Journal of the Nigerian Association of Mathematical Physics
Volume 11 (November 2007), 199 - 212

The control operator for the optimal control model of higher order non-dispersive waves

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Abstract:

The control operator of the Extended Conjugate Gradient Algorithm for the control of two- dimensional higher order non-dispersive waves was constructed in the paper. Explicit expressions of each elements, $R_{i,j}$, of the operator, R, were computed. These elements are useful for the implementation of the Optimal Control Formulation of the model.

Keywords: *optimal control problem, extended conjugate gradient method, non-dispersive wave, fundamental partial differential equation, penalty function.*

1.0 Introduction

Significant contributions have been made on the control of diffusion processes [9], and the control of energized wave equation [10]. Work on lower order non-dispersive wave has also been done, [6]. The optimal control model for the higher order non-dispersive waves is given as:

P1: $\min J\,(z(x,y,t),\, u(x,y,t)) = \min \int\int\int \left[z^2(x,y,t) + u^2(x,y,t) \right] \, dxdydt$

such that:

$$\frac{\partial^2}{\partial t^2} Z(x,y,t) = C_0^2 \left(\frac{\partial^2}{\partial x^2} Z(x,y,t, + \frac{\partial^2}{\partial y^2} Z(x,y,t) \right) + U(x,y,t) \qquad (1.1)$$

where $u(x,y,t)$ is the control function and z(x,y,t) is the state function.

The function space algorithm [2], of problem **PI** is too cumbersome to implement and also it has a line search procedure that is less exact [5]. We shall therefore use the Extended Conjugate Gradient Algorithm [5], which has a lot of improvement over the Function Space Algorithm.

1.1 The unconstrained problem

To make the expressions look simpler and easier to write, we adopt the following notations: $z = z(x,y,t),\; u = u(x,y,t)$

Problem (**P1**) is constrained; we therefore change it to unconstrained problem by introducing the penalty function μ . Thus we have:

P2:

$$\min J(Z,U,\mu)=\min \int\iint\left[(Z^2+U^2)+\mu \,//\, Z_{tt}-C_0Z_{xx}-c_0Z_{yy}-U\,//^2\right] dtdxdy \qquad (1.2)$$

$$0\le t\le 1, \qquad 0\le x\le 1, \qquad 0\le y\le 1.$$

1.2 The Bilinear form

We therefore rewrite equation (1.2) in the bilinear form as:

$$<v,Av>_H = \min J(z,u,\mu) = \min\iiint\Big[(Z^2+U^2)+\mu / Z_{tt}(Z_{tt}-C_0Z_{xx}-Z_{yy})-U)$$
$$-C_0Z_{xx}(Z_{tt}-C_0Z_{xx}-C_0Z_{yy}-u)-C_0Z_{yy}(Z_{tt}-C_0Z_{xx}$$
$$-C_0Z_{yy}-u)-U(Z_{tt}-C_0Z_{xx}-C_0Z_{yy}-u)\}\Big]\, dtdxdy$$
$$=\iiint[Z_1Z_2+U_1U_2+\mu\{(Z_{1tt}Z_{2tt}-Z_0Z_{1tt}Z_{2xx}-C_0Z_{1tt}Z_{2yy}-Z_{1tt}U_2)-(C_0Z_{1xx}Z_{2tt}$$
$$-C_0^2Z_{1xx}Z_{2xx}-C_0^2Z_{1xx}Z_{2yy}-C_0Z_{1xx}U_2)-C_0^2Z_{1xx}Z_{2yy}-C_0Z_{1xx}U_2)-(C_0Z_{1yy}Z_{2tt}$$
$$-C_0^2Z_{1yy}Z_{2xx}-C_0^2Z_{1yy}Z_{2yy}-C_0Z_{1yy}U_2)-(U_1Z_{2tt}-C_0U_1Z_{2xx}$$
$$-C_0Z_1Z_{2yy}-U_1U_2)\}\, dtdxdy \qquad (1.3)$$
$$=\iiint[Z_1Z_2+U_1U_2(1+\mu)+Z_{1tt}(\mu Z_{2tt}-\mu C_0Z_{2xx}-\mu C_0Z_{2yy}-\mu U_2)+Z_{1xx}(-C_0Z_{2tt}+C_0^2Z_{2xx}$$
$$+\mu C_0^2Z_{2yy}+\mu C_0U_2)+Z_{1yy}(-\mu C_0Z_{2tt}+\mu C_0^2Z_{2xx}+\mu C_0^2Z_{2yy}+\mu C_0U_2)+U_1(-\mu Z_{2tt}+\mu C_0Z_{2xx}$$
$$+\mu C_0Z_{2yy})]dtdxdy.$$

By inspection the operator *A* associated with the unconstrained optimal control problem, equation (1.3) is:

$$A=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1+\mu & -\mu & \mu C_0 & \mu C_0 \\ 0 & -\mu & \mu & -\mu C_0 & -\mu C_0 \\ 0 & \mu C_0 & -\mu C_0 & \mu C_0^2 & \mu C_0^2 \\ 0 & \mu C_0 & -\mu C_0 & \mu C_0^2 & \mu C_0^2 \end{bmatrix} \qquad (1.4)$$

2.0 Construction of the operator, *R*, for the extended conjugate gradient method

The operator A above cannot be used to implement the extended conjugate gradient algorithm as proposed by [5], because explicit knowledge of the elements of operator A will be required [6]. Howbeit, the operator A is well suited for minimizing a quadratic function, which requires a line search procedure. Further, the conjugate gradient method cannot conveniently deal with the problem, that’s equation (1.1).

From economic point of view, the operator A used for the minimization of the quadratic function is supposed to be stored in the computer’s memory thereby making demand on memory spaces. But on the other hand, the elements of operator R, which is to be used to implement the Extended Conjugate Gradient Method, have explicit expressions of its elements that are used individually in the algorithm. We construct the first two elements R_{11} and R_{21} of the operator R from the first and second, fourth orders fundamental partial differential equations from equation (1.4).

Rewrite equation (1.2) as:

$$\min <V,RV>_H \;=\; \min J(Z,U,\mu)$$

where **H** is a Hilbert space of continuous functions, square integrable of equivalence classes, [5]. Hence from equation (1.4) and equation (1.6), we can write:

$$\min < V, RV >_H = \iiint \left\{ Z_1 \overline{Z}_2 + Z_{itt} \overline{Z}_{2xx} + Z_{1xx} \overline{Z}_{2xx} + Z_{1yy} \overline{Z}_{2xx} + U_1 \overline{Z}_{2xx} \right\} dtdxdy \quad (2.1)$$

Then the functions α₁, α₂, α₄, $(\alpha_1 - \underline{Z}_2)$, $(\alpha_2 - \overline{Z}_{2xx})$ *and* $(\alpha_4 - \overline{Z}_{244})$ are contained in the space $C_n[0,1]$, where $C_n[0,1]$ is a space of continuous functions *n*-times ifferentiable. Hence by Fomin and Gelfand [3]:

$$\int_0^1 \int_0^1 \left[Z_1(\alpha_1 - \overline{Z}_2) + Z_{1tt}(\alpha \ - \overline{Z}_{2xx}) + Z_{1yy}(\alpha_4 - \overline{Z}_{2xx}) \right] dxdt = 0$$

Putting U_2=0 in equation (1.4) and setting:

$$\alpha_1 = Z_2$$

$$\alpha_2 = -\mu C_0 Z_{2tt} + \mu C_0^2 Z_{2xx} + \mu C_0^2 Z_{2yy}$$

$$\alpha_4 = \mu C_0 Z_{2tt} - \mu C_0^2 Z_{2xx} - \mu C_0^2 Z_{2yy}$$

then we have

$$\frac{\partial^2}{\partial x^2}(\alpha_2 - \overline{Z}_{2xx}) = \alpha_1 - \overline{Z}_2$$

$$\frac{\partial^2}{\partial x^2}(\alpha_4 - \overline{Z}_{2xx}) = \alpha_1 - \overline{Z}_2$$

$$\therefore \quad \frac{\partial^2}{\partial x^2}(\alpha_2 - \overline{Z}_{2xx}) = \frac{\partial^2}{\partial x^2}(\alpha_4 - \overline{Z}_{2xx})$$

$$\therefore \quad \frac{\partial^2}{\partial x^2}(\alpha_2 - \overline{Z}_{2xx}) = \frac{\partial^2}{\partial x^2}(\alpha_4 - \overline{Z}_{2xx})$$

Hence we now have to solve the partial differential equation.

$$\overline{Z}_{2tt} - \overline{Z}_{2xx} = \mu Z_{2tttt} - kZ_{2yytt} - \mu Z_{2ttxx} - \mu C_0^2 Z_{2xxxx} - \mu C_0^2 Z_{2yyxx} = 0 \qquad (2.2)$$

where $k = \mu(C_0^2 - 1)$. The initial conditions are given as:

$$\begin{aligned} &Z(0,y,t)\ Z(1,y,t) = Z(x,0,t) = Z(x,1,t) = 0 \\ &Z_x(0,y,t) = Z_x(1,y,t) = Z_y(x,0,t) = Z_y(x,1,t) = 0 \qquad (2.3) \\ &Z_{xx}(0,y,t) = Z_{xx}(1,y,t) = Z_{yy}(x,0,t) = Z_{yy}(x,1,t) = 0 \end{aligned}$$

while the boundary conditions are: $Z(x,y,0) = Z_0(x,y), \ \ Z(x,y,1) = Z_1(x,y)$

Taking the Laplace transform of equation (2.2) in x-space we have:

$$\begin{aligned} &Z_{tt}(s,y,t) - S^2 Z(s,y,t) + SZ(0,y,t) + Z_x(0,y,t) \ - \mu Z_{tttt}(s,y,t) + KZ_{yytt}(s,y,t) \\ &+ \mu S^2 Z_{tt}(s,y,t) - \mu SZ_{tt}(0,y,t) - \mu Z_{tt}(0,y,t) + \mu C_0 S^4 Z(s,y,t) - \mu C_0 S^3 Z(0,y,t) \\ &- \mu C_0 S^2 Z_y(0,y,t) - \mu C_0 S Z_{xx}(0,y,t) - \mu C_0 Z_{xxx}(0,y,t) + \mu S^2 Z_{yy}(s,y,t) \\ &- \mu S Z_{yy}(0,y,t) - \mu Z_{yyx}(0,y,t) = 0 \qquad (2.4) \end{aligned}$$

Applying equation (2.3) to equation (2.4) yields:

$$\begin{aligned} &Z_{tt}(s,y,t) - S^2 ZC_0, y,t) - \mu Z_{yyyy}(s,y,t) + KZ_{yytt}(s,y,t) + \mu S^2 Z_{tt}(s,y,t) \\ &+ \mu C_0 S^4 Z(s,y,t) + \mu S^2 Z_{yy}(s,y,t) = 0 \end{aligned} \qquad (2.5)$$

The Laplace transform of equation (2.5) in y-space gives:

$$Z_{tt}(s,p,t)-S^2Z(s,p,t)-\mu Z_{tttt}(s,p,t)+p^2 kZ_{tt}(s,p,t)$$
$$-PKZ_{tt}(s,0,t)-KZ_{tty}(s,0,t)+\mu S^2Z_{tt}(s,p,t)$$
$$+\mu C_0S^4Z(s,p,t)+\mu S^2P^2Z(s,p,t)-\mu S^2PZ(s,0,t)$$
$$-\mu S^2Z_y(s,0,t)=0 \quad (2.6)$$

Again applying equation (2.3) on equation (2.5) gives:

$$Z_{tt}(s,p,t)-S^2Z(s,p,t)-\mu Z_{tttt}(s,p,t)+P^2KZ_{tt}(s,p,t)$$
$$+\mu S^2Z_{tt}(s,p,t)+\mu C_0S^4Z(s,p,t)+\mu S^2P^2Z(s,p,t)=0 \quad (2.7)$$

Taking the Laplace transform of equation (2.7) in t-space yields:

$$q^2Z(s,p,q)-qZ(s,p,0)-Z_1(s,p,0)-S^2Z(s,p,q)-\mu q^4Z(s,p,q)$$
$$+\mu q^3Z(s,p,0)+\mu q^2Z_t(s,p,0)+\mu qZ_{tt}(s,p,0)+\mu Z_{ttt}(s,p,0)$$
$$+P^2q^2KZ(s,p,q)-P^2qKZ(s,p,0)-P^2KZ_t(s,p,0)+\mu S^2q^2Z(s,p,q)$$
$$-\mu S^2qZ(s,p,0)-\mu S^2Z_t(s,p,0)+\mu C_0S^4Z(s,p,q)+\mu S^2P^2Z(s,p,q)=0 \quad (2.8)$$

Applying equation (2.3) on equation (2.8) we have:

$$q^2Z(s,p,q)-qZ_0(s,p,0)-Z_1(s,p,0)-S^2Z(s,p,q)$$
$$-\mu q^4Z(s,p,q)+\mu q^3Z_0(s,p,0)+\mu q^2Z_t(s,p,0)+\mu qZ_{tt}(s,p,0)$$
$$+\mu Z_{ttt}(s,p,0)+P^2q^2KZ(s,p,q)-P^2qkZ(s,p,0)-p^2kZ_1(s,p,0)$$
$$+\mu S^2q^2Z(s,p,q)-\mu S^2qZ(s,p,0)-\mu S^2Z_t(s,p,0)+\mu C_0S^4Z(s,p,q)$$
$$+\mu S^2P^2Z(s,p,q)=0 \quad (2.9)$$

Dividing equation (2.9) throughout with $s^4p^2q^4$ we get:

$$\frac{1}{S^4P^2q^2}Z(s,p,q)-\frac{1}{S^4P^2q^2}Z(s,p,0)-\frac{1}{S^4P^2q^4}Z_t(s,p,0)-\frac{1}{S^4P^2q^4}Z(s,p,q)$$
$$-\frac{\mu}{S^4P^2}(s,p,q)+\frac{\mu}{S^4P^2q}Z(s,p,0)+\frac{\mu}{S^4P^2q^2}kZ(s,p,0)-\frac{\mu}{S^4p^2q^3}Z_{tt}(s,p,0)$$
$$+\frac{\mu}{S^4p^2q^4}Z_{ttt}(s,p,0)+\frac{1}{S^4q^2}KZ(s,p,q)-\frac{k}{S^4q^3}Z(s,p,0)-\frac{k}{S^4q^4}Z_t(s,p,0)$$
$$+\frac{\mu}{S^2p^2q^2}Z(s,p,q)-\frac{\mu}{S^2p^2q^3}Z(s,p,0)-\frac{\mu}{S^2p^2q^4}Z(s,p,0)$$
$$+\frac{\mu C_0}{P^2q^4}Z(s,p,q)+\frac{\mu}{S^2q^4}Z(s,p,q)=0 \quad (2.10)$$

The inverse transform of equation (2.10) in q-parameter:

$$\frac{1}{S^4p^2}\left[t\ Z\ (s,p,t)\right]-\frac{1}{S^4p^2}tZ(s,p,0)-\frac{1}{S^4p^2}\frac{t^3}{6}Z_t(s,p,0)-\frac{1}{S^2p^2}\left[\frac{t^3}{6}*Z(s,p,t)\right]$$
$$-\frac{\mu}{S^4p^2}Z(s,p,t)+\frac{\mu}{S^4p^2}Z(s,p,0)+\frac{\mu k}{S^4p^2}tZ(s,p,0)-\frac{\mu}{S^4p^2}\frac{t^2}{2}Z_{tt}(s,p,0)$$
$$+\frac{\mu}{S^4p^2}\frac{t^3}{6}Z_{ttt}(s,p,0)+\frac{k}{S^4}\left[t*Z(s,p,t)\right]-\frac{k}{S^4}\frac{t^2}{2}Z(s,p.0)-\frac{k}{S^4}\left[\frac{t^3}{6}(s,p,0)\right]$$

$$+\frac{\mu}{S^2p^2}\left[t*Z(s,p,t)\right]-\frac{\mu}{S^3p^2}\frac{t^2}{2}Z(s,p,0)-\frac{\mu}{S^2p^2}\frac{t^3}{6}Z_t(s,p,0)+\frac{\mu C_0}{p^2}\left[\frac{t^3}{6}*Z(s,p,t)\right]$$

$$+\frac{\mu}{S^2}\left[\frac{t^3}{6}*Z(s,p,t)\right]=0 \qquad (2.11)$$

Simplifying (2.11) and taking the inverse in p-parameter:

$$\frac{t}{S^2}\left[y*\int_0^t Z(s,y,t)\,dl\right]-\frac{t^2}{S^2}\left[y*Z(s,y,0)\right]-\frac{t}{S^4}\left[y*Z(s,y,0)\right]$$

$$-\frac{t^3}{6}\frac{1}{S^4}\left[y*Z_t(s,y,0)\right]-\frac{t^3}{6}\frac{1}{S^2}\left[y*\int_0^t Z(s,y,l)\,dl\right]$$

$$+\frac{t^4}{6}\frac{1}{S^2}\left[y*Z(s,y,0)\right]-\frac{\mu}{S^4}\left[y*Z(s,y,t)+\frac{\mu tk}{S^4}\left[y*Z(s,y,0)\right]\right]$$

$$+\frac{\mu}{S^4}\left[y*Z(s,y,0)\right]-\frac{t^2}{2}\frac{\mu}{S^4}\left[y*Z_{tt}(s,y,0)\right]$$

$$+\frac{t^3}{6}\frac{\mu}{S^4}\left[y*Z_{ttt}(s,y,0)\right]+\frac{kt}{S^4}\int_0^t Z(s,y,l)\,dl-\frac{kt^2}{S^4}Z(s,y,0)$$

$$-\frac{k}{S^4}\frac{t^2}{2}Z(s,y,0)-\frac{k}{S^4}\frac{t^3}{6}Z_t(s,y,0)+\frac{\mu t}{S^2}\left[y*\int_0^t Z(s,y,l)\,dl\right]$$

$$-\frac{\mu t}{S^2}\left[y*Z(s,y,0)\right]-\frac{t^2}{2}\frac{\mu}{S^2}\left[y*Z(s,y,0)\right]-\frac{\mu}{S^2}\left[y*Z_t(s,y,0)\right]$$

$$+\frac{\mu C_0 t^3}{6}\left[y*\int_0^t Z(s,y,l)\,dl\right]-\frac{t^4}{6}Z(s,y,0)=0 \qquad (2.12)$$

Similarly by simplifying equation (2.12), after taking note of the comments of [8], we have:

$$Yxt^2\int_0^x\int_0^y\int_0^t Z(h,k,l)\,dldkdt-yxt^2\int_0^x\int_0^y Z(h,k,0')\,dkdh$$

$$-yt\frac{X^3}{6}\int_0^x\int_0^y Z(h,k,0')\,dkdh-y\frac{t^3X^3}{36}\int_0^x\int_0^y Z_t(h,k',0)\,dkdh$$

$$-yx\frac{t^3}{6}\int_0^x\int_0^y\int_0^t Z(h,k,0)\,dl'dkdh+yx\frac{t^4}{6}\int_0^x\int_0^y Z_t(h,k',0)\,dkdh$$

$$-\mu y\frac{x^3}{6}\int_0^x\int_0^y Z(h,k,t')\,dhdk-\mu yx\frac{t^2}{2}\int_0^x\int_0^y Z(h,k',0)\,dkdh$$

$$+\mu kyt\frac{x^3}{6}\int_0^x\int_0^y Z(h,k',0)\,dkdh = \mu y\frac{x^3}{6}\int_0^x\int_0^y Z(h,k,0')\,dkdh$$

$$+\mu y\frac{t^2x^3}{12}\int_0^x\int_0^y Z(h,k',0)\,dkdh + \mu y\frac{t^3x^3}{36}\int_0^x\int_0^y Z_{tt}(h',k,0)\,dkdh$$

$$+kt\frac{x^3}{6}\int_0^x\int_0^y Z(h,y,l)\,d'ldh - k\frac{tx^3}{6}\int_0^x Z(h,y,0)\,dh - k\frac{t^2x^3}{12}\int_0^x Z(h,y,0)\,dh$$

$$-kx\frac{t^3}{6}\int_0^x Z_t(h,y,0')\,dh = \mu xyt\int_0^x\int_0^y Z(h,k,0)\,dkdh$$

$$-\mu yx\frac{t^2}{2}\int_0^x\int_0^y Z(h,k,0)dkdh - yx\int_0^x\int_0^y Z_t(h,k,0)\,dkdh$$

$$+\mu C_0\frac{t^3}{6}y\int_0^y\int_0^t Z(x,k',0)\,dldk - \mu yx\int^x\int^y Z_t(h,k,0)\,dkdh$$

$$+\mu C_0 y\frac{t^3}{6}\int_0^y\int_0^t Z(x,k,l)\,dldk - \frac{t^4}{6}Z(x,y,0) = 0 \qquad 2.13)$$

Taking the inverse transform of equation (2.13) in s-parameter:

$$yt^2\left[x*\int_0^y\int_0^t Z(x,k,l)\,dldk\right] - yt^2\left[x*\int_0^y Z(s,k,0)\,dk\right]$$

$$-yt\left[\frac{x^3}{6}*\int_0^y Z(x,k,0)\,dk - y\frac{t^3}{6}\left[\frac{x^3}{6}*\int_0^y Z_t(x,k,0)\,dk\right.\right.$$

$$-\frac{yt^3}{6}\left[x*\int_0^y\int_0^t Z(x,k,0)\,dldk\right] + \frac{t^4}{6}y\left[x*\int_0^y Z_t(x,k,0)\,dk\right.$$

$$-\mu y\left[\frac{x^3}{6}*\int_0^y Z(x,k,t)\,dk\right] - \mu y\frac{t^2}{2}\left[X\ 0*\int_0^y Z_{tt}(x,k,0)\,dk\right.$$

$$+\mu k\ yt\left[\frac{x^3}{6}*\int_0^y Z(x,k,0)\,dk\right] + uy\left[\frac{x^3}{6}*\int_0^y Z(x,k,0)\,dk\right]$$

$$+\mu y\frac{t^2}{2}\left[\frac{x^3}{6}*\int_0^y Z(x,k,0)\,dk + \mu y\frac{t^3}{6}\left[\frac{x^3}{6}*\int_0^y Z_{ttt}(x,k,0)\,dk\right.\right.$$

$$+kt\left[\frac{x^3}{6}*\int\ Z(x,y,l)\,dl - kt\left[\frac{x^3}{6}*Z(x,y,0)\right]\right.$$

$$-k\frac{t^2}{2}\left[\frac{x^3}{6}*Z(x,y,0)\right] - k\frac{t^3}{6}\left[x*Z_t(x,y,0)\right]$$

$$+\mu ty\left[x*\int_0^y Z(x,k,0)\,dk - \mu\frac{t}{2}y\left[x*\int\ Z(x,k,0)\,dk\right]\right.$$

$$-\mu y\left[x*\int_0^y Z_t(x,k,0)\,dk\right] + \mu C_0 y\frac{t^3}{6}\int_0^y\int_0^t Z(x,k,0)\,dkdl$$

$$-\mu y\left[x*\int_0^y Z_t(x,k,0)\,dk + \mu C_0 y\frac{t^3}{6}\int_0^y\int_0^t Z(x,k,l)\,dldk\right. \qquad (2.14)$$

$$-\frac{t^4}{6}Z(x,y,0)=0$$

Differentiating equation (2.14) with respect to t, we have the expression:

$$\frac{t^2}{S^2}\,y\int_0^y\int_0^t Z(s,k,l)\,dldk-\frac{t^2}{S^2}\,y\int_0^y Z(s,k,0)dk$$
$$-\frac{t}{S^4}\,y\int_0^y Z(s,k,0)\,dk-\frac{t^3}{6S^4}\,y\int_0^y Z_t(s,k,0)\,dk-\frac{t^3}{6S^2}\,y\int_0^y\int_0^t Z(s,k,l)\,dldk$$
$$+\frac{t^4}{6S^2}\,y\int_0^t Z_t(s,k,0)\,dk-\frac{\mu}{S^4}\,y\int_0^y Z(s,k,t)\,dk-\frac{\mu t^2}{2S^2}\,y\int_0^y Z_{tt}(s,k,0)\,dk$$
$$+\mu k\frac{t}{S^4}\,y\int_0^y Z(s,k,0)\,dk\quad+\quad\mu\frac{y}{S^4}\int_0^y Z(s,k,0)\,dk$$

$$xt^2\int_0^x\int_0^y\int_0^t Z(h,k',l)\,dldh+xy^2\int_0^x\int_0^y Z(h,y,l')\,dldh-xt^2\int_0^x\int_0^y Z(h,k,0)\,dkdh$$
$$-yxt^2\int_0^x Z(h,y,0)\,dh-\frac{x^3}{6}t\int_0^x\int_0^y Z(h,k,0)\,dhdk-yt\frac{x^3}{6}\int_0^x Z(h,y,0)\,dh$$
$$-\frac{t^3x^3}{36}\int_0^x\int_0^y Z_t(h,k,0)\,dkdl-y\frac{t^3x^3}{36}\int_0^x Z_t(h,y,0)\,dh$$
$$-\frac{xt^3}{6}\int_0^x\int_0^y\int_0^t Z(h,k,0')\,dldkdh-y\frac{xt^3}{6}\int_0^x\int_0^t Z(h,y,0)\,dldh$$
$$+\frac{xt^4}{6}\int_0^x\int_0^y Z_t(h,k,0)\,dkdh+y\frac{t^4}{6}x\int_0^x Z_t(h,y,0)\,dh$$
$$-\frac{\mu x^3}{6}\int_0^x\int_0^y Z(h,k,t)\,dkdh-\mu y\frac{x^3}{6}\int_0 Z(h,y,t)\,dh$$

$$+\mu\frac{t^2}{2S^4}\,y\int_0^t Z(s,k,0)\,dk+\mu\frac{t^3}{6S^4}\,y\int_0^y Z_{ttt}(s,k,0)\,dk$$

$$+k\frac{t}{S^4}\int_0^t Z(s,y,l)\,dl-k\frac{t}{S^4}Z(s,y,0)-k\frac{t^2}{2S^4}Z(s,y,0)$$

$$-k\frac{t^3}{6S^2}Z_t(s,y,0)+\frac{\mu t}{S^2}\,y\int_0^t Z(s,k,0)\,dk-\frac{\mu t^2}{2S^2}\,y\int_0^y Z(s,k,0)\,dk$$

$$-\mu\frac{y}{S^2}\int_0^y Z_t(s,k,0)\,dk+\mu C_0\frac{t^3}{0}y\int_0^y\int_0^t Z(s,k,0)\;dk-\frac{\mu y}{S^2}\int_0^y Z_t(s,k,0)\,dk$$

$$+\mu C_0\frac{t^3}{0}\,y\int_0^y\int_0^t Z(s,k,l)\,dldk-\frac{t^4}{6}Z(s,y,0)=0$$

$$-\mu x\frac{t^3}{2}\int_0^x\int_0^y Z_{tt}(h,k,0)\,dkdh-\mu yx\frac{t^2}{2}\int_0^x Z_{tt}(h,y,0)\,dh$$

$$+\mu kt\frac{x^3}{6}\int_0^x\int_0^y Z(h,k.0)\,dkdh+\mu kyt\frac{x^3}{6}\int_0^x Z(h,y,0)\,dh$$

$$+\mu\frac{x^3}{6}\int_0^x\int_0^y Z(h,k,0)\;dkdh+\mu y\frac{x^3}{6}\int_0 Z(h,y,0)\,dh$$

$$+\frac{\mu t^2x^3}{12}\int_0^x\int_0^y Z(h,k,0)\,dkdh+\mu y\frac{t^2x^3}{12}\int_0^x Z(h,y,0)\,dh$$

$$+\mu\frac{t^3x^3}{36}\int_0^x\int_0^y Z_{ttt}(x,k',0)\,dkdh+\mu y\frac{t^3x^3}{36}\int_0^x Z_{ttt}(x',y,0)\,dh$$

$$kt\frac{x^3}{6}\int_0^x Z(h,y,t)\,dh - k\frac{tx^3}{6}\int_x^0 Z_y(h,y,0)\,dh - k\frac{t^2x^3}{12}\int_0^x Z_y(h,y,0)\,dh$$
$$-kx\frac{t^3}{6}\int_0^x Z_{ty}(h,y,0)\,dh + \mu xt\int_0^x\int_0^y Z(h,k,0)\,dkdh$$
$$+\mu yxt\int_0^x Z(h,y,0)\,dh - \mu xt\frac{t^2}{2}\int_0^x\int_0^y Z(h,k,0)\,dkdh$$
$$-\mu yx\frac{t^2}{2}\int_0^x Z(h,y,0)\,dh - x\int_0^x\int_0^y Z_t(h,k,0)\,dkdh$$
$$-yx\int_0^x Z_t(h,y,0)\,dh + \mu C_0\frac{t^3}{6}\int_0^y\int_0^t Z(x,k,0)\,dkdh$$
$$+\mu C_0 y\frac{t^3}{6}\int_0^t Z(x,y,0)\,dl - \mu x\int_0^x\int_0^y Z_t(h,k,0)\,dkdh$$
$$-\mu xy\int_0^x Z_t(h,y,0)\,dh + \mu C_0\frac{t^3}{6}\int_0^y\int_0^t Z(x,k,l)\,dldk$$
$$+\mu C_0 y\frac{t^3}{6}\int_0^t Z(x,y',l)\,dl + -\frac{t^4}{6}Z_y(x,y',0) = 0 \qquad (2.15)$$

Simplifying (2.15), we have:

$$\mu C_0 y\frac{t^3}{6}\int_0^t Z(x,y',l)\,dl - \frac{t^4}{6}Z_y(x,y',0) + \mu C_0\frac{t^3}{6}\int_0^y\int_0^t Z(x,k',l)\,dldk$$
$$+\mu C_0\frac{t^3}{6}\int_0^y\int_0^t Z(x,k,0')\,dldk + \mu\frac{t^3x^3}{36}\int_0^x\int_0^y Z_{ttt}(x,k',0)\,dkdh$$
$$+\mu y\frac{t^3x^3}{36}\int_0^x Z(x,y,0')\,dh - \mu\frac{x^3}{6}\int_0^x\int_0^y Z(h,k',t)\,dkdh$$
$$-\mu y\frac{x^3}{6}\int_0^x Z(h,y',t)\,dh - \frac{xt^3}{6}\int_0^x\int_0^y\int_0^t Z(h,k',0)\,dldkdh$$
$$-y\frac{x^3t^3}{36}\int_0^x Z_t(h,y,0')\,dldh + x\frac{t^2}{2}\int_0^x\int_0^y\int_0^t Z(h',k,l)dldkdh + xyt^2\int_0^x\int_0^t Z(h,y',l)\,dldh$$
$$-kx\frac{t^3}{6}\int_0^x Z_{ty}(h,y',0)\,dh + (-x-\mu x - xyt^2 + x\frac{t^4}{6})\int_0^x\int_0^y Z_t(h,k,0)\,dkdh$$
$$-\mu yx\frac{t^2}{2}\int_0^x Z_{tt}(h,y,0)\,dh + (\frac{-x^3}{6}t - xt^2 + \mu kt\frac{x^3}{6} + \frac{\mu x^3}{6} + \frac{\mu t^2x^3}{12}$$
$$+\mu xt)\int_0^x\int_0^y Z(h,k,0)\,dkdh + (-yxt^2 - y\frac{tx^3}{6} + \mu kyt\frac{x^3}{6} + \mu y\frac{x^3}{6} + \mu t\frac{t^2x^3}{12}$$
$$+\mu yxt - \mu yx\frac{t^2}{2})\int_0^x Z(h,y',0)\,dh + \mu C_0 y\frac{t^3}{6}\int_0^t Z(x,y,0')\,dl$$
$$-yx\frac{t^3}{6}\int_0^x\int_0^t Z(h,y,0')\,dldh + kt\frac{x^3}{6}\int_0^x Z(h,y',t)\,dh + (yx\frac{t^4}{6} - \mu xy - yx)\int_0^x Z_t(h,y,0)\,dh \qquad (2.16)$$
$$+(-kt\frac{x^3}{6} - k\frac{t^2x^3}{12})\int_0^x Z_y(h,y,0)\,dh = 0$$

Differentiating (1.24) with respect to t gives:

$$\mu y C_0^2 \frac{t^2}{3}\int_0^t Z(x,y,l')\,dl + \mu y C_0 \frac{t^3}{6} Z(x',y,t) - \frac{2t^3}{3} Z_y(x',y,0)$$

$$+ \mu C_0^2 \frac{t^2}{2}\int_0^y\int_0^t Z(x,k,l')\,dldk + \mu C_0^2 \frac{t^3}{6}\int_0^y Z(x',k,t)\,dk + 2C_0^2 \frac{t^3}{3}\int_0^y Z(x',k,0)\,dk$$

$$+\mu \frac{t^2 x^3}{12}\int_0^x\int_0^y Z_{ttt}(x,k',0)\,dkdh + \mu y \frac{t^2 x^3}{12}\int_0^x Z(x,y',0)\,dh$$

$$-\mu \frac{x^3}{6}\int_0^x\int_0^y Z_t(h,k,t)\,dkdh - \mu \frac{x^3}{6} y\int_0^x Z_t(h,y,t)\,dh - \frac{2t^3}{3} x\int_0^x\int_0^y Z(h,k,0)\,dkdh$$

$$-y\frac{x^3 t^2}{12}\int_0^x Z_t(h,y',0)\,dh + xt\int_0^x\int_0^y\int_0^t Z(h,k',l)\,dldkdh$$

$$+ x\frac{t^2}{2}\int_0^x\int_0^y Z(h,k,t')\,dkdh + 2xyt\int_0^x\int_0^t Z(h,y',l)\,dldh$$

$$+ xy\frac{t^2}{2}\int_0^x Z(h,y,t')\,dh - kx\frac{t^2}{2}\int_0^x Z_{tt}(h,y',0)\,dh$$

$$+(-2xyt + 4\frac{xt^3}{6})\int_0^x\int_0^y Z_t(h',k,0)\,dkdh - \mu xt\int_0^x\int_0^y Z_{tt}(h,k,0)\,dkdh$$

$$-\mu yxt\int_0^x Z_{tt}(h,y,0)\,dh + (-\frac{x^3}{6} - 2xt + \mu k\frac{x^3}{6} + \mu k\frac{x^3}{6} + \mu x)\int_0^x\int_0^y Z(h,k,0)\,dkdh$$

$$+(-2xyt - y\frac{x^3}{6} + \mu ky\frac{x^3}{6} + \mu yt\frac{x^3}{6} + \mu xy - \mu yxt)\int_0^x Z(h,y,0)\,dh$$

$$+ \mu C_0^2 y\frac{t^3}{3} Z(x,y,0') - 2xy\frac{t^3}{3}\int_0^x Z(h,y',0)\,dh + \frac{kx^3}{6}\int_0^x Z(h,y,t')\,dh + k\frac{tx^3}{6}\int_0^x Z_t(h',y,t)\,dh$$

$$+2xy\frac{t^2}{3} x\int_0^x Z_t(h,y,0')\,dh \qquad (2.17)$$

Simplifying and re-arranging equation (1.25) we have:

$$R_{11}\bar{Z}_2 = Z(x,y,t)$$

$$= \frac{4}{\mu C_0^2 y} z_y(x,y,0) - \frac{1}{t}\int_0^t Z(x,y,l) - \frac{3}{sty}\int_0^y\int_0^t Z(x,k,l)\,dldk - \frac{1}{y}\int_0^y Z(x,k,t)\,dk$$

$$- \frac{4}{\mu y}\int_0^y Z(x,k,0)\,dk - \frac{x^3}{2ytC_0^2}\int_0^x\int_0^y Z_{ttt}(x,k,0)\,dkdh - \frac{x^3}{2tC_0^2}\int_0^x Z(x,y,0)\,dh$$

$$+ \frac{x^3}{yC_0^2 t^3}\int_0^x\int_0^y Z_t(h,k,t)\,dkdh + \frac{x^3}{C_0^2 t^3}\int_0^x Z_t(h,y,t)\,dh + \frac{4x}{\mu C_0^2 y}\int_0^x\int_0^y Z(h,k,0)\,dkdh$$

$$+ \frac{x^3}{2tC_0^2}\int_0^x Z_t(h,y,0)\,dh - \frac{6x}{\mu C_0^2 t^2 y}\int_0^x\int_0^y\int_0^t Z(h,k,l)\,dldkdh$$

$$- \frac{3x}{\mu ytC_0^2}\int_0^x\int_0^y Z(h,k,t)\,dkdh - \frac{12x}{\mu C_0^2 t^2}\int_0^x\int_0^t Z(h,y,l)\,dldh$$

$$- \frac{3x}{\mu C_0^2 t}\int_0^x Z(h,y,t)\,dh + \frac{3kx}{\mu C_0^2 t}\int_0^x Z_{tt}(h,y,0)\,dh$$

$$-(-\frac{12x}{\mu C_0^2 t^2} + \frac{4x}{\mu y C_0^2})\int_0^x\int_0^y Z_t(h,k,0)\,dkdh + \frac{6x}{yC_0^2 t^2}\int_0^x\int_0^y Z_{tt}(h,k,0)\,dkdh$$

$$+ \frac{6x}{C_0^2 t^2}\int_0^x Z_{tt}(h,y,0)\,dh - (-\frac{x^3}{\mu y C_0^2 t^2} - \frac{12x}{\mu C_0^2 t^2} + \frac{kx^3}{yC_0^2 t^2} + \frac{6x}{yC_0^2 t^2} + \frac{x^3}{yC_0^2 t^2})\int_0^x\int_0^y Z(h,k,0)\,dkdh$$

$$- (-\frac{12x}{\mu C_0^2 t^2} - \frac{x^3}{\mu C_0^2 t^3} + \frac{kx^3}{C_0^2 t^3} + \frac{x^3}{C_0^2 t^2} + \frac{6x}{C_0^2 t^3} - \frac{6x}{C_0^2 t^2})\int_0^x Z(h,y,0)\,dh$$

$$-2Z(x,y,0) + \frac{4x}{\mu C_0^2}\int_0^x Z(h,y,0)\,dh - \frac{kx^3}{\mu y C_0^2 t^3}\int_0^x Z(h,y,t)\,dh$$

$$- \frac{kx^3}{\mu C_0^2 y t^3}\int_0^x Z_t(h,y,t)\,dh - \frac{4x}{\mu C_0^2}\int_0^x Z_t(h,y,0)\,dh$$

and the remaining terms give: $R_{21}\overline{Z}_2 = -\mu Z_{tt} + \mu c_0 Z_{xx} + \mu c_0 Z_{yy}$ (2.18)

Further we also construct the elements R_{12} and R_{22} of the operator *R* from equation (1.4) as follows: Putting $Z_2 = 0$ in equation (1.4) yields:

$$\min(J,u,\mu) = \int_0^1\int_0^1\int_0^1 \left[Z_{1tt}(-\mu U_2) + Z_{1xx}(\mu C_0 U_2) + Z_{1yy}(\mu C_0 U_2) + u_1 u_2 (1+\mu)\right] dxdtdy \qquad (2.19)$$

Let there exist $\overline{U}_2$ *and* $\hat{U}_2$ such that equation (2.19) be equivalent to the equation:

$$\min J(z,u,\mu) = \min \int_0^1\int_0^1\int_0^1 (Z_{1tt}\overline{U}_2 + Z_{1xx}\overline{U}_2 + U_1\hat{U}_2 + Z_{1yy}\overline{U}_2)\,dxdtdy. \qquad (2.20)$$

Putting $\beta_1 = \overline{U}_2 - \mu U_{2tt}$, $\beta_2 = \overline{U}_2 - \mu C_0 U_{2xx}$, $\beta_3 = \hat{U}_2 - \mu C_0 U_{2yy}$ therefore $\beta_1 - \bar{u}_{2tt}$ and $\beta_3 - u_{2xx}$ are continuous functions. We assume that these functions contained

in $D_n[0,1]$ where $D_n[0.1]$ is a set of n-times differentiable functions and the derivatives are also continuous in the set. Then by [3] we have:

$$\int_0^1\int_0^1\int_0^1 \Big(Z_{1tt}(\beta_1 - \overline{U}_{2tt}) + Z_{1xx}(\beta_2 - \overline{U}_{2xx}\Big) + Z_{1yy}(\beta_3 - \overline{U}_2)$$
$$+ Z_{1yy}(\beta_3 - \overline{U}_{2yy})\ dxdydt = 0 \qquad (2.21)$$

Therefore

$$\frac{\alpha^2}{\alpha y^2}(\beta_3 - \overline{U}_{2yy}) = \frac{\alpha^2}{\alpha t^2}(\beta_1 - \overline{U}_{2tt})$$

Hence

$$\beta_{3yy} - \overline{U}_{2yyyy} = \beta_{1tt} - \overline{U}_{2tttt}$$
$$\Rightarrow \quad \beta_{3yy} - \beta_{1tt} = \overline{U}_{2yyyy} - \overline{U}_{2tttt} \qquad (2.22)$$

Therefore, since $\beta_3 = -\beta_1$ by virtue of equation (2.22), we have to solve the partial differential equation:

$$\frac{\mu\partial^4}{\partial y^4}U(x,y,t) - \mu\frac{\partial^4}{\partial t^4}U(x,y,t) = \mu U_{yy}(x,y,t) + \mu U_{tt}(x,y,t) \qquad (2.23)$$

with the initial and boundary conditions:

$$U(0,y,t) = U(1,y,t) = U(x,0,t) = U(x,1,t) = 0$$
$$U_x(0,y,t) = U_x(1,y,t) = U_y(x,0,t) = U_y(x,1,t) = 0$$
$$U_{xx}(0,y,t) = U_{xx}(1,y,t) = U_{yy}(x,0,t) = U_{yy}(x,1,t) = 0$$
(2.24)
$$U(x,y,0) = U_0(x,y)$$
$$U_t(x,y,0) = U_1(x,y)$$

From [10] equation (2.24) with its initial and boundary conditions equation (2.24) becomes:

$$\frac{t^3}{6}\int_0^t U(x,y,t)\,dt - \frac{t^4}{6}U(x,y,t) - \frac{y^3}{6}\int_0^y U(x,y,t)\,dy$$
$$+\frac{t^3}{6}\int_0^y U(x,y,0)\,dy + \frac{ty^3}{6}\int U_t(x,y,0)\,dy + \frac{y^3t^2}{12}\int_0^y U_{tt}(x,y,0)\,dy$$
$$+\frac{y^3t^3}{36}\int_0^y U_{ttt}(x,y,0)\,dy - \frac{yt^3}{6}\int_0^y\int_0^t U(x,y,t)\,dtdy + \frac{yt^4}{6}\int_0^y U(x,y,t)\,dy = 0 \qquad (2.25)$$

When equation (2.25) is first re-arranged and later differentiated with respect to t, we obtain our control operator element, R_{12} as:

$$R_{12}U_2 = 4U(x,y,t) - \frac{3}{t}\int_0^t U(x,y,t)dt + \int_0^y U(x,y,t)\,dy$$
$$+\frac{2}{t}\int_0^y U(x,y,0)\,dy + \int_0^t U_t(x,y,0)\,dy + t0\int^y U_{tt}(x,y,0,)\,dy$$
$$+\frac{t^2}{2}\int_0^y U_{tt}(x,y,t)\,dy - 3ty\int_0^y\int_0^t U(x,y,t)\,dtdy \qquad (2.26)$$
$$-y\int_0^y U(x,y,t)\,dy + 4y\int_0^y U(x,y,0)\,dy$$

By comparing equation (1.3) and equation (2.19), we obtain the last element of R as:

$$R_{22}U_2 = U_2(1+\mu) \qquad (2.27)$$

3.0 Conclusion

The algorithm of the extended conjugate gradient method for the implementation of problem **P1**, using the derived control operator *R*, is:

Guess: $z_0(x,y,t), u_0(x,y,t)$

Compute:

$g_i = <\Delta_z J(z,u,\mu), \Delta_u J(z,u,\mu)>, g_i = p_i$, where J(z,u,μ) is given as:

$$\min J(z,u,\mu) = \min\int_0^1\int_0^1\int_0^1[(z^2+u^2) + \mu // Z_{tt} - C_0Z_{xx} - c_0Z_{yy} - u)]dtdxdy$$

Update state and control variables:

$$z_i(x,y,t), \quad u_i(x,y,t)$$

$$z_{i+1}(x,y,t)=z_i(x,y,t)+\alpha_i(x,y,t)\rho_i(x,y,t)$$
$$u_{i+1}(x,y,t)=u_i(x,y,t)+\alpha_i(x,y,t)\rho_i(x,y,t)$$
where $\alpha_i(x,y,t)=<g_i(x,y,t),g_i(x,y,t)>/<\rho_i(x,y,t),R_i(x,y,t),\rho_i(x,y,t)$

Update the gradient:

$$g_{i+1}(x,y,t)=g_i(x,y,t)+\beta_i(x,y,t)R\rho_i(x,y,t)$$

where $\beta_i(x,y,t)=<g_{i+1}(x,y,t),g_{i+1}(x,y,t)>/\ g_i(x,y,t),g_i(x,y,t)$

Update the descent direction:

$$\rho_{i+1}(x,y,t)=-g_{i+1}(x,y,t)+\beta_i(x,y,t)\rho_i(x,y,t)$$

where $\beta_i(x,y,t)=<g_{i+1}(x,y,t),g_{i+1}(x,y,t)>/\ g_i(x,y,t),g_i(x,y,t)$

The explicit expressions of the components of operator, R, used in the algorithm above are summarized below:

1. $$R_{11}\bar{Z}_2 = \frac{4}{\mu C_0^2 y} z_y(x,y,0) - \frac{1}{t}\int_0^t Z(x,y,l) - \frac{3}{sty}\int_0^y\int_0^t Z(x,k,l)\,dldk - \frac{1}{y}\int_0^y Z(x,k,t)\,dk$$
$$- \frac{4}{\mu y}\int_0^y Z(x,k,0)\,dk - \frac{x^3}{2ytC_0^2}\int_0^x\int_0^y Z_{ttt}(x,k,0)\,dkdh - \frac{x^3}{2tC_0^2}\int_0^x Z(x,y,0)\,dh$$
$$+ \frac{x^3}{yC_0^2t^3}\int_0^x\int_0^y Z_t(h,k,t)\,dkdh + \frac{x^3}{C_0^2t^3}\int_0^x Z_t(h,y,t)\,dh + \frac{4x}{\mu C_0^2 y}\int_0^x\int_0^y Z(h,k,0)\,dkdh$$
$$+ \frac{x^3}{2tC_0^2}\int_0^x Z_t(h,y,0)\,dh - \frac{6x}{\mu C_0^2t^2y}\int_0^x\int_0^y\int_0^t Z(h,k,l)\,dldkdh$$
$$- \frac{3x}{\mu ytC_0^2}\int_0^x\int_0^y Z(h,k,t)\,dkdh - \frac{12x}{\mu C_0^2t^2}\int_0^x\int_0^t Z(h,y,l)\,dldh$$
$$- \frac{3x}{\mu C_0^2 t}\int_0^x Z(h,y,t)\,dh + \frac{3kx}{\mu C_0^2 t}\int_0^x Z_{tt}(h,y,0)\,dh$$
$$-\left(-\frac{12x}{\mu C_0^2t^2} + \frac{4x}{\mu yC_0^2}\right)\int_0^x\int_0^y Z_t(h,k,0)\,dkdh + \frac{6x}{yC_0^2t^2}\int_0^x\int_0^y Z_{tt}(h,k,0)\,dkdh$$

$$+ \frac{6x}{C_0^2t^2}\int_0^x Z_{tt}(h,y,0)\,dh - \left(-\frac{x^3}{\mu yC_0^2t^2} - \frac{12x}{\mu C_0^2t^2} + \frac{kx^3}{yC_0^2t^2} + \frac{6x}{yC_0^2t^2} + \frac{x^3}{yC_0^2t^2}\right)$$
$$\left(\int_0^x\int_0^y Z(h,k,0)\,dkdh\right) - \left(-\frac{12x}{\mu C_0^2t^2} - \frac{x^3}{\mu C_0^2t^3} + \frac{kx^3}{C_0^2t^3} + \frac{x^3}{C_0^2t^2} + \frac{6x}{C_0^2t^3} - \frac{6x}{C_0^2t^2}\right)$$
$$\left(\int_0^x Z(h,y,0)\,dh\right) - 2Z(x,y,0) + \frac{4x}{\mu C_0^2}\int_0^x Z(h,y,0)\,dh - \frac{kx^3}{\mu yC_0^2t^3}\int_0^x Z(h,y,t)\,dh - \frac{kx^3}{\mu C_0^2yt^3}$$

$$(\int_0^x Z(h,y,0)\,dh) - 2Z(x,y,0) + \frac{4x}{\mu C_0^2}\int_0^x Z(h,y,0)\,dh - \frac{kx^3}{\mu y C_0^2 t^3}\int_0^x Z(h,y,t)\,dh - \frac{kx^3}{\mu C_0^2 yt^3}$$

$$(\int_0^x Z_t(h,y,t)\,dh - \frac{4x}{\mu C_0^2}\int_0^x Z_t(h,y,0)\,dh)$$

2. $R_{21}\bar{Z}_2 = -\mu Z_{tt} + \mu c_0 Z_{xx} + \mu c_0 Z_{yy}$

3. $$R_{12}U_2 = 4U(x,y,t) - \frac{3}{t}\int_0^t U(x,y,t)\,dt + \int_0^y U(x,y,t)\,dy$$
$$+\frac{2}{t}\int_0^y U(x,y,0)\,dy + \int_0^t U_t(x,y,0)\,dy + t0\int^y U_{tt}(x,y,0,)\,dy$$
$$+\frac{t^2}{2}\int_0^y U_{tt}(x,y,t)\,dy - 3ty\int_0^y\int_0^t U(x,y,t)\,dtdy$$
$$-y\int_0^y U(x,y,t)\,dy + 4y\int_0^y U(x,y,0)\,dy$$

4. $R_{22}U_2 = U_2(1+\mu)$

With this information, it is possible to implement the formulated constrained optimization problem, equation (1.2), using the extended conjugate gradient algorithm.

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