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Stiffly stable continuous extension of second derivative linear multi-step methods with an off-step point for initial value problems in ordinary differential equations.

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Abstract

***In this paper, we introduce a continuous extension of second derivative linear multi-step methods with a hybrid point for the numerical solution of initial valued stiff ordinary differential equations. The continuous extension is based on the Gear's fixed step size backward differential methods* [7]. *The intervals of absolute stability of methods of step number $k \le 7$ are determined using the root locus plot. Numerical results of the methods solving a non-linearly stiff initial value problem in ordinary differential equations are compared with that from the state-of-the-art ordinary differential equations code of* MATLAB *discussed in Higham et al* [9].**

Keywords: Continuous Linear Multi step Methods, off-step point, Hybrid Method, Stability, Root-Locus.

1.0 Introduction

A class of methods for finding numerical solution to stiff initial value problems in ordinary differential equations of the type.

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0 \tag{1.1}$$

where $f(x, y(x))$ and $y(x)$ may be vectors, can be based on second derivative continuous linear multi-step methods (CLMM) with off-step points in the interval (x_n, x_{n+k}). Methods in this regard are those of Gear [7] and Butcher [3, 4]. A method in this class produces reasonable approximate solution y_{n+k} to $y(x_{n+k})$, while using the off-step solution y_{n+v} approximation to $y(x_{n+v})$. In this regard consider the continuous extension LMM.

$$y_{n+k} = \sum_{j=o}^{k-1} \alpha_j(x) y_{n+j} + \alpha_v(x) y_{n+v} + h\beta_{0,y}(x) f_{n+v} + h\beta_{0,k}(x) f_{n+k}, \ \alpha_k \ne 0 \tag{1.2}$$

and the hybrid y_{n+v} solution in the above is given by the continuous hybrid LMM

$$y_{n+v} = \sum_{j=o}^{k} \alpha_j(x) y_{n+j} + h\beta_{1`,k}(x) f_{n+k} + h^2 \beta_{2,k}(x) f'_{n+k} \tag{1.3}$$

which employs the second derivatives f'_{n+k}, where $h = x_{n+k} - x_{n+k-1}$. The use of the second

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derivatives in the hybrid predictor is motivated from the need to have enhanced stability characteristics of (1.2). The order of the methods (1.2) and (1.3) is k + 2 respectively. The value of $v \in (k-1, k)$ is taken to be $k - \frac{1}{2}$. Similar methods like the backward differentiation methods (BDM)

$$y_{n+k} = \sum_{j=o}^{k-1} \alpha_j y_{n+j} = h\beta_k f_{n+k},\ \alpha_k \neq 0 \tag{1.4}$$

in Gear [7] is known to be stiffly stable for $k \leq 6$. Others have been also reported by Butcher [1], Lambert [11], and how they can be realized in continuous form is found in Otunta, Ikhile and Okuonghae [8], [9], see also Arevalo et al [1]. Burrage and Tian [2], Butcher [3, 4], Onumanyi et al [12], Hairer and Lubeich [8], Otunta et al [13, 14], and Sirisena et al [15]. In the case of (1.2) we have found stiffly stable methods for $k = 2, 3, \ldots, 7$, but for k = 1, the method is not zero stable. Plots of the region of interval of absolute stability for the methods corresponding to the formulas for $k \leq 7$ are given in Figures (4.1 – 4.7). The motivation to derive the hybrid method (1.2) is the fact that, it offers the means to by-pass the Dahlquist order barrier for A-stable conventional LMM and the fact that continuous solution of the initial value problem in ordinary differential equation can be obtained. The modification in (1.2) consists of the addition of the terms $\alpha_v(x)y_{n+v}$ to the left hand side and $h\beta_v(x) f_{n+v}$ to the right hand side of the Gear's fixed step size backward differentiation methods [7], in (1.4), where $\alpha_v(x)$ and $\beta_v(x)$ are additional parameters to be determined when compared to (1.4). To make use of (1.2) in a practical computing procedure requires us to first compute y_{n+v} so that the two extra terms on both sides of (1.2) could be evaluated. Considerations as to how this might be done appear in section 5. In the next section the procedure of obtaining the unknown constants $v, \{\alpha_j\}_{j=0}^{k-1}, \beta_v(x),$ and $\beta_k(x)$ where $t = \frac{x - x_{n+1}}{h}$ and $x_{n+1} = x_n + h$ which yield a method of order as high as possible is presented. The results of computation which indicate for which values of t the method is stable, is given in section 4. Section 2 contains the derivation of the class of methods in (1.2) in continuous form and in section 3 we derive the hybrid method (1.3). In section 4 the stability of the methods in (1.2) are determined, using the root locus and, in section 5 the results of some numerical experiments are presented.

2.0 Derivation of the second derivative CLMM

We seek the numerical solution of (1.1) in the form of the polynomial interpolant

$$y(x) = \sum_{j=0}^{k+2} a_j x^j \tag{2.1}$$

where $\{a_j\}_{j=1}^{k+2}$ are the real parameter constants to be determined. Putting (2.1) in (1.2) results then in the matrix (2.2) gives the solution of $\{\alpha_j(t)\}_{j=1(1)n}$ and substituting the resulting values with $t = \frac{x - x_{n+1}}{h}$, into (2/1) yield the coefficients $\{\alpha_j(t)\}_{j=0}^{k-1}, \alpha_v(t), \alpha_k(t) = 1, \beta_{i,k}(t),\quad \beta_{0,v}(t);\ i = 1, 2$ accordingly, for a fixed value of k with t = k – 1. Table (1.2) shows the coefficients of the methods in (1.2), while table (5.1) in Appendix A shows continuous coefficient of the methods for $k \leq 7$.

$$\begin{pmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{k+2} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{k+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_{n+k-1} & x_{n+k-1}^2 & \cdots & x_{n+k-1}^{k+2} \\ 0 & 1 & 2x_{n+v} & \cdots & x_{n+v}^{k+2} \\ 0 & 1 & 2x_{n+v} & \cdots & (k+2)x_{n+v}^{k+1} \\ 0 & 1 & 2x_{n+k} & \cdots & (k+2)x_{n+k}^{k+1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \\ a_k \\ a_{k+1} \\ a_{k+2} \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ \vdots \\ y_{n+k-1} \\ y_{n+v} \\ f_{n+v} \\ f_{n+k} \end{pmatrix} \tag{2.2}$$

3.0 The derivation of the second derivative continuous hybrid LMM

Similarly, the unknown coefficients in hybrid solution in (1.3),

$$y_{n+v} = \sum_{j=o}^{k} \alpha_j(x) y_{n+j} = h\beta_{1,k}(x) f_{n+k} + h^2 \beta_{2,k}(x) f'_{n+k} \tag{3.1}$$

for (1.2) at the hybrid point x_{n+v} are derived using the following polynomial interpolant

$$y_{n+v}(x) = \sum_{j=0}^{k+2} b_j x^j , \tag{3.2}$$

where b_j*'s* are now the real parameter constants to be determined to obtain a particular method of (1.3) for a fixed k. Following the same procedure in section 2 we obtain a family of continuous hybrid predictor from (3.1) for (1.2). Table (1.3) shows the coefficient of the methods in (1.3) and table (5.2) in Appendix A gives explicit continuous expression for the coefficients of the methods for $k = 1(1)7$.

4.0 Determining the stability of the methods by root locus

In this section, the interest is on the stability of the family of methods in (1.2), using root locus approach. On substituting the hybrid solution y_{n+v} at point x_{n+v} into equation (1.2) for corresponding k, while applying the resultant method to the scalar test problem $y' = \lambda y, \quad \mathrm{Re}(\lambda h) < 0,\ z = \lambda h$ with an arbitrary initial value yield the stability polynomial to be

$$\begin{aligned} \pi(r,z) = r^k - \sum_{j=0}^{k-1} \alpha_j r^{j-1} - \alpha_v \left(\sum_{j=0}^{k} \alpha_j r^j + z\beta_{1,k} r^k + z^2 \beta_{2,k} r^k \right) - \\ z\beta_k r^k - z\beta_v \left(\sum_{j=0}^{k} \alpha_j r^j + z\beta_{1,k} r^k + z^2 \beta_{2,k} r^k \right) \end{aligned} \tag{4.1}$$

Plotting $|r_j|$ against z reveals the interval of absolute stability of the methods. The general graphical root locus plot has been discussed in Lambert [34], and finds application in Otunta,

Table 1.2: The Coefficient of the Second Derivative Continuous Linear Multi-step Method (1.2)

K	t	V	α_0	α_1	α_2	α_3	α_4	α_5
1	0	$\frac{1}{2}$	$\frac{1}{5}$	1	0	0	0	0
2	1	$\frac{3}{2}$	$-\frac{1}{99}$	$\frac{4}{11}$	1	0	0	0
3	2	$\frac{5}{2}$	$\frac{2}{875}$	$-\frac{1}{35}$	$\frac{18}{35}$	1	0	0
4	3	$\frac{7}{2}$	$-\frac{5}{8192}$	$\frac{16}{1825}$	$-\frac{4}{73}$	$\frac{48}{73}$	1	0
5	4	$\frac{9}{2}$	$\frac{4}{10179}$	$-\frac{75}{18473}$	$\frac{8}{377}$	$-\frac{100}{1131}$	$\frac{300}{377}$	1
6	5	$\frac{11}{2}$	$-\frac{10}{46827}$	$\frac{8}{3483}$	$-\frac{25}{2107}$	$\frac{16}{387}$	$-\frac{50}{387}$	$\frac{40}{43}$
7	6	$\frac{13}{2}$	$\frac{-152460}{118908901}$	$\frac{-91}{98304}$	$\frac{28056028}{356726703}$	$\frac{-32207175}{118908901}$	$\frac{-210420210}{118908901}$	$\frac{-210420210}{118908901}$

K	t	V	α_6	α_7	α_v	β_v	β_k	EC
1	0	$\frac{1}{2}$	0	0	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$-\frac{1}{480}$
2	1	$\frac{3}{2}$	0	0	$\frac{64}{99}$	$\frac{16}{33}$	$\frac{2}{11}$	$-\frac{1}{1320}$
3	2	$\frac{5}{2}$	0	0	$\frac{64}{125}$	$\frac{96}{175}$	$\frac{6}{35}$	$-\frac{1}{2800}$
4	3	$\frac{7}{2}$	0	0	$\frac{34816}{89425}$	$\frac{1536}{2555}$	$-\frac{1715}{6144}$	$-\frac{1}{5110}$
5	4	$\frac{9}{2}$	0	0	$\frac{137216}{498771}$	$\frac{5120}{7917}$	$\frac{60}{377}$	$-\frac{1}{42224}$
6	5	$\frac{11}{2}$	1	0	$\frac{3457024}{20650707}$	$\frac{20480}{29799}$	$\frac{20}{129}$	$-\frac{1}{65016}$
7	6	$\frac{13}{2}$	$\frac{1262521260}{11890890}$	1	$\frac{231784448}{356726703}$	$\frac{861020160}{118908901}$	$\frac{180360180}{118908901}$	$-\frac{7}{132912}$

Table 1.3: The coefficient of the second derivative Hybrid predictor continuous linear multi-step method (1.3)

K	t	V	α_0	α_1	α_2	α_3	α_4	α_5
1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{7}{8}$	0	0	0	0
2	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{-1}{128}$	$\frac{3}{16}$	$\frac{105}{128}$	0	0	0
3	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{1}{576}$	$-\frac{5}{256}$	$\frac{15}{64}$	$\frac{1805}{2304}$	0	0
4	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{-5}{8192}$	$\frac{7}{1152}$	$-\frac{35}{1024}$	$\frac{576}{3343}$	$-\frac{81095}{36864}$	0
5	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{7}{25600}$	$-\frac{45}{16384}$	$\frac{7}{512}$	$-\frac{105}{2048}$	$\frac{315}{1024}$	$\frac{300013}{409600}$
6	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{7}{24576}$	$\frac{77}{51200}$	$-\frac{495}{65536}$	$\frac{77}{3072}$	$-\frac{1155}{16384}$	$\frac{693}{2048}$
7	$\frac{11}{2}$	$\frac{13}{2}$	$\frac{33}{401408}$	$-\frac{91}{98304}$	$\frac{1001}{204800}$	$-\frac{2145}{16384}$	$\frac{1001}{3072}$	$-\frac{3003}{4096}$

K	t	V	α_6	α_7	α_v	$\beta_{1,k}$	$\beta_{2,k}$	EC
1	$\frac{-1}{2}$	$\frac{1}{2}$	0	0	1	$\frac{-3}{8}$	$\frac{1}{16}$	$-\frac{1}{384}$
2	$\frac{1}{2}$	$\frac{3}{2}$	0	0	1	$\frac{-21}{64}$	$\frac{3}{64}$	$-\frac{1}{1280}$
3	$\frac{3}{2}$	$\frac{5}{2}$	0	0	1	$\frac{-115}{384}$	$\frac{5}{128}$	$-\frac{1}{3072}$
4	$\frac{5}{2}$	$\frac{7}{2}$	0	0	1	$\frac{-1715}{6144}$	$\frac{35}{1024}$	$-\frac{1}{6144}$
5	$\frac{7}{2}$	$\frac{9}{2}$	0	0	1	$\frac{-1799}{10240}$	$\frac{63}{2048}$	$-\frac{3}{32768}$
6	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{3505733}{4915200}$	0	1	$\frac{20559}{81920}$	$\frac{231}{8192}$	$-\frac{11}{196608}$
7	$\frac{11}{2}$	$\frac{13}{2}$	$\frac{3003}{1024}$	$\frac{335572523}{481689600}$	1	$\frac{-275847}{1146880}$	$\frac{429}{16384}$	$-\frac{143}{3932160}$

Ikhile and Okuonghae [13,14]. Applying the root locus method to $\pi(r.z) = 0$, shows that the methods in (1.2) are stiffly stable for $k \leq 7$. The graphs below show the root loci and thus the interval of absolute stability of each method can be deduced for any given value of *k*, see Table 4.1

Table 4.1: The step number, interval of absolute stability, and the range of t for which method (1.2, 1.3) is zero stable

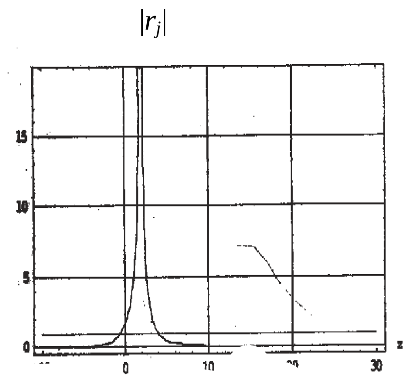
k	***Interval of absolute stability***	***t***	***The range of t for which the methods (1.2, 1.3) is zero stable.***	***Order***	
	SDCLMM(1.2) + Hybrid(1.3)	**SDCLMM (1.2) + Hybrid (1.3)**	**SDCLMM(1.2)+Hybrid(1.3):t**	**SDCLMM**	**HCLMM**
1	(- ∞, - 0.551365) ∪(4, ∞)	0	$\{t : t \in [-0.75, \infty]\}$	3	3
2	(- ∞, 0) ∪(5, ∞)	1	$\{t : t \in (-\infty, -1.3) \cup (0, \infty)\}$	4	4
3	(- ∞,0) ∪(6, ∞)	2	$\{t : t \in (-\infty, -1.1056) \cup (0, \infty)\}$	5	5
4	(- ∞, 0) ∪(6.85, ∞)	3	$\{t : t \in [1, 7.4]\}$	6	6
5	(- ∞, 5.7) ∪(9.75, ∞)	4	$\{t : t \in (-\infty, -1.0749) \cup (2,4)$ $\cup (4.3, \infty)\}$	7	7
6	(- ∞, 0.04) ∪(8.98, ∞)	5	$\{t : t \in (-\infty, -1.075) \cup (4.5, \infty)\}$	8	8
7	(- ∞, 0) ∪(9.07, ∞)	6	$\{t : t \in (-\infty, -1.2168) \cup (5.95, 6.042)$ $\cup (7.568, \infty)\}$	9	9

5.0 Numerical experiment and conclusion

To demonstrate the effectiveness of the SDCLMM (1.2), stiff IVP in Higham et al [9], with $x \in [0,10]$:

$$y_1' = -0.04 y_1 + 10^4 y_2 y_3, \quad y_2' = 400 y_1 + 10^4 y_2 y_3 - 3 \text{ x } 10^7 y_2^2,$$

$$y_3' = 3 \text{ x } 10^7 y_2^2, \quad y(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$



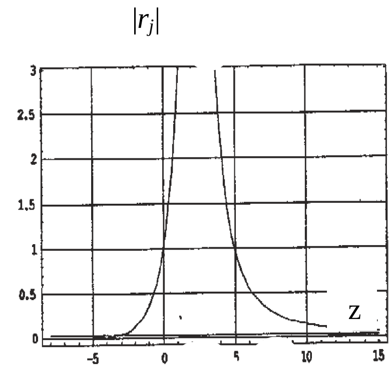
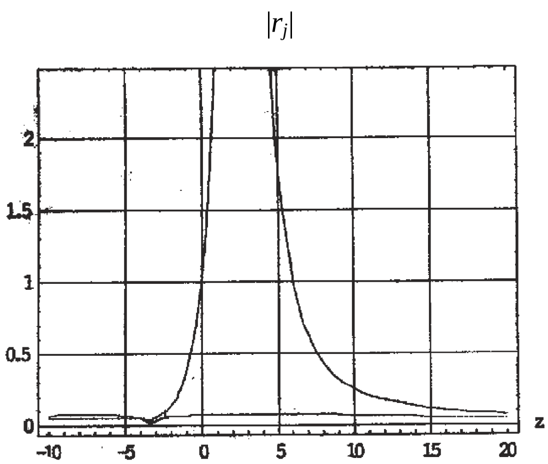


Figure 4.1: k = 1

Figure 4.2: k = 2



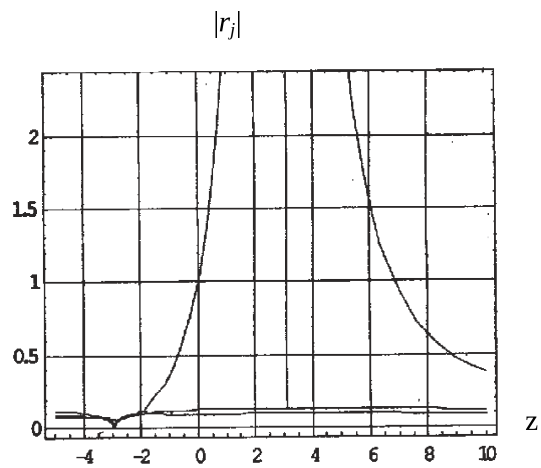
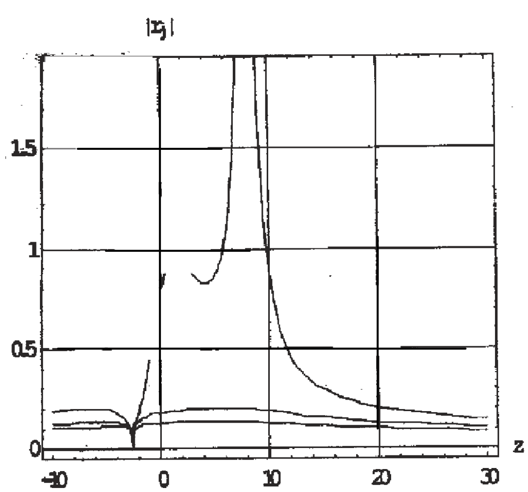


Figure 4.3: k = 3

Figure 4.3: k = 4



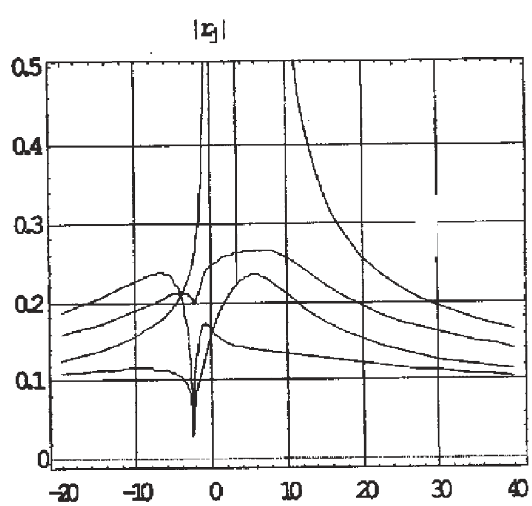


Figure 4.4: k = 5

Figure 4.5: k = 6

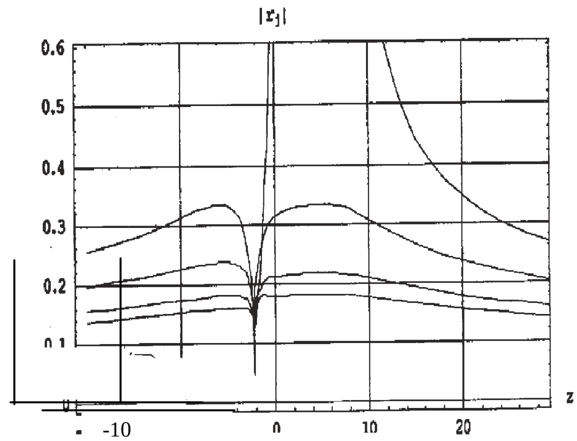


Figure 4.6: k = 7.

is solved using the method (1.2) when k = 2. The application of SDCLMM (1.2) on the above problem on requires the need to solve the system of non-linear equation for y_{n+k},

$$F(y_{n+k}^{[s]}) = -y_{n+k}^{[s+1]} + \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \alpha_v(x) y_{n+v}^{[s]} - h\beta_v(x) f_{n+v}^{[s]} - h\beta_{1,k}(x) f_{n+k}^{[s]} = 0, \ \alpha_k \neq 0 \tag{5.1}$$

where

$$y_{n+v}^{[s]} = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + y_{n+k}^{[s]} + h\beta_{1,k}(x) f_{n+k}^{[s]} + h\beta_{2,k}(x) f_{n+k}'^{[s]} \tag{5.2}$$

This implicit system of equations in y_{n+k} is resolved by applying the Newton-Raphson iterative scheme

$$y_{n+k}^{[s+1]} = y_{n+k}^{[s]} - F'(y_{n+k}^{[s]})^{-1} F(y_{n+k}^{[s]}), s = 0, 1, 2, \cdots \tag{5.3}$$

in (5.1) where, $F'(y_{n+k}^{[s]})$ is the Jacobian matrix of the vector ODE systems above, as suggested by Enright [5], Fatunla [6] and Lambert [11]. The trapezoidal rule

$$y_{n+1}^{[0]} = y_n + \frac{h}{2}(f_{n-1} + f_n) \tag{5.4}$$

is used to generate the starting values for the iterative schemes (5.3). The new scheme (1.2) for k = 2, is

$$y_{n+2} = \frac{1}{99}(-y_n + 36y_{n+1} + 64y_{n+\frac{3}{2}}) + \frac{h}{99}(18f_{n+2} + 48f_{n+\frac{3}{2}}) \tag{5.5}$$

$$y_{n+\frac{3}{2}} = \frac{1}{128}(-y_n + 24y_{n+1} + 105y_{n+2}) + \frac{h}{128}(-42f_{n+2} + 6hf_{n+2}')$$

The results in the graphs below of the second component of the initial value problem compares with that from ODES code in MATLAB discussed in Higham et al [9]. The plot is as in Figure 5.1.

Finally, in this paper, a class of stiffly stable hybrid SDCLMM of step number $k \leq 7$ is considered. The stability graphs in Figures (4.1) – (4.7) shows that, the methods are stiffly stable.

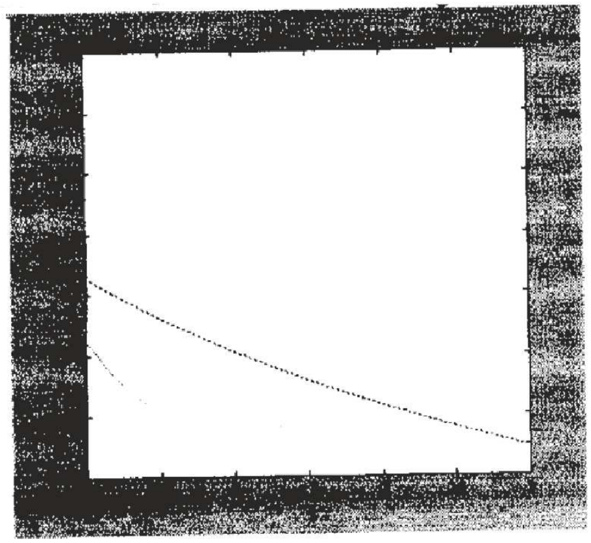


Figure 5.1

The order of the methods, the values of t for which the methods are stable, and the error constant for the SDCLMM (1.2) and its corresponding hybrid counter part is shown in Table 4.1 above. The graph in Figure 5.1 shows the accuracy of the methods which compare to results from the state-of-the-art ODES code discussed in [9, p. 158].

Appendix A

Table 5.1: The coefficients of second derivative continous linear multi-step method of (1.2)

k	***t***	***j***	$\alpha_j(t)$	$\alpha_j(k\text{-}1)$	$\beta_j(t)$	$\beta_j(k\text{-}1)$
1	0	0	$\frac{1}{5}-\frac{12t^2}{5}-\frac{16t^3}{5}$	$\frac{1}{5}$	0	0
		$\frac{1}{2}$	$\frac{4}{5}-\frac{12t^2}{5}-\frac{16t^3}{5}$	$\frac{4}{5}$	$\frac{2}{5}-\frac{14t^2}{5}-\frac{12t^3}{5}$	$\frac{2}{5}$
		1	1	1	0	0
2	1	0	$-\frac{22t}{33}-\frac{19t^2}{99}-\frac{4t^3}{99}+\frac{20t^4}{99}$	$\frac{-1}{99}$	0	0
		1	$1+\frac{46t}{11}+\frac{43t^2}{11}-\frac{28t^3}{11}-\frac{36t^4}{11}$	$\frac{4}{11}$	0	0
		$\frac{3}{2}$	$\frac{128t}{33}-\frac{272t^2}{99}-\frac{32t^3}{9}+\frac{304t^4}{99}$	$\frac{64}{99}$	$\frac{12t}{11}-\frac{64t^2}{33}-\frac{4t^3}{3}-\frac{56t^4}{33}$	$\frac{16}{33}$
		2	1	1	$\frac{t}{11}+\frac{5t^2}{11}+\frac{8t^3}{3}+\frac{4t^4}{33}$	$\frac{2}{11}$

3	2	0	$-\frac{108t}{875}+\frac{603t^2}{1750}-\frac{132t^3}{350}+\frac{136t^4}{875}-\frac{22t^5}{875}$	$\frac{2}{875}$	0	0

k	*t*	*j*	$\alpha_j(t)$	$\alpha_j(k-1)$	$\beta_j(t)$	$\beta_j(k-1)$
		1	$1-\frac{188t}{105}+\frac{17t^2}{315}+\frac{100t^3}{63}-\frac{332t^4}{315}+\frac{64t^5}{315}$	$\frac{-1}{35}$	0	0
		2	$\frac{288t}{35}-\frac{453t^2}{70}-\frac{85t^3}{14}+\frac{244t^4}{35}-\frac{58t^5}{35}$	$\frac{18}{35}$	0	0
		$\frac{5}{2}$	$-\frac{2368t}{375}-\frac{6832t^2}{1125}+\frac{1088t^3}{225}-\frac{6832t^4}{1125}+\frac{1664t^5}{1125}$	$\frac{64}{125}$	$\frac{416t}{175}-\frac{1384t^2}{525}-\frac{176t^3}{105}+\frac{1384t^4}{525}-\frac{368t^5}{525}$	$\frac{96}{175}$
		3	1	1	$\frac{9t}{35}+\frac{12t^2}{35}+\frac{t^3}{7}-\frac{12t^4}{35}-\frac{4t^5}{35}$	$\frac{6}{35}$
4	3	0	$-\frac{925t}{7154}+\frac{14515t^2}{42924}-\frac{3587t^3}{10731}+\frac{2269t^4}{14308}-\frac{391t^5}{10731}+\frac{t^6}{1533}$	$\frac{-5}{1825}$	0	0
		1	$1-\frac{1179t}{730}-\frac{111t^2}{3650}+\frac{4959t^3}{3650}-\frac{3447t^4}{3560}+\frac{468t^5}{1825}-\frac{46t^6}{1825}$	$\frac{16}{1825}$	0	0
		2	$\frac{1775t}{438}-\frac{6545t^2}{2628}-\frac{191t^3}{73}+\frac{7591t^4}{2628}-\frac{205t^5}{219}=\frac{67t^6}{657}$	$\frac{-4}{73}$	0	0
		3	$-\frac{1625t}{146}+\frac{5425t^2}{438}+\frac{2875t^3}{438}-\frac{1727t^4}{146}+\frac{1000t^5}{219}-\frac{122t^6}{2219}$	$\frac{48}{73}$	0	0
		$\frac{7}{2}$	$-\frac{473344t}{53655}-\frac{8211721t^2}{804825}-\frac{444608t^3}{89425}+\frac{7827776t^4}{804825}-\frac{1032896t^5}{268275}+\frac{54848t^6}{114975}$	$\frac{3481}{8942}$	$\frac{1696t}{511}+\frac{30736t^2}{7665}+\frac{4432t^3}{2555}-\frac{29168^4}{7665}+\frac{4048t^5}{2555}-\frac{224t^6}{1095}$	$\frac{6}{35}$
		4	1	1	$\frac{25t}{73}-\frac{65t^2}{146}-\frac{11t^3}{73}+\frac{61^4}{146}-\frac{14t^5}{73}+$	$\frac{-1715}{6144}$
5	4	0	$-\frac{3724t}{30537}+\frac{1181951t^2}{366444}-\frac{241915t^3}{7322888}+\frac{62521t^4}{36644}-\frac{34861t^5}{732888}+\frac{1253t^6}{183222}-\frac{73t^7}{183222}$	$\frac{4}{10179}$	0	0
		1	$-\frac{13022t}{7917}+\frac{13679t^2}{221676}-\frac{247t^3}{192}-\frac{223651t^4}{221676}+\frac{74021t^5}{221676}-\frac{418t^6}{7917}+\frac{181t^7}{55419}$	$\frac{-75}{18473}$	0	0

		2	$-\frac{36456\,t}{9425}-\frac{9401\,t^{2}}{3770}-\frac{85039\,t^{3}}{37700}+\frac{5282\,t^{4}}{1885}-\frac{41459\,t^{5}}{37700}+\frac{361\,t^{6}}{1885}-\frac{119\,t^{7}}{9425}$	$\frac{8}{377}$	0	0
		3	$-\frac{17738\,t}{3393}+\frac{243593\,t^{2}}{40716}+\frac{108785\,t^{3}}{40716}-\frac{223921\,t^{4}}{40716}+\frac{102683\,t^{5}}{40716}-\frac{4918\,t^{6}}{10179}+\frac{347\,t^{7}}{10179}$	$\frac{-100}{1131}$	0	0

k	***t***	***j***	$\alpha_j(t)$	$\alpha_j(k-1)$	$\beta_j(t)$	$\beta_j(k-1)$
		4	$\frac{16268\,t}{1131}-\frac{85253\,t^{2}}{4524}-\frac{49051\,t^{3}}{9048}+\frac{76891\,t^{4}}{4524}-\frac{79825\,t^{5}}{9048}+\frac{4181\,t^{6}}{2262}-\frac{317\,t^{7}}{2262}$	$\frac{300}{377}$	0	0
		$\frac{9}{2}$	$-\frac{60157952\,t}{5343975}+\frac{336025216\,t^{2}}{22444695}+\frac{451982912\,t^{3}}{112223475}-\frac{302116544\,t^{4}}{22444695}+\frac{798337472\,t^{5}}{112223475}-\frac{4844096\,t^{6}}{3206385}+\frac{12996608\,t^{7}}{1112223475}$	$\frac{137216}{498771}$	$\frac{69632\,t}{16965}-\frac{396352\,t^{2}}{71253}-\frac{483872\,t^{3}}{356265}+\frac{354464\,t^{4}}{71253}-\frac{961952\,t^{5}}{356265}+\frac{5984\,t^{6}}{10179}-\frac{16448\,t^{7}}{356265}$	$\frac{5120}{7917}$
		5	1	1	$-\frac{147t}{377}+\frac{413t^{2}}{754}-\frac{81t^{3}}{754}-\frac{365t^{4}}{754}+\frac{209t^{5}}{754}-\frac{24t^{6}}{377}+\frac{12^{7}}{377}$	$\frac{60}{377}$
6	5	0	$-\frac{1167\,t}{10406}+\frac{190567\,t^{2}}{624360}-\frac{334981\,t^{3}}{102180}+\frac{6198737\,t^{4}}{33715440}-\frac{132389\,t^{5}}{2247696}+\frac{366857\,t^{6}}{33715440}-\frac{23\,t^{7}}{21285}+\frac{377\,t^{8}}{8428860}$	$-\frac{10}{46827}$	0	
		1	$1-\frac{2665\,t}{1548}+\frac{5627\,t^{2}}{27864}+\frac{26837\,t^{3}}{20898}-\frac{139703\,t^{4}}{125388}+\frac{5965\,t^{5}}{13932}-\frac{21913\,t^{6}}{250776}+\frac{193t^{7}}{20898}-\frac{25t^{8}}{62694}$	$\frac{8}{3483}$	0	
		2	$\frac{8640\,t}{2107}-\frac{6234\,t^{2}}{2107}-\frac{26939\,t^{3}}{12642}+\frac{237799\,t^{4}}{75852}-\frac{9061\,t^{5}}{6321}+\frac{24427\,t^{6}}{75852}-\frac{229\,t^{7}}{6321}+\frac{31t^{8}}{18963}$	$\frac{-25}{2107}$	0	
		3	$\frac{-125\,t}{27}+\frac{1775\,t^{2}}{324}+\frac{335\,t^{3}}{162}-\frac{1583\,t^{4}}{324}+\frac{202\,t^{5}}{81}-\frac{191\,t^{6}}{324}+\frac{11t^{7}}{162}-\frac{367\,t^{8}}{87075}$	$\frac{19}{387}$	0	

		4	$\frac{615t}{86}-\frac{10243t^2}{1032}-\frac{34693t^3}{18576}+\frac{477071t^4}{55728}-\frac{31649t^5}{6192}+\frac{75575t^6}{55728}-\frac{200t^7}{1161}+\frac{119t^8}{13932}$	$\frac{-50}{387}$	0	0
		5	$\frac{-3051t}{172}+\frac{44871t^2}{1720}+\frac{3823t^3}{1290}-\frac{170567t^4}{7740}+\frac{7345t^5}{516}-\frac{62269t^6}{15480}+\frac{679t^7}{1290}-\frac{109t^8}{3870}$	$\frac{40}{43}$	0	0

k	*t*	*j*	$\alpha_j(t)$	$\alpha_j(k-1)$	$\beta_j(t)$	$\beta_j(k-1)$
		$\frac{11}{2}$	$\frac{156127232t}{11472615}-\frac{1041328230\ 4t^2}{516267675}-\frac{293451775t^3}{140800275}+\frac{7890272844\ 8t^4}{4646409075}-\frac{1145633792t^5}{103253535}+\frac{1471199897\ 6t^6}{4646409075}-\frac{60427264t^7}{140800275}+\frac{104813312t^8}{4646409075}$	$\frac{3457024}{2065070[illegible]}$	$\frac{-15872t}{3311}+\frac{1069184t^2}{148995}+\frac{26176t^3}{40635}-\frac{8061568t^4}{1340955}+\frac{118912t^5}{29799}-\frac{1549696t^6}{1340955}+\frac{6464t^7}{40635}-\frac{11392t^8}{1340955}$	$\frac{20480}{29799}$
		6	1	1	$\frac{18t}{43}-\frac{55t^2}{86}+\frac{61t^3}{1548}+\frac{2461t^4}{4644}-\frac{563t^5}{1548}+\frac{505t^6}{4644}-\frac{2t^7}{129}+\frac{t^8}{1161}$	$\frac{20}{129}$
7	6	0	$-\frac{122062017t}{118+08901}+\frac{6858513893t^2}{2378178020}-\frac{1546970117t^3}{475635604}+\frac{1865626170\ 1t^4}{9512712080}-\frac{6666329593t^5}{9512712080}+\frac{1459185343t^6}{9152712080}-\frac{38485139t^7}{1902542416}+\frac{878339t^8}{594544505}-\frac{109263t^9}{2378178020}$	$\frac{152460}{118908901}$	0	0
		1	$\frac{1189088901}{118908901}-\frac{43031297309t}{278178020}+\frac{12804594439t^2}{3567267030}+\frac{358975608901t^3}{28538136240}-\frac{29200056431t^4}{2378178020}+\frac{5072485145t^5}{951271208}-\frac{1542479393t^6}{1189089010}+\frac{1737544263t^7}{9152712080}-\frac{99477287t^8}{71345340606}+\frac{3192917t^9}{7134534060}$	$\frac{-91}{98304}$	0	0
		2	$\frac{4771743476\quad 5t}{1070180109}-\frac{4634708388\quad 31t^2}{1284216130\quad 8}-\frac{2579547080\quad 11t^3}{1284216130\quad 8}+\frac{6164271503\quad 39t^4}{1712288174\quad 4}-\frac{3209768491\quad 93t^5}{1712288174\quad 4}+\frac{9568680121\quad t^6}{1902542416}-\frac{1291393203\quad 1t^7}{1712288174\quad 4}+\frac{386342957\quad t^8}{6421080654}-\frac{25622597\quad t^9}{1284216130\quad 8}$	$\frac{28056028}{356726703}$	0	0
		3	$-\frac{5073611614\quad 5t}{832362307}+\frac{2657292133\quad 21t^2}{3329449228}+\frac{3001817089\quad 7t^3}{1664724614}-\frac{1609019340\quad 5t^4}{237817802}+\frac{4876309438\quad t^5}{118908901}-\frac{5705291449\quad t^6}{475635604}+\frac{3176649905\quad t^7}{1664724614}-\frac{132366377\quad t^8}{832362307}+\frac{4539678\quad t^9}{832362307}$	$\frac{-32207175}{118908901}$	0	0

		4	$\frac{46747497797t}{594544505}-\frac{4139903546779t^{2}}{35672670300}-\frac{373680384077t^{3}}{1664724614}+\frac{4538178331687t^{4}}{47563560400}-\frac{3078604127599t^{5}}{47563560400}+\frac{967563513917t^{6}}{47563560400}-\frac{162453420129t^{7}}{47563560400}-\frac{2649290644t^{8}}{8918167575}-\frac{376322947t^{9}}{35672670300}$	$\frac{84168084}{118908901}$	0	0
		$\frac{13}{2}$	$-\frac{5948684251136t}{37456303815}+\frac{144230158885888t^{2}}{561844557225}-\frac{4177013190656t^{3}}{561844557225}-\frac{5383859794688t^{4}}{26754502725}+\frac{4177735346176t^{5}}{26754502725}-\frac{162078029312t^{6}}{2972722525}+\frac{1884737615872t^{7}}{187281519075}-\frac{536355657472t^{8}}{561844557225}+\frac{20621840384t^{9}}{561844557225}$	$\frac{231784448}{356726703}$	$\frac{6439786496t}{118908901}-\frac{157064647168t^{2}}{1783633515}+\frac{587719296t^{3}}{1783633515}+\frac{40862197376t^{4}}{594544505}-\frac{32038918912t^{5}}{594544505}+\frac{11290895616t^{6}}{594544505}-\frac{2104740352t^{7}}{594544505}+\frac{605368192t^{8}}{1783633515}-\frac{23538944t^{9}}{1783633515}$	$\frac{86102}{11890}$

k	***t***	***j***	$\alpha_j(t)$	$\alpha_j(k-1)$	$\beta_j(t)$	$\beta_j(k-1)$
		7	1	1	$-\frac{519609090\ t}{118908901}+\frac{1711560851\ t^{2}}{237817802}-\frac{189950761\ t^{3}}{475635604}-\frac{661320660\ t^{4}}{118908901}+\frac{1061119059\ t^{5}}{237817802}-\frac{381762381\ t^{6}}{237817802}+\frac{145576431\ t^{7}}{475635604}-\frac{3578575\ t^{8}}{118908901}+\frac{143143\ t^{9}}{118908901}$	$\frac{180360180}{118908901}$

Table 2: The Coefficients of the Second Derivative Hybrid Predictor Continuous Linear Multi-step Method of (1.3)

k	***t***	***j***	$\alpha_j(t)$	$\alpha_j(t_{value})$	$\beta_{1,j}(t_{value})$	$\beta_{1,j}(t_{values})$	$\beta_{2,k}(t)$	$\beta_{2,k}(t_{values})$
1	$\frac{-1}{2}$	0	$-t^3$	$\frac{1}{8}$	0	0	0	0
		$\frac{1}{2}$	1	1	0	0	0	0
		1	$1+t^3$	$\frac{7}{8}$	$t-t^3$	$\frac{-3}{8}$	$\frac{t^2}{2}+\frac{t^3}{2}$	$\frac{1}{16}$
2	$\frac{1}{2}$	0	$-\frac{t}{8}+\frac{3t^2}{8}-\frac{3t^3}{8}+\frac{3t^3}{8}$	$\frac{-1}{128}$	0	0	0	0
		1	$1-2t+2t^3-t^4$	$\frac{3}{16}$	0	0	0	0
		$\frac{3}{2}$	1	1	0	0	0	0
		2	$\frac{17}{8}-\frac{3t^2}{8}-\frac{13t^3}{8}+\frac{7t^4}{8}$	$\frac{105}{128}$	$-\frac{5t}{4}+\frac{3t^2}{4}+\frac{5t^3}{4}-\frac{3t^4}{4}$	$\frac{-21}{64}$	$\frac{t}{4}-\frac{t^2}{4}-\frac{t^3}{4}+\frac{t^4}{4}$	$\frac{3}{64}$
	$\frac{3}{2}$	0	$-\frac{4t}{27}+\frac{10t^2}{27}-\frac{t^3}{3}+\frac{7t^3}{54}-\frac{t^5}{54}$	$\frac{1}{576}$	0	0	0	0
		1	$1-\frac{3t}{2}-\frac{t^2}{4}+\frac{11t^3}{8}-\frac{3t^3}{4}+\frac{t^5}{8}$	$\frac{-5}{256}$	0	0	0	0

		2	$4t-2t^2-3t^3+\frac{5t^4}{2}-\frac{t^5}{2}$	$\frac{15}{64}$	0	0	0	0
		$\frac{5}{2}$	1	1	0	0	0	0
		3	$-\frac{127t}{54}+\frac{203t^2}{108}+\frac{47t^3}{24}-\frac{203t^3}{108}+\frac{85t^5}{216}$	$\frac{1805}{2304}$	$\frac{14t}{9}-\frac{25t^2}{18}-\frac{5t^3}{4}+\frac{25t^3}{18}-\frac{11t^5}{36}$	$\frac{-115}{384}$	$-\frac{t}{3}+\frac{t^2}{3}+ -\frac{t^3}{3}+\frac{t^5}{12}$	$\frac{5}{128}$
4	$\frac{5}{2}$	0	$-\frac{9t}{64}+-\frac{21t^3}{128}+\frac{7t^3}{48}-\frac{t^5}{32}+\frac{t^6}{384}$	$\frac{-5}{8192}$	0	0	0	0

k	***t***	***j***	$\alpha_j(t)$	$\alpha_j(t_{value})$	$\beta_{1,j}(t_{value})$	$\beta_{1,j}(t_{values})$	$\beta_{2,k}(t)$	$\beta_{2,k}(t_{values})$
		1	$1-\frac{3t}{2}-\frac{t^2}{6}+\frac{35t^3}{27}-\frac{22t^3}{27}$	$\frac{7}{1152}$	0	0	0	0
		2	$\frac{27t}{8}-\frac{27t^2}{16}-\frac{9t^3}{4}+\frac{17t^3}{8}-\frac{5t^5}{8}+\frac{t^6}{16}$	$\frac{-35}{1024}$	0	0	0	0
		3	$-\frac{9t}{2}+\frac{9t^2}{2}+3t^3-\frac{13t^4}{3}+\frac{3t^5}{2}-\frac{t^6}{2}$	$\frac{576}{3343}$	0	0	0	0
		$\frac{7}{2}$	1	1	0	0	0	0
		4	$\frac{177t}{64}-\frac{9529t^2}{3456}-\frac{7183t^3}{3456}+\frac{9529t^3}{3456}+\frac{2375t^5}{3456}$	$\frac{-81095}{36864}$	$-\frac{29t}{16}+\frac{65t^2}{32}+\frac{155t^3}{144}-\frac{35t^3}{18}+\frac{53t^5}{72}-\frac{25t^6}{288}$	$\frac{-1715}{6144}$	$\frac{3t}{8}-\frac{7t^2}{16}-\frac{5t^3}{24}+\frac{5t^4}{12}-\frac{t^5}{6}+\frac{t^6}{48}$	$\frac{35}{1024}$
5	$\frac{7}{2}$	0	$-\frac{16t}{125}+\frac{124t^2}{375}-\frac{41t^3}{125}+\frac{49t^3}{300}-\frac{131t^5}{3000}+\frac{3t^6}{500}-\frac{t^7}{3000}$	$\frac{7}{25600}$	0	0	0	0
		1	$1-\frac{19t}{12}-\frac{t^2}{48}+\frac{247t^3}{192}-\frac{359t^3}{27}+\frac{113t^5}{384}-\frac{17t^6}{384}+\frac{t^7}{384}$	$\frac{-45}{16384}$	0	0	0	0
		2	$\frac{32t}{9}-\frac{56t^2}{27}-\frac{58t^3}{27}+\frac{131t^3}{54}-\frac{97t^5}{108}+\frac{4t^6}{27}-\frac{t^7}{108}$	$\frac{7}{512}$	0	0	0	0

		3	$-4t+\frac{13t^2}{2}+\frac{9t^3}{4}-\frac{193t^3}{48}+\frac{83t^5}{48}-\frac{5t^6}{16}+\frac{t^7}{48}$	$\frac{-105}{2048}$	0	0	0	0
		4	$\frac{16t}{3}-\frac{20t^2}{3}-\frac{7t^3}{3}+\frac{73t^3}{12}-\frac{71t^5}{24}+\frac{7t^6}{12}-\frac{t^7}{24}$	$\frac{315}{1024}$	0	0	0	0
		$\frac{9}{2}$	1	1		0	0	0

k	***t***	***j***	$\alpha_j(t)$	$\alpha_j(t_{value})$	$\beta_{1,j}(t_{value})$	$\beta_{1,j}(t_{values})$	$\beta_{2,k}(t)$	$\beta_{2,k}(t_{values})$
		5	$-\frac{14299t}{4500}+\frac{221269t^2}{54000}+\frac{274973t^3}{216000}-\frac{321137t^3}{86400}+\frac{810739t^5}{432000}-\frac{164467t^6}{432000}+\frac{12019t^7}{43200}$	$\frac{300013}{409600}$	$-\frac{76t}{9}+\frac{59t^2}{72}+\frac{5729t^3}{7200}-\frac{1921t^3}{2400}+\frac{1567t^5}{7200}-\frac{137t^6}{7200}$	$\frac{-1799}{10240}$	$-\frac{2t}{5}+\frac{8t^2}{15}+\frac{17t^3}{120}-\frac{23t^4}{48}+\frac{61t^5}{240}-\frac{13t^6}{240}+\frac{t^7}{240}$	$\frac{63}{2048}$
6	$\frac{9}{2}$	0	$\frac{5t}{216}-\frac{-149t^2}{2592}+\frac{1399t^3}{25920}-\frac{65t^3}{2592}+\frac{t^5}{162}-\frac{t^6}{1296}+\frac{t^7}{25920}$	$\frac{7}{24576}$	0	0	0	0
		1	$1-\frac{101t^2}{60}+\frac{29t^3}{200}+\frac{959t^3}{750}-\frac{3199t^4}{3000}+\frac{119t^5}{300}-\frac{47t^6}{600}+\frac{t^7}{125}-\frac{t^8}{3000}$	$\frac{77}{51200}$	0	0	0	0
		3	$\frac{-125t^2}{32}-\frac{1025t^3}{384}-\frac{1615t^3}{768}+\frac{2221t^4}{768}-\frac{163t^5}{128}+\frac{53t^6}{192}-\frac{23t^7}{768}+\frac{t^8}{768}$	$\frac{-495}{65536}$	0	0	0	0

		4	$\frac{125t^2}{24}-\frac{22t^3}{32}-\frac{305t^3}{192}+\frac{1177t^4}{192}-\frac{337t^5}{96}+\frac{43t^6}{48}-\frac{7t^7}{764}+\frac{t^8}{192}$	$\frac{-1155}{16384}$		0	0	0
		5	$-\frac{25t}{4}+\frac{215t^2}{24}+\frac{4t^3}{3}-\frac{919t^4}{120}+\frac{19t^5}{4}-\frac{31t^6}{24}+\frac{t^7}{6}-\frac{t^8}{120}$	$\frac{693}{2048}$	0	0	0	0
		$\frac{11}{2}$	1	1	0	0	0	0

k	***t***	***j***	$\alpha_j(t)$	$\alpha_j(t_{value})$	$\beta_{1,j}(t_{value})$	$\beta_{1,j}(t_{values})$	$\beta_{2,k}(t)$	$\beta_{2,k}(t_{values})$
		6	$\frac{15397t}{4320}-\frac{1345709t^2}{259200}-\frac{1713679t^3}{2592000}+\frac{11428061t^4}{259200}-\frac{725941t^5}{259200}+\frac{100777t^6}{129600}-\frac{265111^7}{259200}+\frac{13489t^8}{1259200}$	$\frac{3505733}{4915200}$	$\frac{-53t}{24}+\frac{4669t^2}{1440}+\frac{1813t^3}{4800}-\frac{4389t^4}{1600}+\frac{847t^5}{480}-\frac{119}{240}+\frac{317t^7}{4800}-\frac{49t^8}{14400}$	$\frac{20559}{81920}$	$\frac{5t}{12}-\frac{89t^2}{144}-\frac{191^3}{1440}+\frac{749t^4}{1440}-\frac{49t^5}{144}+\frac{7t^6}{72}-\frac{19t^7}{1440}+\frac{t^8}{1440}$	$\frac{231}{8192}$
7	$\frac{11}{2}$	0	$-\frac{36t}{343}+\frac{501t^2}{1715}-\frac{558t^3}{1715}+\frac{142t^3}{735}-\frac{299t^5}{4410}+\frac{57t^6}{3920}-\frac{463t^7}{246960}+\frac{11t^8}{82320}-\frac{t^9}{246960}$	$\frac{33}{401408}$	0	0	0	0
		1	$1-\frac{107t}{60}+\frac{19t^2}{60}+\frac{34t^3}{27}-\frac{3871t^3}{3240}+\frac{13153t^5}{25920}-\frac{391t^6}{3240}+\frac{43t^7}{2592}-\frac{t^8}{810}+\frac{t^9}{25920}$	$\frac{77}{51200}$	0	0	0	0
		2	$\frac{108t}{25}-\frac{432t^2}{125}-\frac{252t^3}{125}+\frac{428t^3}{125}-\frac{1303t^5}{25920}+\frac{2729t^6}{6000}-\frac{133t^7}{2000}+\frac{31t^8}{6000}-\frac{t^9}{6000}$	$\frac{1001}{16384}$	0	0	0	0
		3	$-\frac{5t}{192}+\frac{47t^2}{2304}+\frac{t^3}{48}-\frac{23t^3}{1152}+\frac{t^5}{192}+\frac{t^6}{2304}$	$\frac{-2145}{16384}$	0	0	0	0
		4	$\frac{5t}{162}-\frac{19t^2}{648}-\frac{29t^3}{1296}+\frac{37t^3}{1296}-\frac{11t^5}{1296}+\frac{t^6}{1296}$	$\frac{1001}{3072}$	0	0	0	0

		5	$-\frac{5t}{32}+\frac{31t^2}{960}+\frac{t^3}{48}-\frac{t^3}{32}+\frac{t^5}{96}+\frac{t^6}{960}$	$-\frac{3003}{4096}$	0	0	0	0
		6	$-\frac{t}{30}-\frac{13t^2}{360}-\frac{t^3}{48}+\frac{5^3}{144}-\frac{t^5}{80}+\frac{t^6}{720}$	$\frac{3003}{1024}$	0	0	0	0
		$\frac{13}{2}$	1	1	0	0	0	0

k	***t***	***j***	$\alpha_j(t)$	$\alpha_j(t_{value})$	$\beta_{1,j}(t_{value})$	$\beta_{1,j}(t_{values})$	$\beta_{2,k}(t)$	$\beta_{2,k}(t_{values})$
		7	$-\frac{807431t}{205800}+\frac{8606723t^2}{137200}-\frac{6290717t^3}{74880000}-\frac{31479321t^3}{6350400}-\frac{479219107t^5}{12700800}-\frac{41097743t^6}{31752000}+\frac{104165237t^7}{444528000}-\frac{9670603t^8}{44452800}+\frac{726301t^9}{889056000}$	$\frac{335572523}{481689600}$	$\frac{1159t}{490}-\frac{37241t^2}{9800}+\frac{136133t^2}{176400}+\frac{150823t^4}{50400}-\frac{231541t^5}{10080}+\frac{200099t^6}{25200}-\frac{511331t^7}{352800}+\frac{47899t^{85}}{352800}-\frac{1211t^9}{10080}$	$-\frac{275847}{1146880}$	$-\frac{3t}{7}+\frac{97t^2}{140}-\frac{17t^2}{840}-\frac{391t^4}{720}+\frac{607t^5}{1440}-\frac{53t^6}{360}+\frac{137t^7}{5040}-\frac{13t^8}{5040}-\frac{t^9}{10080}$	$\frac{429}{16384}$

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