*

$$[M]\{\ddot{\phi}\}_{t=0} = \{F\}_{t=0} - [K]\{\phi\}_{t=0} \tag{2.18}$$

Equations (2.15, 2.16, and 2.17) were then used to step ahead in time using the unconditionally stable Newmark's algorithm. The algorithm consists of:

Given the initial conditions $\phi_{t=0}, \dot{\phi}_{t=0}$,

Compute $\ddot{\phi}_{t=0}$ using equation (2.18), and $\phi_{t=n}, \dot{\phi}_{t=n}, \ddot{\phi}_{t=n}$, $n = 1, 2, \ldots$ using equations (2.15), (2.16), and (2.17) thus:

$$\left([M] + \alpha\Delta t^2 [K]\right)\{\phi\}_{n+1} = [M]\left(\{\phi\}_n + \Delta t\{\dot{\phi}\}_n + d_1\Delta t^2\{\ddot{\phi}\}_n\right) + d_2\Delta t^2\{F\}_{n+1}$$

$$\{\dot{\phi}_{n+1}\} = \{\dot{\phi}\}_n + \frac{\delta}{\alpha\Delta t}\left(\{\phi\}_{n+1} - \{\phi\}_n - \Delta t\{\dot{\phi}\}_n\right) + d_2\Delta t\{\phi\}_n$$

$$\{\ddot{\phi}\}_{n+1} = \frac{1}{\alpha\Delta t^2}\left(\{\phi\}_{n+1} - \{\phi\}_n - \Delta t\{\dot{\phi}\}_n\right) - \frac{d_1}{\alpha}\{\ddot{\phi}\}_n$$

3.0 Numerical example

Consider the Boussinesq wave equation with the following parameters; $\beta = 0.8$, $h' = 1.2m$, $g = 9.8ms^2$, $L = 20m$, $h = 10m$, therefore, $a = gh' = 9.8 \times 1.2 = 11.76m^2/s^2$, $b = \alpha h'^2 = = \frac{-1}{3}(1+\beta)h'^2 = -\frac{1}{3}(1+0.8)\times 1.2^2 = -0\cdot 864m^2, d = -\alpha g h'^3 = \frac{\beta}{3} g h'^3 = \frac{0\cdot 8}{3}\times 9\cdot 8\times 1\cdot 2^3 = 4\cdot 5158m^4/s^2$ and $f = 9\cdot 8F_t$.

Therefore let

$$[M] = \int_{x_A}^{x_B}\left(\psi_i\psi_j + 0.864\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x}\right)dx \tag{3.1}$$

$$[K] = \int_{x_A}^{x_B}\left(11.76\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x} + 4.5158\frac{\partial^2\psi_i}{\partial x^2}\frac{\partial^2\psi_j}{\partial x^2}\right)dx \tag{3.2}$$

$$\{F_i\} = -\int_{x_A}^{x_B}(9.81\psi_i F_t)dx \tag{3.3}$$

The finite element model can thus be written as;

$$[M]\{\ddot{\varphi}\} + [K]\{\varphi\} = \{F\} + \{Q\} \tag{3.4}$$

The Hermite interpolation functions are:

$$\psi_1 = 1 - 3\left(\frac{x}{h}\right)^2 + 2\left(\frac{x}{h}\right)^3, \qquad \psi_2 = -x\left(1 - \frac{x}{h}\right)^2$$
$$\psi_3 = 3\left(\frac{x}{h}\right)^2 - 2\left(\frac{x}{h}\right)^3, \qquad \psi_4 = -x\left[\left(\frac{x}{h}\right)^2 - \frac{x}{h}\right] \tag{3.5}$$

$$let\ h_1 = h_2 = \frac{L}{2}\ \ where\ L = 10m$$

The mass and stiffness matrices are obtained thus:

$$m_{11} = 2.0645 \quad m_{12} = -1.3959 \quad m_{13} = 0.4355 \quad m_{14} = 0.6874$$
$$m_{21} = -1.3959 \quad m_{22} = 1.7665 \quad m_{23} = -0.6874 \quad m_{24} = -1.0369$$
$$m_{31} = 0.4355 \quad m_{32} = -0.6874 \quad m_{33} = 2.0645 \quad m_{34} = 1.3959 \tag{3.6}$$
$$m_{41} = 0.6874 \quad m_{42} = -1.0369 \quad m_{43} = 1.3959 \quad m_{44} = 1.7665$$

$$k_{11} = 3.2559 \quad k_{12} = -2.2598 \quad k_{13} = -3.2559 \quad k_{14} = -2.2598$$
$$k_{21} = -2.2598 \quad k_{22} = 11.4526 \quad k_{23} = 2.2598 \quad k_{24} = -0.1537$$
$$k_{31} = -3.2559 \quad k_{32} = 2.2598 \quad k_{33} = 3.2559 \quad k_{34} = 2.2598 \tag{3.7}$$
$$k_{41} = -2.2598 \quad k_{42} = -0.1537 \quad k_{43} = 2.2598 \quad k_{44} = 11.4526$$

Therefore

$$M = \begin{bmatrix} 2.0645 & -1.3959 & 0.4355 & 0.6874 \\ -1.3959 & 1.7665 & -0.6874 & -1.0369 \\ 0.4355 & -0.6874 & 2.0645 & 1.3959 \\ 0.6874 & -1.0369 & 1.3959 & 1.7665 \end{bmatrix} \tag{3.8}$$

$$K = \begin{bmatrix} 3.2559 & -2.2598 & -3.2559 & -2.2598 \\ -2.2598 & 11.4526 & 2.2598 & -0.1537 \\ -3.2559 & 2.2598 & 3.2559 & 2.2598 \\ -2.2598 & -0.1537 & 2.2598 & 11.4526 \end{bmatrix} \tag{3.9}$$

The assembled equation of the finite element model is:

$$\begin{bmatrix} m_{11}^1 & m_{12}^1 & m_{13}^1 & m_{14}^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{21}^1 & m_{22}^1 & m_{23}^1 & m_{24}^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{31}^1 & m_{32}^1 & m_{33}^1+m_{11}^2 & m_{34}^1+m_{12}^2 & m_{13}^2 & m_{14}^2 & 0 & 0 & 0 & 0 \\ m_{41}^1 & m_{42}^1 & m_{43}^1+m_{21}^2 & m_{44}^1+m_{22}^2 & m_{23}^2 & m_{24}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{31}^2 & m_{32}^2 & m_{33}^2+m_{11}^3 & m_{34}^2+m_{12}^3 & m_{13}^3 & m_{14}^4 & 0 & 0 \\ 0 & 0 & m_{41}^2 & m_{42}^2 & m_{43}^2+m_{21}^3 & m_{44}^2+m_{22}^3 & m_{23}^3 & m_{24}^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{31}^3 & m_{32}^3 & m_{33}^3+m_{11}^4 & m_{44}^3+m_{12}^4 & m_{13}^4 & m_{14}^4 \\ 0 & 0 & 0 & 0 & m_{41}^3 & m_{42}^3 & m_{43}^3+m_{21}^4 & m_{44}^3+m_{22}^4 & m_{23}^4 & m_{24}^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_{31}^4 & m_{32}^4 & m_{33}^4 & m_{34}^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_{41}^4 & m_{42}^4 & m_{43}^4 & m_{44}^4 \end{bmatrix} \begin{Bmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \\ \ddot{\varphi}_3 \\ \ddot{\varphi}_4 \\ \ddot{\varphi}_5 \\ \ddot{\varphi}_6 \\ \ddot{\varphi}_7 \\ \ddot{\varphi}_8 \\ \ddot{\varphi}_9 \\ \ddot{\varphi}_{10} \end{Bmatrix} +$$

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 & k_{13}^1 & k_{14}^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_{21}^1 & k_{22}^1 & k_{23}^1 & k_{24}^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_{31}^1 & k_{32}^1 & k_{33}^1+k_{11}^2 & k_{34}^1+k_{12}^2 & k_{13}^2 & k_{14}^2 & 0 & 0 & 0 & 0 \\ k_{41}^1 & k_{42}^1 & k_{43}^1+k_{21}^2 & k_{44}^1+k_{22}^2 & k_{23}^2 & k_{24}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{31}^2 & k_{32}^2 & k_{33}^2+k_{11}^3 & k_{34}^2+k_{12}^3 & k_{13}^3 & k_{14}^4 & 0 & 0 \\ 0 & 0 & k_{41}^2 & k_{42}^2 & k_{43}^2+k_{21}^3 & k_{44}^2+k_{22}^3 & k_{23}^3 & k_{24}^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{31}^3 & k_{32}^3 & k_{33}^3+k_{11}^4 & k_{44}^3+k_{12}^4 & k_{13}^4 & k_{14}^4 \\ 0 & 0 & 0 & 0 & k_{41}^3 & k_{42}^3 & k_{43}^3+k_{21}^4 & k_{44}^3+k_{22}^4 & k_{23}^4 & k_{24}^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_{31}^4 & k_{32}^4 & k_{33}^4 & k_{34}^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_{41}^4 & k_{42}^4 & k_{43}^4 & k_{44}^4 \end{bmatrix} \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \\ \varphi_8 \\ \varphi_9 \\ \varphi_{10} \end{Bmatrix} = \begin{Bmatrix} F_1^1 \\ F_2^1 \\ F_3^1+F_1^2 \\ F_4^1+F_2^2 \\ F_3^2+F_1^3 \\ F_4^2+F_2^3 \\ F_3^3+F_1^4 \\ F_4^3+F_2^4 \\ F_3^4 \\ F_4^4 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1+Q_1^2 \\ Q_4^1+Q_2^2 \\ Q_3^2+Q_1^3 \\ Q_4^2+Q_2^3 \\ Q_3^3+Q_1^4 \\ Q_4^3+Q_2^4 \\ Q_3^4 \\ Q_4^4 \end{Bmatrix} \tag{3.10}$$

In this paper we assume that $F=0$, also, due to the balance of internal fluxes it follows that;

$$Q_3^1+Q_1^2=Q_4^1+Q_2^2=Q_3^2+Q_1^3=Q_4^2+Q_2^3=Q_3^3+Q_1^4=Q_4^3+Q_2^4=0 \qquad (3.11)$$

The assembled equations become:

$$\begin{bmatrix} 2.0645 & -1.3959 & 0.4355 & 0.6874 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.3959 & 1.7665 & -0.6874 & -1.0369 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4355 & -0.6874 & 4.1290 & 0 & 0.4355 & 0.6874 & 0 & 0 & 0 & 0 \\ 0.6874 & -1.0369 & 0 & 3.5338 & -0.6874 & -1.0369 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4355 & -0.6874 & 4.1290 & 0 & 0.4355 & 0.6874 & 0 & 0 \\ 0 & 0 & 0.6874 & -1.0369 & 0 & 3.5338 & -0.6874 & -1.0369 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4355 & -0.6874 & 4.1290 & 0 & 0.4355 & 0.6874 \\ 0 & 0 & 0 & 0 & 0.6874 & -1.0369 & 0 & 3.5338 & -0.6874 & -1.0369 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4355 & -0.6874 & 2.0645 & 1.3959 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6874 & -1.0369 & 1.3959 & 1.7665 \end{bmatrix} \begin{Bmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \\ \ddot{\varphi}_3 \\ \ddot{\varphi}_4 \\ \ddot{\varphi}_5 \\ \ddot{\varphi}_6 \\ \ddot{\varphi}_7 \\ \ddot{\varphi}_8 \\ \ddot{\varphi}_9 \\ \ddot{\varphi}_{10} \end{Bmatrix} +$$

$$\begin{bmatrix} 3.2559 & -2.2598 & -3.2559 & -2.2598 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.2598 & 11.4526 & 2.2598 & -0.1537 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3.2559 & 2.2598 & 6.5118 & 0 & -3.2559 & -2.2598 & 0 & 0 & 0 & 0 \\ -2.2598 & -0.1537 & 0 & 22.9052 & 2.2598 & -0.1537 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3.2559 & 2.2598 & 6.5118 & 0 & -3.2559 & -2.2598 & 0 & 0 \\ 0 & 0 & -2.2598 & -0.1537 & 0 & 22.9052 & 2.2598 & -0.1537 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.2559 & 2.2598 & 6.5118 & 0 & -3.2559 & -2.2598 \\ 0 & 0 & 0 & 0 & -2.2598 & -0.1537 & 0 & 22.9052 & 2.2598 & -0.1537 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3.2559 & 2.2598 & 3.2559 & 2.2598 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2.2598 & -0.1537 & 2.2598 & 11.4526 \end{bmatrix} \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \\ \varphi_8 \\ \varphi_9 \\ \varphi_{10} \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ Q_3^4 \\ Q_4^4 \end{Bmatrix} \qquad (3.12)$$

we now consider the boundary conditions, $\varphi_1=\varphi(0,t)=0$ and $\varphi_2=\dfrac{\partial\varphi(0,t)}{\partial x}=0$

We take $\hat{Q}_3^4=1000(N)$ i.e. the force applied on a flow control structure by the wave and $\widehat{Q}_4^4=0$. The condensed equations therefore become;

$$\begin{bmatrix} 4.1290 & 0 & 0.4355 & 0.6875 & 0 & 0 & 0 & 0 \\ 0 & 3.5338 & -0.6874 & -1.0369 & 0 & 0 & 0 & 0 \\ 0.4355 & -0.6874 & 4.1290 & 0 & 0.4355 & 0.6874 & 0 & 0 \\ 0.6874 & -1.0369 & 0 & 3.5338 & -0.6874 & -1.0369 & 0 & 0 \\ 0 & 0 & 0.4355 & -0.6874 & 4.1290 & 0 & 0.4355 & 0.6874 \\ 0 & 0 & 0.6874 & -1.0369 & 0 & 3.5338 & -0.6874 & -1.0369 \\ 0 & 0 & 0 & 0 & 0.4355 & -0.6874 & 2.0645 & 1.3959 \\ 0 & 0 & 0 & 0 & 0.6874 & -1.0369 & 1.3959 & 1.7665 \end{bmatrix} \begin{Bmatrix} \ddot{\varphi}_3 \\ \ddot{\varphi}_4 \\ \ddot{\varphi}_5 \\ \ddot{\varphi}_6 \\ \ddot{\varphi}_7 \\ \ddot{\varphi}_8 \\ \ddot{\varphi}_9 \\ \ddot{\varphi}_{10} \end{Bmatrix} +$$

$$\begin{bmatrix} 6.5118 & 0 & -3.2559 & -2.2598 & 0 & 0 & 0 & 0 \\ 0 & 22.9052 & 2.2598 & -0.1537 & 0 & 0 & 0 & 0 \\ -3.2559 & 2.2598 & 6.5118 & 0 & -3.2559 & -2.2598 & 0 & 0 \\ -2.2598 & -0.1537 & 0 & 22.9052 & 2.2598 & -0.1537 & 0 & 0 \\ 0 & 0 & -3.2559 & 2.2598 & 6.5118 & 0 & -3.2559 & -2.2598 \\ 0 & 0 & -2.2598 & -0.1537 & 0 & 22.9052 & 2.2598 & -0.1537 \\ 0 & 0 & 0 & 0 & -3.2559 & 2.2598 & 3.2559 & 2.2598 \\ 0 & 0 & 0 & 0 & -2.2598 & -0.1537 & 2.2598 & 11.4526 \end{bmatrix} \begin{Bmatrix} \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \\ \varphi_8 \\ \varphi_9 \\ \varphi_{10} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1000 \\ 0 \end{Bmatrix} \qquad (3.13)$$

From this point the problem can be solved by two different methods viz:

1. Eigenvalue method.
2. Newmark algorithm.

3.1 Eigenvalue method

Let the solution be in the form $\varphi = \varphi(x)e^{i\omega t}$ So that

$$\frac{\partial^2\varphi}{\partial t^2} = \ddot{\varphi} = i^2\omega^2\varphi(x)e^{i\omega t} = -\omega^2\varphi e^{i\omega t} \quad but \quad [M]\ddot{\varphi}+[K]\varphi = f+Q.$$

the homogeneous form of the finite element model is $[M]\ddot{\varphi}+[K]\varphi=0$ Putting equations (2.18) and (3.1) into equation (3.2), we have

$$[M]\left(-\omega^2\varphi e^{i\omega t}\right)+[K]\varphi e^{i\omega t} = 0 \quad or \quad [M]\left(-\omega^2\varphi\right)+[K]\varphi = 0$$

$$\left(K-\omega^2 M\right) = 0$$

$$Let \ \omega^2 = \lambda$$

$$\det\left(K-\lambda M\right) = 0 \tag{3.14}$$

This gives the following equation

$$2737.38\lambda^8 - 132245.06\lambda^7 + 2462697.43\lambda^6 - 22625823.11\lambda^5 + 108981172.84\lambda^4 - 269569722.30\lambda^3 + 309790369.97\lambda^2 - 128518851.44\lambda + 8321393.49 = 0 \tag{3.15}$$

Solving the above equation gives the following eigenvalues;

$$\lambda_1 = 0.0787,\ \lambda_2 = 0.6899,\ \lambda_3 = 1.8355,\ \lambda_4 = 3.4138$$
$$\lambda_5 = 5.5293,\ \lambda_6 = 8.1634,\ \lambda_7 = 11.7636 \ and \ \lambda_8 = 16.8383 \tag{3.16}$$

The corresponding eigenvectors are then obtained thus;

$$v_1 = [1 \quad -0.205 \quad 2.002 \quad -0.173 \quad 2.686 \quad -0.093 \quad 2.909 \quad 0.026]^T$$
$$v_2 = [1 \quad -0.124 \quad 0.891 \quad 0.181 \quad -0.385 \quad 0.270 \quad -1.172 \quad 0.030]^T$$
$$v_3 = [1 \quad 0.084 \quad -0.561 \quad 0.343 \quad -0.456 \quad -0.373 \quad 1.046 \quad -0.077]^T$$
$$v_4 = [1 \quad 0.638 \quad -1.202 \quad -0.409 \quad 1.338 \quad 0.211 \quad -1.595 \quad 0.268]^T$$
$$v_5 = [1 \quad 4.14 \quad 1.284 \quad -3.893 \quad -2.988 \quad 2.540 \quad 5.297 \quad -1.998]^T \tag{3.17}$$
$$v_6 = [1 \quad -5.225 \quad -3.436 \quad -0.482 \quad 0.565 \quad 5.292 \quad 5.777 \quad -4.460]^T$$
$$v_7 = [1 \quad -2.631 \quad -0.247 \quad -3.864 \quad -1.310 \quad -1.374 \quad -2.798 \quad 3.926]^T$$
$$v_8 = [1 \quad -2.132 \quad 1.318 \quad -5.563 \quad 2.050 \quad -10.473 \quad 11.246 \quad -24.520]^T$$

The homogeneous solution can thus be written as:

$$\varphi_h = \sum_{i=1}^{8} v_i\left(a_i\cos\left(\sqrt{\lambda_i}\cdot t\right)+b_i\sin\left(\sqrt{\lambda_i}\cdot t\right)\right) \tag{3.18}$$

The particular solution is obtained thus:

$$[M]\ddot{\varphi}_p + [K]\varphi_p = \left\{\hat{Q}\right\} \tag{3.19}$$

But $\varphi_p = \text{constant}$ and $\ddot{\varphi}_p = 0$

So that $$[K]\varphi_p = \left\{\hat{Q}\right\} \text{ and } \varphi_p = [K]^{-1}\left\{\hat{Q}\right\} \tag{3.20}$$

$$\varphi_p = \begin{bmatrix} 6.5118 & 0 & -3.2559 & -2.2598 & 0 & 0 & 0 & 0 \\ 0 & 22.9052 & 2.2598 & -0.1537 & 0 & 0 & 0 & 0 \\ -3.2559 & 2.2598 & 6.5118 & 0 & -3.2559 & -2.2598 & 0 & 0 \\ -2.2598 & -0.1537 & 0 & 22.9052 & 2.2598 & -0.1537 & 0 & 0 \\ 0 & 0 & -3.2559 & 2.2598 & 6.5118 & 0 & -3.2559 & -2.2598 \\ 0 & 0 & -2.2598 & -0.1537 & 0 & 22.9052 & 2.2598 & -0.1537 \\ 0 & 0 & 0 & 0 & -3.2559 & 2.2598 & 3.2559 & 2.2598 \\ 0 & 0 & 0 & 0 & -2.2598 & -0.1537 & 2.2598 & 11.4526 \end{bmatrix}^{-1} \times \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \end{Bmatrix}$$

$$\varphi_p = \begin{bmatrix} 0.180 \\ -0.039 \\ 0.390 \\ -0.042 \\ 0.603 \\ -0.042 \\ 0.815 \\ -0.043 \end{bmatrix} \tag{3.21}$$

The general solution can thus be written as:

$$\varphi = \varphi_p + \varphi_h \tag{3.22}$$

$$\varphi = \varphi_p + \sum_{i=1}^{8} v_i \left(a_i \cos\left(\sqrt{\lambda_i} \cdot t\right) + b_i \sin\left(\sqrt{\lambda_i} \cdot t\right)\right)$$

In order to obtain the constants a_i and b_i we now consider the initial conditions thus: $\varphi(x,0) = 0$,

i.e. $\varphi_p + \sum_{i=1}^{8} v_i \left(a_i \cos\left(\sqrt{\lambda_i} \cdot t\right) + b_i \sin\left(\sqrt{\lambda_i} \cdot t\right)\right)\Big|_{t=0} = 0$

or $\sum_{i=1}^{8} v_i \left(a_i \cos\left(\sqrt{\lambda_i} \cdot t\right) + b_i \sin\left(\sqrt{\lambda_i} \cdot t\right)\right)\Big|_{t=0} = -\varphi_p$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -0.205 & -0.124 & 0.084 & 0.638 & 4.140 & -5.225 & -2.631 & -2.132 \\ 2.002 & 0.891 & -0.561 & -1.202 & 1.284 & -3.436 & -0.247 & 1.318 \\ -0.173 & 0.181 & 0.343 & -0.409 & -3.893 & -0.482 & -3.864 & -5.563 \\ 2.686 & -0.385 & -0.456 & 1.338 & -2.988 & 0.565 & -1.310 & 2.050 \\ -0.093 & 0.270 & -0.373 & 0.211 & 2.540 & 5.292 & -1.374 & -10.473 \\ 2.909 & -1.172 & 1.046 & -1.595 & 5.297 & 5.777 & -2.798 & 11.246 \\ 0.026 & 0.030 & -0.077 & 0.268 & -1.998 & -4.460 & 3.926 & -24.520 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{Bmatrix} = \begin{Bmatrix} -0.180 \\ 0.039 \\ -0.390 \\ 0.042 \\ -0.603 \\ 0.042 \\ -0.815 \\ 0.043 \end{Bmatrix}$$

$$
\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -0.205 & -0.124 & 0.084 & 0.638 & 4.140 & -5.225 & -2.631 & -2.132 \\ 2.002 & 0.891 & -0.561 & -1.202 & 1.284 & -3.436 & -0.247 & 1.318 \\ -0.173 & 0.181 & 0.343 & -0.409 & -3.893 & -0.482 & -3.864 & -5.563 \\ 2.686 & -0.385 & -0.456 & 1.338 & -2.988 & 0.565 & -1.310 & 2.050 \\ -0.093 & 0.270 & -0.373 & 0.211 & 2.540 & 5.292 & -1.374 & -10.473 \\ 2.909 & -1.172 & 1.046 & -1.595 & 5.297 & 5.777 & -2.798 & 11.246 \\ 0.026 & 0.030 & -0.077 & 0.268 & -1.998 & -4.460 & 3.926 & -24.520 \end{bmatrix}^{-1} \begin{Bmatrix} -0.180 \\ 0.039 \\ -0.390 \\ 0.042 \\ -0.603 \\ 0.042 \\ -0.815 \\ 0.043 \end{Bmatrix}
$$

$$
\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{Bmatrix} = \begin{Bmatrix} -0.226 \\ 0.065 \\ -0.027 \\ 0.011 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
$$

To obtain the constants b_i we consider the other initial condition: $\dfrac{\partial\varphi(x,0)}{\partial t} = \{\dot{\varphi}\}_{t=0} = 0$

$$
\sum_{i=1}^{8} \sqrt{\lambda_i} \cdot v_i \left(-a_i \sin\left(\sqrt{\lambda_i} \cdot t\right) + b_i \cos\left(\sqrt{\lambda_i} \cdot t\right)\right)\Big|_{t=0} = 0
$$

It follows therefore that, $b_i = 0$. The general solution therefore is given as:

$$
\varphi = \varphi_p + \sum_{i=1}^{4} \left(a_i v_i \cos\left(\sqrt{\lambda_i} \cdot t\right)\right) \tag{3.33}
$$

The solutions at the various nodes are given below:

$at \;\; x = 0.25L$

$$\varphi = 0.180 - 0.226\cos(0.281t) + 0.065\cos(0.831t) - 0.027\cos(1.355t) + 0.011\cos(1.848t)$$

$at \;\; x = 0.50L$

$$\varphi = 0.390 - 0.452\cos(0.281t) + 0.058\cos(0.831t) + 0.015\cos(1.355t) - 0.013\cos(1.848t)$$

$at \;\; x = 0.75L$

$$\varphi = 0.603 - 0.607\cos(0.281t) - 0.025\cos(0.831t) + 0.012\cos(1.355t) + 0.015\cos(1.848t)$$

$at \;\; x = L$

$$\varphi = 0.815 - 0.657\cos(0.281t) - 0.076\cos(0.831t) - 0.028\cos(1.355t) - 0.018\cos(1.848t)$$

3.2 Newmark algorithm

For the interval 0 < τ < Δt, the interval corresponding to the time step, the acceleration is expressed as:

$$
\ddot{\phi}_{t+\tau} = \ddot{\phi}_t \left(1 - \frac{\tau}{\Delta t}\right) + \ddot{\phi}_{t+\Delta t}\left(\frac{\tau}{\Delta t}\right) \tag{3.34}
$$

Integrating (3.9) yields

$$\ddot{\phi}_{t+\tau} = \dot{\phi}_t + \ddot{\phi}_t\left(\tau - \frac{\tau^2}{2\Delta t}\right) + \ddot{\phi}_{t+\Delta t}\left(\frac{\tau^2}{2\Delta t}\right) \qquad (3.35)$$

At $\tau = \Delta t$, equation (3.10) yields

$$\dot{\phi}_{t+\Delta t} = \dot{\phi}_t + \left(\frac{\ddot{\phi}_t + \phi_{t+\Delta t}}{2}\right)\Delta t = \dot{\phi}_t + \Delta t\ddot{\phi}_{AVG} \quad (44)$$

that is the increment in the velocity is based on the approximate average acceleration on the interval (0, Δt). Integrating equation (2.8) yields:

$$\phi_{t+\tau} = \phi_t + \tau\dot{\phi}_t + \ddot{\phi}_t\left(\frac{\tau^2}{2} + \frac{\tau^3}{6\Delta t}\right) + \ddot{\phi}_{t+\Delta t}\left(\frac{\tau^3}{6\Delta t}\right) \qquad (3.36)$$

Also with $\tau = \Delta t$, equation 30 yields:

$$\phi_{t+\Delta t} = \phi_t + \Delta t\phi_t + \left(\frac{2\ddot{\phi}_t + \ddot{\phi}_{t+\Delta t}}{6}\right)\Delta t^2 \qquad (3.37)$$

These expressions are employed with the differential equation (3.24) to yield the conditionally stable average acceleration algorithm. The Newmark's generalized equations from equations (3.35) and (3.37) are:

$$\dot{\phi}_{t+\Delta t} = \dot{\phi}_t + \left((1-\delta)\ddot{\phi}_t + \delta\ddot{\phi}_{t+\Delta t}\right)\Delta t^2 \qquad (3.38)$$

$$\phi_{t+\Delta t} = \phi_t + \Delta t\dot{\phi}_t + \left(\left(\frac{1}{2} - \alpha\right)\ddot{\phi}_t + \alpha\ddot{\phi}_{t+\Delta t}\right)\Delta t^2 \qquad (3.39)$$

where δ and α are parameters to be chosen for accuracy and stability. The method is unconditionally stable as long as the parameters δ and α are chosen to satisfy, $\delta = 0.5$ and $\alpha \geq 0.25(\delta + 0.5)^2$.

Equation (3.39) was solved for $\phi_{t+\Delta t}$ and substituted into equation (3.38) to yield:

$$\dot{\phi}_{t+\Delta t} = \dot{\phi}_t + \delta\left(\frac{\phi_{t+\Delta t} - \phi_t - \Delta t\dot{\phi}_t}{\alpha\Delta t}\right) + d_2\Delta t\ddot{\phi}_t$$

where $d_2 = 1 - \frac{\delta}{2\alpha}$ and then into the differential equation evaluated at t + Δt to yield

$$\left([M] + \alpha\Delta t^2[K]\right)\phi_{t+\Delta t} = [M]\left(\phi_t + \Delta t\dot{\phi}_t + d_1\Delta t^2\ddot{\phi}_t\right) + \alpha\Delta t^2\{F\} \qquad (3.40)$$

where $d_1 = \frac{1}{2} - \alpha$. Equation (3.40), together with the two equations for velocity and acceleration at t + Δt, namely

$$\dot{\phi}_{t+\Delta t} = \dot{\phi}_t + \delta\left(\frac{\phi_{t+\Delta t} - \phi_t - \Delta t\dot{\phi}_t}{\alpha\Delta t}\right) + d_2\Delta t\ddot{\phi}_t \qquad (3.41)$$

and

$$\ddot{\phi}_{t+\Delta t} = \left(\frac{\phi_{t+\Delta t} - \phi_t - \dot{\phi}_t}{\alpha \Delta t^2} \right) - \left(\frac{d_1 \phi_t}{\alpha} \right) \tag{3.42}$$

respectively were used to step ahead in time to determine the solution. In order to start the process the acceleration at time $t = 0$ was needed. This was determined using equation (3.4) evaluated at $t = 0$ *i.e.*

$$[M]\{\dot{\phi}\}_{t=0} = \{F\}_{t=0} - [K]\{\phi\}_{t=0} \tag{3.43}$$

Equations (3.39), (3.40), and (3.41) were then used to step ahead in time using the unconditionally stable Newmark's algorithm. The algorithm consists of:

Given the initial conditions $\phi_{t=0}, \dot{\phi}_{t=0}$, Compute $\ddot{\phi}_{t=0}$ using equation (3.18), and

$$\phi_{t=n}, \dot{\phi}_{t=n}, \ddot{\phi}_{t=n} \quad ,n = 1, 2, \ldots$$

using equations (3.15), (3.16), and (3.17) thus:

$$\left([M] + \alpha\Delta t^2 [K]\right)\{\phi\}_{n+1} = [M]\left(\{\phi\}_n + \Delta t\{\dot{\phi}\}_n + d_1\Delta t^2\{\ddot{\phi}\}_n\right) + d_2\Delta t^2\{F\}_{n+1}$$

$$\{\dot{\phi}_{n+1}\} = \{\dot{\phi}\}_n + \frac{\delta}{\alpha\Delta t}\left(\{\phi\}_{n+1} - \{\phi\}_n - \Delta t\{\dot{\phi}\}_n\right) + d_2\Delta t\{\phi\}_n$$

$$\{\dot{\phi}\}_{n+1} = \frac{1}{\alpha\Delta t^2}\left(\{\phi\}_{n+1} - \{\phi\}_n - \Delta t\{\dot{\phi}\}_n\right) - \frac{d_1}{\alpha}\{\dot{\phi}\}_n$$

The matrices M, K and F are obtained from equation (3.13) and substituted into quations (3.38), (3.39) and (3.40) which are then programmed into MathCAD with which successive values of the displacement, velocity and acceleration for the various time intervals are obtained.

4.0 Results

Table 1: Transient displacements at the center and right end of the rod

	Displacement			
	Center ($x = 0.5L$)		Right end ($x = L$)	
Time (s)	**Eigenvalue solution**	**Newmark solution**	**Eigenvalue solution**	**Newmark solution**
0	-0.002	0	0.036	0.040
5	0.311	0.340	0.740	0.700
10	0.788	0.820	1.437	1.450
15	0.676	0.650	1.066	1.100
20	-0.017	-0.012	0.341	0.350
25	0.033	0.042	0.390	0.400
30	0.672	0.650	1.118	1.200
35	0.754	0.800	1.502	1.450
40	0.254	0.245	0.693	0.650
45	-0.011	-0.018	0.094	0.140
50	0.313	0.253	0.815	0.850
60	0.649	0.670	0.997	1.000
70	0.102	0.150	0.363	0.400
80	0.751	0.810	1.480	1.500
90	-0.008656	0	0.145	0.120
100	0.841	0.800	1.491	1.500
110	-0.056	-0.050	0.329	0.300
120	0.747	0.750	1.187	1.200

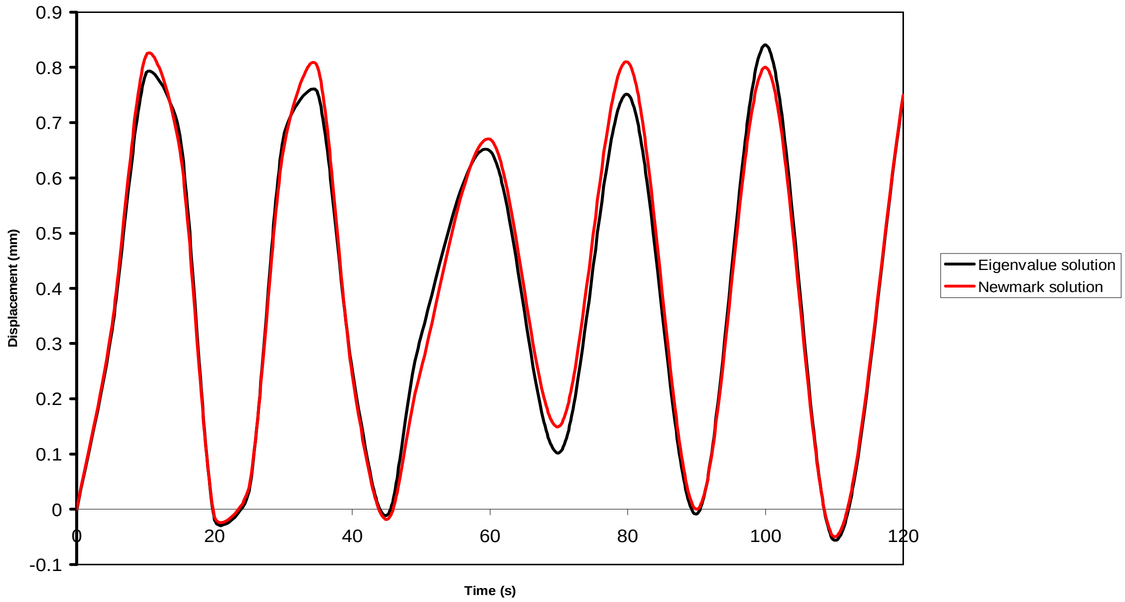

Figure 1: Graph of Displacement (mm) against Time (s) at the center of the rod

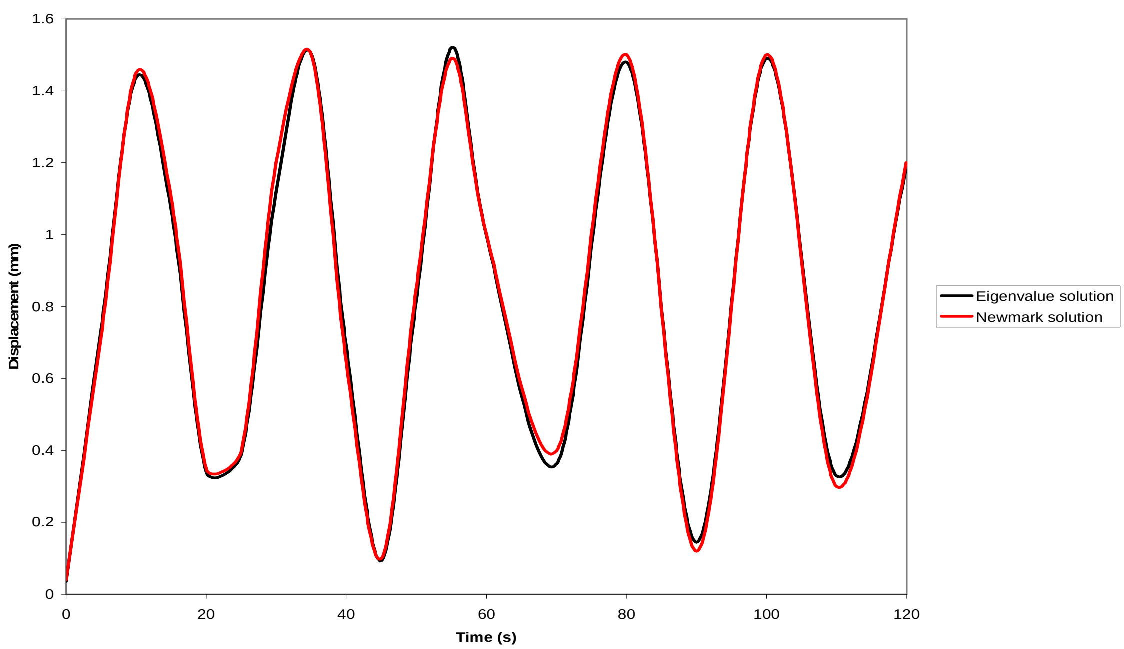

Figure 2: Graph of Displacement (mm) against Time (s) at the right end of the rod

5.0 Discussion

The results displayed in Table 1 show that the solutions obtained by both the finite element-Eigenvalue method and those obtained by the Finite element-Newmark algorithm method maintain very close proximity which demonstrates that these solutions are accurate in characterizing the behaviour of waves as described by the Boussinesq wave equation.

A careful examination solutions obtained using the Finite Element Method (presented in figures 1 and 2) show that the trajectories described by the transient displacements of the waves at the various points in the domain of interest are very accurate and have approximately the same amplitudes as the exact solution.

6.0 Conclusion

We have presented in this work a model for solving the Boussinesq wave equation using the finite element method. The results obtained are accurate and efficient in characterizing the behaviour of waves. The solution to the wave equation obtained can be used in the design of the structures that are subjected to vibrations for example machine tool foundations, chatter analysis, wave control structures such as railway embankments , offshore and deep off shore oil rigs e.t.c.

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