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Constructing the operator matrix for the optimal control of linear lower order non-dispersive waves

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Abstract

Non-dispersive waves propagate with constant phase velocities therefore they are indispensable in communication and some other areas of application. This paper constructed the Operator, R, for the control of linear lower order non-dispersive wave equation using the extended conjugate gradient Method proposed in* [2]. *The work of Ibiejugba et al,* [3] *was on the role of multipliers in the multiplier method which was applied to dynamical system. This work involves system governed by first order partial differential equation, namely linear lower order non-dispersive wave.

Keywords: *optimal control problem, extended conjugate gradient method, non-dispersive wave, fundamental partial differential equation*

1.0 Introduction

The optimal problem for linear lower order non-dispersive waves was formulated by [6] and given as:
P1

$$\min J(x,u) = \int_0^1 \int_0^1 (z^2 + u^2)\, dxdt \qquad 1.1)$$

$$\frac{\partial z}{\partial t} + c_0 \frac{\partial z}{\partial x} = u(x,t), \quad c_0 \text{ is a constant}$$

with the initial conditions: $z(0,t) = z(l,t) = 0$ and the boundary condition: $z(x,0) = z_0(x)$

The paper [6], used the Hamiltonian of the problem to determine the control u(x,t) and the state z(x,t). A follow up of the work is presented here. The present work will use the Extended Conjugate Gradient Method, ECGM, due to [2] to solve the problem. The work of Ibiejugba et al (1992) was on the role of the multipliers in the multiplier methods and was applied to dynamical systems. The system of the present work is that governed by partial differential equation and was inspired by the works of Ibiejugba et al [5], Otunta [7], Reju et al [10] and others. The construction of the control operator is central in the algorithm of the ECGM. Our approach is to convert the constrained optimization problem, equation (1.1), into an unconstrained optimization problem through the introduction of a penalty function, μ>0, following the approach of [2], [5] and [6].

The mathematical model of lower order non-dispersive waves was given by;

$$\frac{\partial z}{\partial t} + c_0 \frac{\partial z}{\partial x} = u(x,t), \quad c_0 \text{ is a constant}$$

the initial conditions

$$z(0,t) = z(l,t) = 0 \tag{1.2}$$

the boundary condition

$$z(x,0) = z_0(x)$$

The function u(x,t) is the external force causing the disturbances.

1.1 The Bilinear form

Following Ibiejugba (1986) [2], Otunta (1998) [6], Reju (2000) [8] and others, the problem P1 shall be converted into a bilinear form: that is we need continuous functions w_1 and w_2 in $C^1[0,1]$ such that:

$$\min J(z,u,\mu) = \langle w_1, \ Aw_2 \rangle_H \tag{1.3}$$

where $w_1 = \{z_1, z_{1x}, z_{1t}, u_1\}$; $w_2 = \{z_2, z_{2x}, z_{2t}, u_2\}$.

A is the control operator in equation (1.3), while the space H is a Hilbert space of continuous functions, square integrable and of equivalence classes. Let $z_t = \frac{\partial z}{\partial t}$ and $z_x = \frac{\partial z}{\partial x}$.

Expanding equation (2.6) bilinearly, we have:

$$J\ (z,u,\mu) = \int_0^1 \int_0^1 \{(z_1 z_2 + u_1 u_2) + \mu(u_1 u_2 - u_1 z_{2t} - c_0 u_1 z_{2x}) - \mu(z_{11} u_2 - z_{11} z_{2t} - c_0 z_{1t} z_{2x})$$
$$- \mu(c_0 z_{1x} u_2 - c_0 z_{1x} z_{2t} - c_0^2 z_{1x} z_{2x})\} dxdt$$
$$= \int_0^1 \int_0^1 \{z_1 z_2 + \mu_1 u_2 + \mu u_1 u_2 - \mu u_1 z_{2t} - \mu c_0 u_1 z_{2x} - z_{1t} u_2 + z_{1t} z_{2t} + c_0 z_{1t} z_{2x}$$
$$- c_0 z_{1x} u_2 + c_0 z_{1x} z_{2t} + c_0 z_{1x} z_{2t} + c_0^2 z_{1x} z_{2x}\} dxdt \tag{1.4}$$

This gives the control operator A associated with equation (1.3) as:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mu c_0 & \mu c_0 & -\mu \\ 0 & \mu c_0 & \mu & -\mu \\ 0 & -\mu c_0 & -\mu & 1+\mu \end{pmatrix}$$

Despite the fact that the operator A is symmetric, it is not used for Extended Conjugate Gradient Algorithm, because A is not positive semi-definite, bounded and self-adjoint [6].

2. 0 Construction Of Control Operator For The Extended Conjugate Gradient Method

Let R denote the operator for the ECGM, we need a vector V = (z, u) such that equation (1.3) and equation (1.4) are equivalent. That is:

$$\min < v, Rv >_H = \int_0^1 \int_0^1 \ \{ z_1 z_2 + u_1(u_2 + \mu u_2 - \mu z_{2t} - \mu c_0 z_{2x}) + z_{1x}(\mu c_0 z_{2x} - \mu u_2$$
$$- \mu c_0{}^2 z_{2x}) + z_{1t}(\mu z_{2t} - \mu u_2 - \mu c_0 z_{2x})) dxdt \tag{2.1}$$

Let there exist continuous functions z_2, z_{2x} *and* z_{2t}, which are square integrable such that equation (2.1) is satisfied. Therefore, we have:

$$\min < v, Rv > \ = \int_0^1 \int_0^1 \{ z_1 z_2 + u_1 z_{2x} + z_{1x} z_{2x} + z_{1t} z_{2t} \} dxdt \tag{2.2}$$

Let $v_2 = \{z_2, u_2\}$, then

$$Rv_2 = \begin{bmatrix} R_{11}z_2 + R_{12}u_2 \\ R_{21}z_2 + R_{22}u_2 \end{bmatrix} \tag{2.3}$$

Putting $u_2 = 0$, in equation (2.1) and equation (2.3) we have

$$Rv_2 = \begin{bmatrix} R_{11}z_1 \\ R_{21}z_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

and equation (2.1) reduces into

$$\min < v, Rv >_H = \int_0^1 \int_0^1 \left(z_1 z_2 + u_1(-\mu t_{2t} - \mu co z_{2x}) + z_{1x}\left(\mu co\, z_{2x} - \mu co^2 t_{2x}\right)\right.$$
$$\left. + z_{1t}\left(\mu z_2 + \mu co z_{2x}\right) dxdt. \right. \tag{2.4}$$

Let

$\alpha_1 = z_2$

$\alpha_2 = -\mu z_{2t} - \mu c 0 z_{2x}$

$\alpha_3 = \mu z_{2t} + \mu c 0 z_{2x}$

Therefore $(\alpha_1 - \bar{z}_2), (\alpha_2 - \bar{z}_{2x})$ and $(\alpha_3 - \bar{z}_{2t})$ are continuous functions on $L^p(D)$. That is the function is p-times differentiable on (D) where $D = [0,1] \times [0,1]$.

Equip the space $L^P[D]$ with the norm:

$$\left\| y^n \right\| = \max_{0 \le t \le 1} \lfloor y \rfloor + \max_{0 \le x \le 1} \left| y' \right| + \cdots + \max_{0 \le x \le 1} \left\lfloor y^n \right\rfloor$$

where $y^{(i)}$ is the i^{th} derivative of the function y.

Lemma (Gelfand and Fomin (1963))

If $\phi(x)$ and $\psi(x)$ are *continuous in* [a,b] *and if* $\int_a^b \int_a^b [\varphi(x)h(x) + \psi(x)'h(x)']dx = 0$, *for all* $h(x)$ Є C^1[a, b] *with* $h(a) = h(b) = 0$ *then* $\phi(x) = \psi'(x)$, $\forall x$ Є (a, b). Therefore from the lemma, if

$$\int_0^1 \int_0^1 \left[Z_1(\alpha_1 - \bar{Z}_2) + Z_{11}(\alpha_2 - \bar{Z}_{2X}) \right] dxdt = 0$$

then $\frac{\partial}{\partial t}(\alpha_2 - \bar{z}_{2x}) = \alpha_1 - \bar{z}_2$, $\frac{\partial}{\partial x}(\alpha_3 - \bar{z}_{2x}) = \alpha_1 - \bar{z}_2$ .Hence

$$\frac{\partial(\alpha_2 - \bar{z}_{2t})}{\partial t} = \frac{\partial(\alpha_3 - \bar{z}_{2x})}{\partial x}$$

This leads into: $$\frac{\partial^2 \bar{z}_2}{\partial t^2} = \frac{\partial^2 \bar{z}_2}{\partial x^2} + (\alpha_{3x} - \alpha_{2t}) \tag{2.5}$$

$$= \frac{\partial^2 \bar{z}_2}{\partial x^2} + [(\mu z_{2x} + \mu c_0 z_{2xt}) - (-\mu z_{2t} - \mu c_0 z_{2xx})]$$

$$\Rightarrow \quad (1-\mu)\frac{\partial^2 \bar{z}_2}{\partial x^2} = (1 + \mu c_0{}^2)$$

Put $(1+\mu C_0{}^2)/(1-\mu)$ where $|1-\mu| > 0$ and imposing initial and boundary conditions from problem P2, we have

$$\bar{z}_{2tt} = k z_{2xx} \tag{2.6}$$

We impose the following conditions on equation (2.6) as was done in [7] and [8] :

$$z_2(x,0) = z_{2^0}(x)$$
$$z_{2t}(x,0) = z_{21}(x)$$
$$z(0,t) = z_x(c,t) = z_{xx}(0,t) = 0$$
$$z(L,t) = z_x(L,t) = z_{xx}(L,t) = 0 \qquad 0 \le x,t \le 1$$

The Laplace transform of equation (2.6) in t-space is:

$$\int \frac{\partial^2 z}{\partial t^2}(x,t)\ell^{-st}\,dt = k\int \frac{\partial^2 z}{\partial x^2}(x,t)\ell^{-st}dt; s > 0$$

$$\Rightarrow s^2 z(x,s) - sz(x,0) - z_t(x,0) = k\frac{\partial^2 z}{\partial x2}(x,s) \qquad (2.7)$$

where $L[z(x,t)] = z(x,s)$. The left hand side of equation (2.7) indicates that we take another Laplace transform in x-space. That is

$$S^3 Z(\lambda,s) - sz(\lambda,0) - z_1(\lambda,0) = K\left[\lambda^2 z(\lambda,s) - \lambda z(0,s) - z_x\right] \Rightarrow \frac{S^2 Z(\lambda,s) - sz(\lambda,0) - z_1(\lambda,0)}{\lambda^2} = Kz(\lambda,s)$$

$$S^2 Z(\lambda,s) = k\lambda^2 z(\lambda,s) + sz(\lambda,0) + Z_t(\lambda,0)$$

$$\frac{S^2 Z(\lambda,s)}{k} = \lambda^2 Z(\lambda,s) + \frac{s}{k}z(\lambda,0) + \frac{Z_t(\lambda,0)}{k}$$

$$\Rightarrow \frac{S^2 Z(x,s)}{k} = \lambda^2 z(x,s) - sz(\lambda,0) + z_1(\lambda,0) \qquad (2.8)$$

$$\therefore \qquad Z(x,s) = \left[\frac{\lambda^2}{s^2}z(x,s) - \frac{1}{s}Z(\lambda,0) + \frac{1}{s^2}Z_1(\lambda,0)\right]k \qquad (2.9)$$

The inverse transform of equation (2.9) in t-space is:

$$\frac{z(\lambda,t)}{k} = k\left[\lambda^2\left(t * z(\lambda,t)\right) + \frac{z(\lambda,0}{k} + t.z_t\frac{(\partial,0)}{k}\right]$$

$$\Rightarrow \quad \frac{z(\lambda,t)}{\lambda^2} = k\left[\int_0^t (t-s)\,z(\lambda,s) + \frac{z(\lambda,0)}{\lambda^2} + t.\frac{z_t(\lambda,o)}{\lambda^2}\right]$$

Taking the inverse transform in x-space we have:

$$\frac{x^3}{6}z(x,t) = k\left[\int_0^t (t-s)\,z(x,s)\,ds + \frac{x^3}{6}z(x,0) + \frac{x^3}{6}t.z_t(x,0)\right] \qquad (2.10)$$

The last equation is obtained from the expression:

$$\int_0^x \frac{(x-\lambda)^2}{6}z(\lambda,t)d\lambda = k[\int_0^t (t-s)z(x,s)ds + \int_0^x \frac{(x-\lambda)^3}{6}z(\lambda,0)d\lambda + \frac{t}{6}\int_0^x (x-\lambda)^3 z_t(\lambda,0)d\lambda$$

Substituting the initial and boundary conditions, equation (2.10) simplifies into:

$$\frac{x^3}{6}z(x,t) = k\left[\int_0^t (t-s)\,z(x,s)\,ds + \frac{x^3}{6}z_0(x) + \frac{x^3}{6}z_1(x)t\right] \qquad (2.11)$$

The terms λ and s are parameters in Laplace transform in x-space and t-space respectively. Hence we have:

$$\frac{z(\lambda,t)}{k} = \lambda^2[t*z(\lambda,t)] + \frac{z(\lambda,0)}{k} + \frac{z_t(\lambda,0)t}{k} \tag{2.12}$$

where $t*z(\lambda, t)$ is a convolution of the transforms of $\frac{1}{s^2}$ and $z(\lambda,s)$ $z(\lambda,s)$,

Therefore by remarks due to [7], we have:

$$t*z(\lambda,t) = \int_0^t (t-\alpha)z(\lambda,\alpha)d\alpha = -t^2 z(\lambda,0) + t\int_0^t z(\lambda,\alpha)d\alpha \tag{2.13}$$

Substituting equation (2.13) into equation (2.8) yields:

$$\frac{z(\lambda,t)}{k} = \lambda^2\left[t^2 z(\lambda,0) + t\int_0^t z(\lambda,0)\,ds\right] + \frac{z(\lambda,0)}{k} + \frac{z_t(\lambda,0)}{k} \tag{2.14}$$

Dividing through out by λ^2, we have:

$$\frac{1}{k}\left[\frac{z(\lambda,t)}{\lambda^2}\right] = \left[t^2 z(\lambda,0) + t\int_0^t z(\lambda,\alpha)ds\right] + \frac{1}{k}\left[z(\lambda,0) + z_t(\lambda,0)\right]\frac{1}{\lambda^2}$$

The inverse transform of which yields:

$$\Rightarrow \frac{1}{k}[x*z(x,t)] = \left[t^2 z(x,0) + t\int_0^t z(x,\alpha)ds\right] + \frac{x}{k}\left[z(x,0) + z_t(x,0)\right]$$

From equation (2.13), we have a similar result for the convolution of x and $z(x,t)$, that is $x*z(x,t)$. Therefore equation (2.14) becomes:

$$\frac{1}{k}\left[x^2 z(0,t) + x\int_0^x z(h,t)\,dh = \left[t^2 z(x,0) + t\int_0^t z(x,\alpha)ds\right] + \frac{x}{k}\left[z(x,0) + z_t(x,0)\ \right] \tag{2.15}$$

Differentiating equation (2.15) with respect to x we have:

$$\frac{1}{k}\left[2xz(0,t) + \int_0^x z(h,t)dh + xz(x,t) = t^2 z_x(x,0) + t\int_0^t z_x(x,\alpha)d\alpha + [z(x,0) + z_t(x,0)]\right.$$

$$+\frac{d}{dx}[z(x,0) + z_x(x,0)] \tag{2.16}$$

Applying the initial and boundary conditions,

$$z(x,0) = z_0(x)$$
$$z_t(x,0) = z_1(x)$$

and $z(0,t) = z_x(0,t) = z_{xx}(0,t) = 0$
$z(l,t) = z_t(l,t) = z_{tt}(l,t) = 0$

equation (2.16) becomes:

$$\frac{1}{k}\left[0 + \int_0^x z(h,t)dh + xz(x,t)\right] = t^2 z_x(x,0) + t\int_0^t z_x(x,\alpha)d\alpha + z_0(x) + z_1(x)$$

$$\Rightarrow z(x,t) = \frac{k}{x}\left[(t^2+1)z_0(x) + t\int_0^t z_x(x,\alpha)ds - \int_0^x z(h,t)dh + z_1(x)\right].$$

$$= KX^{-1}\left[(t^2+1)z_0(x) + t\int_0^t z_x(x,\alpha)ds - \int_0^x z(h,t)dh + z_1(x)\right] \tag{2.17}$$

and from equation (2.3)

$$\overline{z}_2 = R_{11}z_2 \Rightarrow R_{11}z_2 = kx^{-1}\left[(t^2+1)z_0(x) + t\int_0^t z_x(x,\alpha)\,d\alpha - \int_0^x z(h,t)dh + z_1(x)\right] \tag{2.18}$$

By inspection, the remaining terms in equation (2.4) are:

$$\int\int z_{1x}(-\mu z_2 c_0^2 + \mu z_{2x} c_0^2)dxdt \Rightarrow \hat{z}_2 = R_{21} z_2 = \mu C_0(1-C_0)Z_{2x}$$

where $z_2(x,t)$ is as given in equation (2.18)

The remaining elements of the operator, R are derived from equation (2.6) by setting z_2 to zero.We therefore obtain the third and fourth second orders fundamental partial differential equations. Thus we have:

$$J(z,u,\mu) = \int\int\{u_1u_2 + \mu u_1 u_2 - z_{1t}u_2 - c_0\, z_{1x}u_2\}\, dxdt \qquad (2.19)$$

Let there exists $\bar{u}_2$ *and* $\bar{u}_{2x}$ continuous in $C^1[0,1]$ and have at least first derivatives in both x and *t*, and also satisfies equation (2.19). Then we can have:

$$j(z,u,\mu) = \int\int \; \{z_{1t}\bar{z}_t + z_{1x}\bar{z}_{1x} + (1+\mu)u_1u_2 \qquad (2.20)$$

Let

$$\beta_1 = -\mu\, u_2$$
$$\beta_2 = u_2(1+\mu) \qquad (2.21)$$

Hence, applying due to Gelfand and Fomin, [2], we have

$$\frac{\partial(\beta_1 - \bar{u}_{2x})}{\partial x} = \beta_1 - u_2 \qquad (2.22)$$

Therefore from equation (2.22) we have

$$\bar{u}_2 - \bar{u}_{xx} = -\mu u_2 - \mu u_{2x} \qquad (2.23)$$

We shall impose the following boundary conditions on equation (2.23):

$$\bar{u}_2(0,y) = \bar{u}_{2x}(0,y) = \bar{u}_{2xx}(0,y) = 0$$

Taking the Laplace transform of equation (2.23) in t-space, we have

$$\bar{u}_2(x,p) - -\bar{u}_{2xx}(x,p) = -\mu u_2(x,p) - \mu u_2(x,p) \qquad (2.24)$$

Also the Laplace transform of equation (2.24) in x-space yields:

$$\bar{u}_2(s,p) - s^2\bar{u}_2(x,p) - s\bar{u}_2(0,p) + \bar{u}_{2x}(0,p) = -\mu u_2(s,p) - \mu s u_2(x,p) + u_2(0,p)$$

Simplifying yields:

$$(1-s^2)\bar{u}_2(s,p) = -\mu u_2(s,p) - \mu s u_2(s,p) \qquad (2.25)$$

The inverse transform of equation (2.25) in p-space is:

$$\bar{u}_2(s,t) = \frac{\mu u_2(s,t)}{(1-s^2)} - \frac{\mu s u_2(s,t)}{(1-s^2)} \qquad (2.26)$$

and the inverse transform of equation (2.26) in s-space after simplifying is:

$$\bar{u}_2(x,p) = -\mu \sinh x * u_2(x,p) - \mu \cosh x * u_2(x,p)$$
$$= -\mu\int_0^x \sinh(x-h)u_2(h,t)dh - \mu\int_0^x \cosh(x-h)u_2(h,t)dh$$
$$= -\mu\sinh(x-h)\int_0^x u_2(h,t)dh + \mu\int_0^x \cosh(x-h)dh\int_o^x u_2(h,t)dh$$
$$+\cosh(x-h)\int_0^x u_2(h,t)dh - \mu x\int_0 \cosh(x-h)dh\int_0^x u_2(h,t)dh \qquad (2.27)$$

Hence from equation (2.4) we have therefore,

$$R_{12}u_2 = u(x,t) = -\mu\sinh(x)\int_0^x u_2(h,t)dh + \mu\sinh(x)dh\int_0^x u_2(h,t)dh$$

$$+\mu\cosh(x)\int_0^x u_2(h,t)dh - \mu\cosh(x)dh\int_0^x u_2(h,t)dh \qquad (2.28)$$

The remaining terms in equation (2.20) are: $J(z,u,\mu) = \iint \{z_{1t}u_1 + z_{1x}u_1 + u_1 z_{1x}\}dxdt$.

Comparing with equation (2.19) we have: $R_{22}u_2 = \overline{u}_2 = u_2(1+\mu)$.

3.0 Conclusion

The paper constructed the elements of the operator of the linear lower order non-dispersive wave equation. These elements will be useful at the implementation stage of problem, P_4.

The algorithm of the extended conjugate gradient method for the implementation of problem P1, using the derived control operator R, is:

Guess: $z_0(x,t), u_0(x,t)$

Compute: $g_i = <\Delta_z J(z,u,\mu), \Delta_u J(z,u,\mu)>, g_i = p_i$

where J(z,u,μ) is given as:

$$\min J(z,u,\mu) = \min\int_0^1\int_0^1\int_0^1[(z^2+u^2)+\mu(z_{tt}-c_0 z_{xx}-c_0 z_{yy}-u)]dtdxdy$$

Update the state and control variables: $z_i(x,t),\ and\ u_i(x,t)$. That is:

$$z_{i+1}(x,t) = z_i(x,t) + \alpha_i(x,t)\rho_i(x,t)$$
$$u_{i+1}(x,t) = u_i(x,t) + \alpha_i(x,t)\rho_i(x,t)$$

$where\ \alpha_i(x,t) = < g_i(x,t), g_i(x,t)> / < \rho_i(x,t), R(x,t), \rho_i(x,t)$. Update the gradient: $g_i(x,t)$.

$$g_{i+1}(x,t) = g_i(x,t) + \beta_i(x,t)R\,p_i(x,t)$$

where $\beta_i(x,t) = < g_{i+1}(x,t), g_{i+1}(x,t) > / < g_i(x,t), g_i(x,t) >$

These elements of operator, R, used in the algorithm above are summarized below:

1. $R_{11}\overline{Z}_2 = kx^{-1}\left[(t^2+1)z_0(x) + t\int_0^t z_x(x,\alpha)d\alpha - \int_0^x z(h,t)dh + z_1(x)\right]$
2. $R_{21}z_2 = \mu c_0(1-c_0)z_{2x}$
3. $R_{12}u_2 = -\mu\sinh(x)\int_0^x u_2(h,t)dh + \mu\sinh(x)dh\int_0^x u_2(h,t)dh$

$$+\mu\cosh(x)\int_0^x u_2(h,t)dh - \mu\cosh(x)dh\int_0^x u_2(h,t)dh$$

4. $R_{22}u_2 = \overline{u}_2(1+\mu)$

References

[1] Gelfand I. M. and Fomin S. F. (1963): Calculus of Variations, Prentice Hall Inc, Englewood Cliffs, New Jersey.

[2] Ibiejugba M. A., Rubio J. E., and Orisamobi R. J.(1986): A penalty optimization techniques for a class of regulator problems. ABACUS (Journal of Mathematical Association of Nigeria) 17(1): 19 – 50.

[3] Ibiejugba M. A., Rubio J. E., Olorunsola F., Otunta F., Aderibigbe F. M. (1992): The role of the multipliers in the multiplier methods, Part II, Journal of the Nigerian Mathematical Society, vol II, pp81-104.

[4] Iain G. Main (1993): Vibrations and waves in Physics, 3rd Ed. Cambridge, U. K.

[5] Otunta O. Francis (1992): An extended Conjugate Gradient Algorithm for discrete Optimal Control problem, International Conference on Scientific Computing, Benin. Pp:(183 - 187).

[6] Otunta O. Francis (1998): A Control Operator for a class of regulator problems, Journal of the Nigerian Association of Mathematical Physics, Pp:(70 - 87).

[7] Reju, Comfort O. (2003): Optimal Control of Quasi-Linear Non-Dispersive Waves, M. Tech Thesis, Federal University of Technology, Minna
[8] Sunday a. Reju, Matthew A. Ibiejugba and David J. Evans (2000):Optimal Control of the wave propagation problem with the Extended Conjugate Gradient Method, International Journal of Computer Mathematics, Vol. 77, pp(425 - 439).
[9] Victor Onomza Waziri and Sunday A. Reju (2000): The control operator for the one-Dimensional Energized Wave Equation, AU Journal of Technology, 9(4): 243 -247.
[10] Whitham G. B. (F.R.S.) (1974): Linear and Non-linear wave. John Wiley and Sons. New York.