Journal of the Nigerian Association of Mathematical Physics
Volume **11 (November 2007), 159 - 174**
© *J. of NAMP*

Second derivative continuous linear multistep methods for the numerical integration of Stiff system of ordinary differential equations.

[1]F. O. Otunta, M. N. O. Ikhile[2] and [3]R. I. Okuonghae
Department of Mathematics, University of Benin,
Benin City, Nigeria
[1]e-mail: Otunta2000@yahoo.co.uk, [2]e-mail: mnoikhilo@yahoo.com, [3]e-mail: okunoghae01@yahoo.co.uk

Abstract

Continuous linear multi-step methods (CLMM) form a super class of linear multi-step methods (LMM), with properties that embed the characteristics of LMM and hybrid methods. This paper gives a continuous reformulation of the Enright* [5] *second derivative methods. The motivation lies in the fact that the new formulation offers the advantage of a continuous solution of the initial value problem (IVP) unlike the discrete solution generated from the Enright's methods. The success of these methods is in their attainable stiff stability characteristics useful for resolving the problem posed by stiffness in the IVP. In this regard we derive a family of variable order continuous second derivative hybrid methods for the solution of stiff initial value problems in ordinary differential equations. A numerical example is given to demonstrate the application of the methods.

Keywords: *Continuous Linear Multi Step Methods, Hybrid Methods, Predictor Methods, Stiff Stability, Root Locus*
AMS: 65L05, 65L06

1.0 Introduction

The general form of the Enright [5] second derivative continuous linear multistep method is

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = \sum_{j=0}^{k} \beta_{1j} f_{n+j} + h^2 \sum_{j=0}^{k} \beta_{2,j} f_{n+j}; \qquad \alpha_k \neq 0 \qquad (1.1)$$

There has been considerable progress in the development of these methods and its hybrid counter part

$$\sum_{j=0}^{k} \alpha_{1,j} y_{n+j} + \sum_{j=0}^{k} \alpha_{2,j} y_{n} + v_j = h\left(\sum_{j=0}^{k} \beta_{1,j} f_{n+j} + \sum_{j=0}^{k} \beta_{2,j} f_{n+v_j}\right) + h^2\left(\sum_{j=0}^{k} \delta_{1,j} f_{n+j} + \sum_{j=0}^{k} \delta_{2,j} f_{n+v_j}\right); \alpha_k \neq 0$$

. For any given *k*, the parameters $\alpha_{1,j}$, $\alpha_{2,j}$, $\delta_{1,j}$, $\delta_{2,j}$, $\beta_{1,j}$, *and* $\beta_{2,j}$ may be chosen in a variety of ways, but usually the objective is to make the order as high as possible, subject

---
[2]Corresponding author

to the condition of stability, while also desiring small error constant and minimum number of functions evaluation. Interesting special case (see for example the references and that mentioned above) of these methods are known to be stiffly stable for numerical solution of the IVP.

$$y' = f(x,y), \quad y(x_0) = y_0, \quad y \in R \tag{1.2}$$

in ordinary differential equations where $y(x)$ and $f(x, y)$ may be vectors. We seek the numerical solution of this problem by second derivative continuous linear multi-step methods (CLMM) and its hybrid counter part obtained by reformulation of discrete second derivative method in (1.1) into continuous methods. The second derivative continuous linear multi-step methods form a super class of Enright [1] methods and the classical second derivative methods (1.1) in general. Now consider the second derivative CLMM.

$$y(x_{n+k}) = y_{n+k-1} + h\sum_{j=0}^{k} \beta_j(x) f_{n+j} + h^2 \lambda_k(x) f_{n+k}^1 \tag{1.3}$$

where $y''' = \frac{df}{dx}$ is the second derivative of the solution $y(x)$. The new formulation in (1.3) offers an advantage of a continuous solution of the initial value problem in (1.2) over the discrete solution generated from the Enright [1] methods in (1.1). Stiffly stable methods which are useful for resolving the problem of stiffness can be derived from (1.3). In this regard a family of variable orders continuous second derivative hybrid methods for the solution of stiff initial value problems in ordinary differential equations in (1.2) are derived. The parameter v is incorporated to provide off gird collocation point at x_{n+v} in the open interval (x_{n+k-1}, x_{n+k}). By construction, the order of a method is P which is the degree of the basis polynomial function. Also, a family of methods at the hybrid point x_{n+v} which is present in the implicit second derivative scheme is derived to give the numerical values of the solution y_{n+v} at x_{n+v}. The hybrid solution y_{n+v} at the point x_{n+v} is given by continuous method,

$$y(x_{n+k}) = \alpha_j(x)_{n+j} = h\beta_k(x) f_{n+k} + hy^2 \lambda_k(x) f_{n+k}^1 \tag{1.4}$$

In this paper, continuous collocation methods in the literature of Arevalo et al [1], Burrage and Tian [2], Butcher [3, 4], Onumanyi et al [12], Hairer and Lubeich [8], Otunta et al [13, 14], and Sirisena el at [15] for the solution of (1.2) are proposed such that the collocation is done at all grid points x_{n+j} = 0(1) and one off grid point x_{n+v}, with $x_{n+v} \in (x_{n+k-1}, x_{n+k})$ with $v = k - \frac{1}{2}$. The paper is arranged as follows, section 2 will contain a discussion of the derivation of the methods. The way the methods are derived differs from Taylor's series expansion approach. Section 3 contains the determination of absolute stability of the schemes, see Lambert [4]. In section 5, the results of some numerical experiments using one of the methods are presented.

2.0 Derivation of the second derivative CLMM

The continuous linear multistep methods which incorporate analytic property of the second derivative like the Enright [5] second derivative methods, to be derived is given in (1.3). Let the continuous solution to (1.1) be in the form,

$$y(x) = \sum_{j=0}^{k+3} a_j \phi_j(x), \quad m = 1, 2, 3\ldots \tag{2.1}$$

where $\{\phi_j(x)\}_{j=0}^{m}$ are the polynomial basis function given by $\phi_j(x) = x^j$, $j = 0(1)m$ and the

a_j's are the real parameter constants to be determined. From (2.1), setting $m = k + 3$,

$$y'(x)=\sum_{j=0}^{k+3} j a_j \phi_j(x), \quad y''(x)=\sum_{j=0}^{k+3} j(j-1)a_j\phi_{j-2}(x) \tag{2.2}$$

Collocating (2.2) at $x=x_{n+j}$, $j=0(1)k$, x_{n+v} and interpolating (2.1) at $x=x_n$ and x_{n+1}, we obtain the linear system of equations

$$\begin{pmatrix} 0 & \phi_o(x_n) & 2\phi_1(x_n) & \cdots & (k+3)\phi_{k+2}(x_n) \\ 0 & \phi_o(x_{n+1}) & 2\phi_1(x_{n+1}) & \cdots & (k+3)\phi_{k+2}(x_{n+1}) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \phi_o(x_{n+k}) & 2\phi_1(x_{n+k}) & \cdots & (k+3)\phi_{k+2}(x_{n+k}) \\ \vdots & \phi_o(x_{n+v}) & 2\phi_1(x_{n+v}) & \cdots & (k+3)\phi_{k+2}(x_{n+v}) \\ 0 & 0 & 2\phi_0(x_{n+k}) & \cdots & (k+3)(k+2)\phi_{k+2}(x_{n+k}) \\ \phi_o(x_{n+k-1}) & \phi_1(x_{n+k-1}) & \phi_2(x_{n+k-1}) & \cdots & \phi_{k+3}(x_{n+k-1}) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \\ a_{k+1} \\ a_{k+2} \\ a_{k+3} \end{pmatrix} = \begin{pmatrix} f_n \\ f_{n+1} \\ \vdots \\ f_{n+k} \\ f_{n+v} \\ f'_{n+k} \\ y_{n+k-1} \end{pmatrix} \tag{2.3}$$

Solving equation (2.3) by Gaussian elimination methods the values of a_j's are determined and substituting the resulting values a_j's into (2.1) with $t=\dfrac{x-x_{n+1}}{h}$, $a_k(t)=1$ and setting $x=x_{n+k}$ on the left hand side of (2.1), a specific scheme will emerge for a fixed value of *k*. The table (1) shows the values of a_j's, the resultant methods, error constants, order *P*, and the values for *k*= 1(1)7.

Table 1: The SDCLMM

k	**Values of a_j's of the methods**	t	C_{o+1}	p
1	$a_0=y_n,\ a_1=f_n,\ a_2=\dfrac{4f_n-8f_{n+1/2}+4f_{n+1}-hf'_{n+1}}{2h}$, $a_3=\dfrac{-5f_n+16f_{n+1/2}-11f_{n+1}+3hf'_{n+1}}{3h^2},\ a_4=-\dfrac{f_n-4f_{n+1/2}+3f_{n+1}-hf'_{n+1}}{2h^3}$ $y_{n+1}=y_n+\dfrac{h}{6}(f_n+4f_{n+1/2}+f_{n+1})$	0	$\frac{-1}{2880}$	4
2	$a_0=-\dfrac{209hf_n+1212hf_{n+1}-1472hf_{n+3/2}+771hf_{n+2}-186h^2f'_{n+2}-720y_{n+1}}{720}$, $a_1=f_n,\ a_2=\dfrac{8f_n-36f_{n+1}+64f_{n+3/2}-36f_{n+2}+9f'_{n+2}}{6h}$, $a_3=-\dfrac{-31f_n-240f_{n+1}-512f_{n+3/2}+303f_{n+2}-78f'_{n+2}}{36h^2}$, $a_4=-\dfrac{13f_n-132f_{n+1}-320f_{n+3/2}201f_{n+2}+54f'_{n+2}}{48h^3}$, $a_5=\dfrac{-f_n+12f_{n+1}-32f_{n+3/2}+21f_{n+2}-6f'_{n+2}}{30h^4}$, $y_{n+2}=y_{n+1}+\dfrac{h}{720}(-f_n+132f_{n+1}+448f_{n+3/2}+141f_{n+2}-6f'_{n+2})$	1	$\frac{-1}{144000}$	5

3	$a_0 = -\frac{1}{675}(199hf_n + 1065hf_{n+1} - 585hf_{n+2} + 1536hf_{n+5/2} - 865hf_{n+3} + 210h^2 f'_{n+3} - 675y_{n+2}),$ $a_1 = f_n,\ a_2 = -\frac{1}{60}(77f_n - 2255f_{n+1} + 675f_{n+2} - 1152f_{n+5/2} + 625f_{n+3} - 150h\ f'_{n+3}),$ $a_3 = -\frac{1}{540h^2}(-466f_n + 2115hf_{n+1} - 8370f_{n+2} + 14976f_{n+5/2} - 8275hf_{n+3} + 2010h^2 f'_{n+3}),$ $a_4 = -\frac{1}{360h^3}(103f_n - 615hf_{n+1} + 2925f_{n+2} + 14976f_{n+5/2} - 8275hf_{n+3} - 780hf'_{n+3}),$ $a_5 = -\frac{1}{900h^4}(-46f_n + 315f_{n+1} - 1710f_{n+2} + 3456f_{n+5/2} - 2015f_{n+3} + 510hf'_{n+3}),$ $a_6 = -\frac{1}{540h^5}(2f_n - 15f_{n+1} + 90f_{n+2} - 192f_{n+5/2} + 115f_{n+3} - 30hf'_{n+3})$ $y_{n+3} = y_{n+2} + \frac{h}{5400}(f_n - 15f_{n+1} + 1035f_{n+2} + 3264f_{n+5/2} + 1115f_{n+3} - 60hf'_{n+3})$	2	$\frac{-13}{60480}$	6
4	$a_0 = -\frac{1}{31360}(9189hf_n + 48832hf_{n+1} + 924hf_{n+2} + 102144hf_{n+3} - 135168hf_{n+7/2}$ $- 61859hf_{n+4} - 15540h^2 f'_{n+4} - 31360y_{n+3}),$ $a_1 = f_n,$ $a_2 = -\frac{1}{210h}(275hf_n - 784f_{n+1} + 1470f_{n+2} - 3920f_{n+3} + 6144f_{n+7/2} - 3285f_{n+4} - 735hf'_{n+4}),$ $a_3 = -\frac{1}{21600h^2}(-1905f_n + 8704hf_{n+1} - 21360f_{n+2} + 61440f_{n+3} - 98304f_{n+7/2} + 51425f_{n+4}$ $-11940hf'_{n+4}),$ $a_4 = -\frac{1}{2688h^3}(905f_n + 8704hf_{n+1} + 14924f_{n+2} - 47264f_{n+3} - 77824f_{n+7/2} - 41237f_{n+4}$ $+ 9660hf'_{n+4}),$ $a_5 = -\frac{1}{1260h^4}(-93f_n + 602f_{n+1} - 1953f_{n+2} + 6762f_{n+3} - 11520f_{n+7/2} + 6202f_{n+4} - 1470hf'_{n+4}),$ $a_6 = -\frac{1}{2880h^6}(25f_n - 176f_{n+1} + 620f_{n+2} - 2320f_{n+3} + 4096f_{n+7/2} - 2245f_{n+4} + 540hf'_{n+4})$ $a_7 = -\frac{1}{35280h^6}(-15f_n + 112f_{n+1} - 420f_{n+2} + 1680f_{n+3} - 3072f_{n+7/2} + 1715f_{n+4} - 420f'_{n+4})$ $y_{n+4} = y_{n+3} + \frac{h}{846720}(-39f_n + 448f_{n+1} - 3444f_{n+2} + 166656f_{n+3} + 503808f_{n+7/2} +$ $17929f_{n+4} - 10500hf'_{n+4})$	3	$\frac{-1}{120960}$	7
5	$a_0 = -\frac{1}{33075}(9506hf_n + 52740hf_{n+1} - 1400hf_{n+2} + 89040hf_{n+3} - 98910\ f_{n+4}$ $+ 163840hf_{n+9/2} - 825161\ f_{n+5} + 18480h^2 f'_{n+5} - 33075y_{n+4}),$ $a_1 = f_n,$ $a_2 = -\frac{1}{2520h}(3409\ f_n - 10125\ f_{n+1} + 18900\ f_{n+2} - 31500\ f_{n+3} + 70875\ f_{n+4}$ $- 102400\ f_{n+9/2} + 50841\ f_{n+5} - 11340hf'_{n+5}),$	4	$\frac{-733}{20321280}$	8

	$a_3 = -\frac{1}{226800 h^2}(-217994 f_n + 1036125 f_{n+1} - 2501100 f_{n+2} + 4483500 f_{n+3} - 10442250 f_{n+4}$ $+ 15257600 f_{n+9/2} - 7615881 f_{n+5} + 1704780 h f'_{n+5}),$ $a_4 = -\frac{1}{604800 h^3}(242144 f_n - 1431225 f_{n+1} + 4038720 f_{n+2} - 7910700 f_{n+3} + 19303200 f_{n+4}$ $-28651520 f_{n+9/2} + 14409381 f_{n+5} - 341980 h f'_{n+5}),$ $a_5 = -\frac{1}{151200 h^4}(-15344 f_n + 102915 f_{n+1} - 322560 f_{n+2} + 683340 f_{n+3} - 1758960 f_{n+4}$ $+ 2662400 f_{n+9/2} - 1351791 f_{n+5} + 306180 f'_{n+5}),$ $a_6 = -\frac{1}{36288 h^5}(560 f_n 4077 f_{n+1} + 13776 f_{n+2} - 31164 f_{n+3} + 84672 f_{n+4}$ $-131072 f_{n+9/2} + 67305 f_{n+5} - 15372 h f'_{n+5}),$ $a_7 = -\frac{1}{1058400 h^6}(-1372 f_n + 10575 f_{n+1} - 37800 f_{n+2} + 90300 f_{n+3} - 258300 f_{n+4}$ $+ 409600 f_{n+9/2} - 213003 f_{n+5} + 49140 h f'_{n+5}),$ $a_8 = -\frac{1}{604800 h^7}(28 f_n - 255 f_{n+1} + 840 f_{n+2} - 2100 f_{n+3} + 6300 f_{n+4}$ $-10240 f_{n+9/2} + 5397 f_{n+5} - 1260 h f'_{n+5}),$ $y_{n+5} = y_{n+4} + \frac{h}{3175200}(49 f_n - 540 f_{n+1} + 3150 f_{n+2} - 16590 f_{n+3} + 635985_{n+4} + 1871360 f_{n+7/2}$ $+ 681786 f_{n+5} - 41580 h f'_{n+5})$			
6	$a_0 = -\frac{1}{9580032}(2701130\, h f_n + 15699640\, h f_{n+1} - 184375\, h f_{n+2} + 27794800\, h f_{n+3} - 15336750\, f_{n+4}$ $+ 54224280\, f_{n+5} - 66764800\, f_{n+11/2} + 31425735\, f_{n+6} 676830\, h^2 f'_{n+6} - 9580032\, y_{n+6}),$ $a_1 = f_n,$ $a_2 = -\frac{1}{9240 h}(12929\, f_n - 40656\, f_{n+1} + 81675\, f_{n+2} - 135520\, f_{n+3} + 190575\, f_{n+4}$ $-365904\, f_{n+5} + 491520\, f_{n+11/2} - 234619\, f_{n+6} + 50820\, h f'_{n+6}),$ $a_3 = -\frac{1}{831600\, h^2}(-870310\, f_n + 4387152\, f_{n+1} - 11263725\, f_{n+2} + 20044640\, f_{n+3} - 29140650\, f_{n+4}$ $+ 57047760\, f_{n+5} - 77168640\, f_{n+9/2} + 36963773\, f_{n+6} - 8024940\, h f'_{n+6}),$ $a_4 = -\frac{1}{475200\, h^3}(223475\, f_n - 1402104\, f_{n+1} + 4180275\, f_{n+2} - 8077520\, f_{n+3} + 12263625\, f_{n+4}$ $-24651000\, f_{n+5} + 33669120\, f_{n+11/2} - 16205871\, f_{n+6} + 3529680\, h f'_{n+6}),$ $a_5 = -\frac{1}{2376000\, h^4}(-317030\, f_n + 2257904\, f_{n+1} - 7442325\, f_{n+2} + 1541040\, f_{n+3} - 24609750\, f_{n+4}$ $+ 51076080\, f_{n+5} - 70615040\, f_{n+11/2} - 2038443\, f_{n+6} + 448140\, f'_{n+6}),$ $a_6 = -\frac{1}{570240\, h^5}(13762\, f_n - 106392\, f_{n+1} + 376695\, f_{n+2} - 829840\, f_{n+3} + 1385010\, f_{n+4}$ $-2978712\, f_{n+5} + 4177920\, f_{n+11/2} - 2038443\, f_{n+6} + 448140\, h f'_{n+6}),$ $a_7 = -\frac{1}{3326400\, h^6}(-8990\, f_n + 73568\, f_{n+1} - 274725\, f_{n+2} + 635360\, f_{n+3} - 2207250\, f_{n+4}$ $+ 2471040\, f_{n+5} - 3522560\, f_{n+11/2} + 1733457\, f_{n+6} - 658350\, h f'_{n+6}),$	5	$\frac{-443}{261273600}$	9

	$a_8 = -\frac{1}{26611200\ h^7}(4550\ f_n - 38808\ f_{n+1} + 150975\ f_{n+2} - 363440\ f_{n+3}$ $+ 658350\ f_{n+4} - 1524600\ f_{n+5} + 2211840\ f_{n+11/2} - 1098867\ f_{n+6}$ $+ 244860\ hf'_{n+6}),$ $a_9 = -\frac{1}{14968800\ h^8}(-70\ f_n + 616\ f_{n+1} - 2475\ f_{n+2} + 6160\ f_{n+3} - 11550\ f_{n+4}$ $+ 27720\ f_{n+5} - 40960\ f_{n+11/2} + 20559\ f_{n+6} - 4620\ hf'_{n+6}),$ $y_{n+6} = y_{n+5} + \frac{1}{1197504000\ h}(-7330\ f_n + 82984\ f_{n+1} - 462825\ f_{n+2}$ $+ 1833040\ f_{n+3} - 7466250\ f_{n+4} + 242759880\ f_{n+5} + 701480960\ f_{n+11/2}$ $+ 259283541\ f_{n+6} - 16165380\ hf'_{n+6})$			
7	$a_0 = -\frac{1}{22425185423600}(659964369105360 hf_n + 3448166411412450 hf_{n+1} +$ $423827814874464 hf_{n+2} + 4480073880857575 hf_{n+3} + 26684764126020 f_{n+4} +$ $3918878883009090 hf_{n+5} - 45730484800 hf_{n+6} + 610460355133440 f_{n+13/2} -$ $205169268185739 f_{n+7} + 27407439079020 h^2 f'_{n+7} + 224251851423600 y_{n+6}),$ $a_1 = f_n,$ $a_2 = -\frac{1}{5491882075680 h}(7042629430188 f_n - 18873101642850 f_{n+1}$ $+ 28481102329680 f_{n+2} - 32893244759375 f_{n+3} + 26684764126020 f_{n+4}$ $- 14056372267938 f_{n+5} + 1020219200 f_{n+6} + 12386551726080 f_{n+13/2}$ $- 8773349161005 f_{n+7} + 2285292309300 f'_{n+7}),$ $a_3 = -\frac{1}{691977141535680 h^2}(-579506835570696 f_n + 2461539471603030 f_{n+1} -$ $4881731880641616 f_{n+2} + 6065807117094725 f_{n+3} - 5083075433892456 f_{n+4}$ $+ 140563722671989846 f_{n+5} + 233347674560 f_{n+6} - 2437808638132224 f_{n+13/2}$ $+ 1729758875223951 f_{n+7} - 45130621667900 hf'_{n+7}),$ $a_4 = -\frac{1}{8387601715588400 h^3}(255227989469760 f_n - 1335847189090050 f_{n+1}$ $+ 3101765825287584 f_{n+2} - 4225971105203575 f_{n+3} + 3703316201151840 f_{n+4}$ $- 2035734844189410 f_{n+5} + 229284328000 f_{n+6} + 1860577565736960 f_{n+13/2}$ $- 1323563727491109 f_{n+7} + 346142333628820 hf'_{n+7}),$ $a_5 = -\frac{1}{12881439150992000 h^4}(-76398972843600 f_n + 448839514182150 f_{n+1}$ $- 1150392949497792 f_{n+2} + 1689209516019025 f_{n+3} - 1555009759583280 f_{n+4}$ $+ 880502472199350 f_{n+5} - 160219259200 f_{n+6} + 826239617925120 f_{n+13/2}$ $+ 589650016708467 f_{n+7} - 154667428575660 hf'_{n+7}),$ $a_6 = -\frac{1}{384431745297600 h^5}(1635508409040 f_n - 1013300583750 f_{n+1}$ $+ 27204400471248 f_{n+2} - 41451511876775 f_{n+3} + 39127705737120 f_{n+4}$ $- 222485029406840 f_{n+5} + 13431017600 f_{n+6} + 21353384263680 f_{n+13/2}$ $- 15264882777573 f_{n+7} + 4009953287340 hf'_{n+7}),$	6	$\frac{-370919}{172821463920}$	10

$$a_7 = -\frac{1}{1345511108 54160 h^6}(81048049038\, f_n - 566052886035\, f_{n+1}$$
$$+171773984964\, f_{n+2} + 2952813318455\, f_{n+3} + 33125134009998\, f_{n+4}$$
$$-1982136369213\, f_{n+5} 819538720 f_{n+6} + 2094359887872\, f_{n+13/2}$$
$$-1151645681 9449\, f_{n+7} + 4034817586 80 h f'_{n+7}),$$

$$a_8 = -\frac{1}{6150907924\ 761600\, h^7}(-9800090467\ 20\, f_n + 6949504475\ 550\, f_{n+1}$$
$$-2145016406\ 3328\, f_{n+2} - 3758080338\ 5125\, f_{n+3} - 4059593345\ 8080\, f_{n+4}$$
$$+2632251448\ 8270\, f_{n+5} + 3979976000\, f_{n+6} - 2876322742\ 2720\, f_{n+13/2}$$
$$+2093253166\ 5903\, f_{n+7} - 5599108454\ 940\, h f'_{n+7}),$$

$$a_9 = -\frac{1}{2075931424\ 6070400\, h^8}(2784806244\ 00\, f_n - 2020912598\ 250\, f_{n+1}$$
$$+6397288305\ 408\, f_{n+2} - 1152065206\ 58023875\, f_{n+3} + 1281779691\ 1920\, f_{n+4}$$
$$-8574887371\ 050\, f_{n+5} - 795995200\, f_{n+6} + 9830947553\ 280\, f_{n+13/2}$$
$$-7207259406\ 633\, f_{n+7} + 1943604002\ 340\, h f'_{n+7}),$$

$$a_{10} = -\frac{1}{1153295235\ 892800\, h^9}(-4772267280\, f_n + 3530008755\ 0\, f_{n+1}$$
$$-1141672371\ 84\, f_{n+2} + 2106068214\ 25\, f_{n+3} - 2406938133\ 60\, f_{n+4}$$
$$+1658941584\ 30\, f_{n+5} - 11211200\, f_{n+6} - 2000078438\ 40\, f_{n+13/2}$$
$$+1478288830\ 59\, f_{n+7} - 4026644622\ 0 h f'_{n+7}),$$

$$y_{n+7} = y_{n+6} = +\frac{1}{1162521597\ 7799400}(-7239906872\ 54208\, f_n$$
$$+6182308958\ 593950\, f_{n+1} - 2410001253\ 6503904\, f_{n+2} + 5725791557\ 7587725\, f_{n+3}$$
$$-9463215415\ 0453920\, f_{n+4} + 1228926152\ 82234798\, f_{n+5} + 8316894185\ 60\, f_{n+6}$$
$$+9875956151\ 66791680\, f_{n+13/2} + 1080484684\ 79527719\, f_{n+7}$$
$$+1498662213\ 2822340\, h f'_{n+7})$$

3.0 Derivation of the second derivative hybrid CLMM

Similarly, the continuous hybrid solution in (1.3) is

$$y_{n+v}(x) = \sum_{j=0}^{k+2} b_j \phi_j(x), \qquad k = 1,\ 2,\ 3 \ldots \tag{3.1}$$

where b_j's are as usual the parameter constants to be determined. Therefore

$$y'_{n+v}(x) = \sum_{j=0}^{k+2} j b_j \phi_{j-1}(x), \qquad y''_{n+v}(x) = \sum_{j=0}^{k+2} j(j-1) b_j \phi_{j-2}(x) \tag{3.2}$$

Collocating (3.2) at $x = x_{n+j}$, $j = 0(1)k$, and interpolating (3.1) at $x = x_n$ and x_{n+1}, we obtain the general solution matrix to be

$$\begin{pmatrix} \phi_o(x_n) & \phi_1(x_n) & \phi_2(x_n) & \cdots & \phi_{k+2}(x_n) \\ \phi_o(x_{n+1}) & \phi_1(x_{n+1}) & \phi_2(x_{n+1}) & \cdots & \phi_{k+2}(x_{n+1}) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \phi_o(x_{n+1}) & \phi_1(x_{n+1}) & \phi_2(x_{n+k}) & \cdots & \phi_{k+2}(x_{n+k}) \\ 0 & \phi_o(x_{n+k}) & \phi_1(x_{n+k}) & \cdots & (k+2)\phi_{k+1}(x_{n+k}) \\ 0 & 0 & 2\phi_0(x_{n+k}) & \cdots & (k+2)(k+1)\phi_k(x_{n+k}) \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_k \\ b_{k+1} \\ b_{k+2} \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ \vdots \\ y_{n+k} \\ f_{n+k} \\ f'_{n+k} \end{pmatrix} \tag{3.3}$$

Solving equation (3.3), gives the values of b_j*'s* and substituting the resulting values b_j*'s* into (3.1) with $t = \frac{x - x_{n+1}}{h}$, $\alpha_k(t) = 1,\ v = k - \frac{1}{2}$ and setting $x = x_{n+v}$ on the left hand side of (3.1), a specific scheme is obtained. The table (2) shows the values of b_j*'s*, the methods, error constants (C_{p+1}), order P, and t-values for $k \le 7$.

Table 2: The hybrid CLMM

k	***values of bj's of the Methods***	***t***	***C_{p+1}***	***p***
1	$b_0 = y_n,\ b_1 = -\frac{1}{2h}(4hf_{n+1} - h^2 f'_{n+1} + 6y_n - 6y_{n+1}),$ $b_2 = -\frac{1}{h^2}(-3hf_{n+1} + h^2 f'_{n+1} - 3y_n + 3y_{n+1}),\ b_3 = -\frac{1}{h^3}(2hf_{n+1} - h^2 f'_{n+1} + 2y_n - 2y_{n+1}),$ $y_{n+1/2} = \frac{1}{16}(2y_n + 14y_{n+1}) + \frac{h}{16}(-6f_{n+1} + hf'_{n+1})$	$\frac{-1}{2}$	$\frac{-1}{384}$	3
2	$b_0 = y_n,\ b_1 = -\frac{1}{2h}(-8hf_{n+2} + 2h^2 f'_{n+2} + 5y_n - 16y_{n+1} + 11y_{n+2}),$ $b_2 = -\frac{1}{4h^2}(30hf_{n+2} - 8h^2 f'_{n+2} - 9y_n + 48y_{n+1} - 39y_{n+2}),$ $b_3 = -\frac{1}{8h^3}(-34hf_{n+2} - 10h^2 f'_{n+2} + 7y_n - 48y_{n+1} + 41y_{n+2}),$ $b_4 = -\frac{1}{8h^4}(6hf_{n+2} - 2h^2 f'_{n+2} - 5y_n + 8y_{n+1} - 7y_{n+2}),$ $y_{n+3/2} = \frac{1}{128}(-y_n + 24y_{n+1} + 105y_{n+2}) + \frac{h}{128}(-42f_{n+2} + 6hf'_{n+2})$	$\frac{1}{2}$	$\frac{-1}{1280}$	4
3	$b_0 = y_n,\ b_1 = -\frac{1}{4h}(26hf_{n+3} - 6h^2 f'_{n3} + 10y_n - 27y_{n+1} + 54y_{n+2} - 37y_{n+3}),$ $b_2 = -\frac{1}{24h^2}(-330hf_{n+3} + 78h^2 f'_{n+3} - 56y_n + 243y_{n+1} - 648y_{n+2} + 461y_{n+3}),$ $b_3 = -\frac{1}{216h^3}(2130hf_{n+3} - 522h^2 f'_{n+3} + 224y_n - 1215y_{n+1} + 3888y_{n+2} - 2897y_{n+3}),$ $b_4 = -\frac{1}{72h^4}(-210hf_{n+3} + 54h^2 f'_{n+3} - 16y_n + 99y_{n+1} - 360y_{n+2} + 277y_{n+3}),$ $b_5 = -\frac{1}{216h^5}(66hf_{n+3} - 18h^2 f'_{n+3} + 4y_n - 27y_{n+1} + 108y_{n+2} - 85y_{n+3}),$ $y_{n+5/2} = \frac{1}{2304}(4y_n - 45y_{n+1} + 540y_{n+2} + 1805y_{n+3}) + \frac{h}{2304}(-690f_{n+3} + 90hf'_{n+3})$	$\frac{3}{2}$	$\frac{-1}{3072}$	5

4	$b_0 = y_n, \; b_1 = -\frac{1}{36h}(-336hf_{n+4} + 72h^2 f'_{n+4} + 93y_n - 256y_{n+1} + 432y_{n+2}$ $- 768y_{n+3} + 499y_{n+4}),$ $b_2 = -\frac{1}{432h^2}(9300hf_{n+4} - 2016h^2 f'_{n+4} + 1107y_n + 4864y_{n+1} - 10800y_{n+2}$ $+ 20736y_{n+3} - 1693y_{n+4}),$ $b_3 = -\frac{1}{1728h^3}(-31020hf_{n+4} + 6840h^2 f'_{n+4} + 2205y_n - 12032y_{n+1} + 31536y_{n+2}$ $- 66816y_{n+3} + 45107y_{n+4}),$ $b_4 = -\frac{1}{1152h^4}(7980hf_{n+4} - 1800h^2 f'_{n+4} - 393y_n + 2432y_{n+1} - 7128y_{n+2}$ $+ 16512y_{n+3} - 11423y_{n+4}),$ $b_5 = -\frac{1}{1728h^5}(-2172hf_{n+4} + 504h^2 f'_{n+4} + 81y_n - 544y_{n+1} + 1728y_{n+2}$ $- 4320y_{n+3} + 3055y_{n+4}),$ $b_6 = -\frac{1}{3456h^6}(300hf_{n+4} - 72h^2 f'_{n+4} - 9y_n + 64y_{n+1} + 216y_{n+2}$ $+ 567y_{n+3} - 415y_{n+4}),$ $y_{n+7/2} = \frac{1}{73728}(-45y_n + 448y_{n+1} - 2520y_{n+2} + 20160y_{n+3} + 55685hf_{n+4})$ $+ \frac{h}{73728}(-20580f_{n+4} + 2520hf'_{n+4})$	$\frac{5}{2}$	$\frac{-1}{6144}$	6
5	$b_0 = y_n, \; b_1 = -\frac{1}{720h}(8940hf_{n+5} - 1800h^2 f'_{n+5} + 1932y_n - 5625y_{n+1} + 10000y_{n+2}$ $- 15000y_{n+3} + 25500y_{n+4} - 13807y_{n+5}),$ $b_2 = -\frac{1}{43200h^2}(-1323420hf_{n+5} + 268200h^2 f'_{n+5} - 122184y_n + 568125y_{n+1}$ $- 1310000y_{n+2} - 2115000y_{n+3} - 3285000y_{n+4} + 2034059y_{n+5}),$ $b_3 = -\frac{1}{432000h^3}(1230979hf_{n+5} - 2158200h^2 f'_{n+5} + 669456y_n - 3864375y_{n+1}$ $+ 104200000y_{n+2} - 18405000y_{n+3} + 29970000y_{n+4} - 18790081y_{n+5}),$ $b_4 = -\frac{1}{8640h^4}(-112980hf_{n+5} + 23400h^2 f'_{n+5} - 4176y_n + 27315y_{n+1}$ $- 81760Y_{n+2} + 156240y_{n+3} - 268560y_{n+4} + 170941Y_{n+5}),$ $b_5 = -\frac{1}{21600h^5}(68460hf_{n+5} - 14400h^2 f'_{n+5} + 1872y_n - 13275y_{n+1}$ $+ 42800y_{n+2} - 87300y_{n+3} + 158400y_{n+4} - 102497Y_{n+5}),$ $b_6 = -\frac{1}{2160h^6}(-840hf_{n+5} + 180h^2 f'_{n+5} - 18y_n + 135y_{n+1}$ $460y_{n+2} + 990y_{n+3} - 1890y_{n+4} + 1243y_{n+5}),$ $b_7 = -\frac{1}{432000h^7}(8220hf_{n+5} - 1800h^2 f'_{n+5} + 144y_n - 31125y_{n+1}$ $+ 4000y_{n+2} - 9000y_{n+3} + 18000y_{n+4} - 12019y_{n+5}),$	$\frac{7}{2}$	$\frac{-3}{32768}$	7

	$y_{n+9/2} = \frac{1}{409600}(112\, y_n - 1125\, y_{n+1} + 56000\, y_{n+2} - 21000\, y_{n+3} + 126000\, hf_{n+4}$ $+ 30013\, y_{n+5}) + \frac{h}{409600}(-107940\, f_{n+5} + 12600\, hf'_{n+5})$			
6	$b_0 = y_n,\ b_1 = -\frac{1}{600h}(-9420hf_{n+6} + 1800h^2 f'_{n+6} + 1670y_n - 5184y_{n+1} + 10125y_{n+2}$ $-16000y_{n+3} + 20250y_{n+4} - 25920y_{n+5} + 15059y_{n+6}),$ $b_2 = -\frac{1}{12000h^2}(490980hf_{n+6} - 94200h^2 f'_{n+6} - 37200y_n + 184896y_{n+1} - 462375y_{n+2}$ $+784000y_{n+3} - 1026000y_{n+4} + 1339200y_{n+5} - 782521y_{n+6}),$ $b_3 = -\frac{1}{216000h^3}(-8945580hf_{n+6} + 1726200h^2 f'_{n+6} + 397600y_n - 2457216y_{n+1} + 7138125y_{n+2}$ $-13152000y_{n+3} + 17982000y_{n+4} - 2105600y_{n+5} + 14197091y_{n+6}),$ $b_4 = -\frac{1}{1296000h^4}(27934940hf_{n+6} - 5430600hf'_{n+6} - 837200y_n + 5868288y_{n+1} - 18842625y_{n+2}$ $-13152000y_{n+3} + 17982000y_{n+4} - 2105600y_{n+5} + 14197091y_{n+6}),$ $b_5 = -\frac{1}{172800h^5}(-1091580hf_{n+6} + 214200h^2 f'_{n+6} + 23940y_n - 182016y_{n+1} + 627525y_{n+2}$ $-1318400y_{n+3} + 1982700y_{n+4} - 2845440y_{n+5} + 1711691y_{n+6}),$ $b_6 = -\frac{1}{2592000\, h^6}(2730420\, hf_{n+6} - 541800\, h^2 f'_{n+6} - 46300\, y_n + 372384\, y_{n+1} - 1353375\, y_{n+2}$ $+ 2984000\, y_{n+3} - 4684500\, y_{n+4} + 6976800\, y_{n+5} + 42490009\, y_{n+6}),$ $b_7 = -\frac{1}{864000\, h^7}(-80580\, hf_{n+6} + 16200\, h^2 f'_{n+6} + 1100\, y_n - 9216\, y_{n+1} + 34875\, y_{n+2}$ $- 80000\, y_{n+3} + 130500\, y_{n+4} - 201600\, y_{n+5} + 124321\, y_{n+6}),$ $b_8 = -\frac{1}{2592000\, h^8}(8820\, hf_{n+6} - 1800\, h^2 f'_{n+6} - 100\, y_n + 864\, y_{n+1} - 3375\, y_{n+2}$ $+ 8000\, y_{n+3} - 13500\, y_{n+4} + 21600\, y_{n+5} - 13489\, y_{n+6}),\backslash$ $y_{n+11/2} = \frac{1}{4915200}(-700\, y_n + 7392\, y_{n+1} - 37125\, y_{n+2} + 123200\, y_{n+3} - 346500\, y_{n+4}$ $+ 1663200\, y_{n+5} + 3505733\, y_{n+6}) + \frac{h}{4915200}(-1233540\, f_{n+6} + 138600\, hf'_{n+6})$	$\frac{9}{2}$	$\frac{-11}{196608}$	8
7	$b_0 = y_n,$ $b_1 = -\frac{1}{25200\, h}(4825\, hf_{n+7} - 88200\, hh^2 f'_{n+7} + 72540\, y_n$ $- 240100\, y_{n+1} + 518616\, y_{n+2} - 900375\, {}_{n+3} + 1200500\, y_{n+4} - 1296540\, y_{n+5}$ $+ 1440600\, y_{n+6} - 795241\, y_{n+7}),$ $b_2 = -\frac{1}{3582000\, h^2}(-184324140\, hf_{n+7} + 33780600\, h^2 f'_{n+7}$ $- 11878000\, y_n + 63146300\, y_{n+1} - 17699128\, y_{n+2} + 320833625\, {}_{n+3} - 441784000\, y_{n+4}$ $+ 486202500\, y_{n+5} - 546947800\, y_{n+6} + 303126503\, y_{n+7}),$ $b_3 = -\frac{1}{55566000\, h^3}(3146107860\, hf_{n+7} - 578768400\, h^2 f'_{n+7}$ $+ 118934325\, y_n - 787888150\, y_{n+1} + 2490069897\, y_{n+2}$ $- 4999782375\, {}_{n+3} + 7172687375\, y_{n+4} - 8093650950\, y_{n+5}$ $+ 9258916275\, y_{n+6} - 5159286397\, y_{n+7}),$	$\frac{11}{2}$	$\frac{-143}{3932160}$	9

	$b_4 = -\frac{1}{3628800h^2}(-117846540hf_{n+7} + 21785400h^2 f'_{n+7} - 2996640y_n$ $+ 22549660y_{n+1} - 78539328y_{n+2} + 168674625_{n+3} - 252912800y_{n+4} + 293906340y_{n+5}$ $- 343244160y_{n+6} + 192562303y_{n+7}),$ $b_5 = -\frac{1}{6350400h^5}(1388143260hf_{n+7} - 258161400h^2 f'_{n+7} + 257076000y_n -$ $210058100y_{n+1} + 784041552y_{n+2} - 1779490125_{n+3} + 2783494000y_{n+4}$ $- 3338855100y_{n+5} + 3993519600y_{n+6} - 2258359427y_{n+7}),$ $b_6 - \frac{1}{9072000h^6}(-20246949hf_{n+7} + 3792600h^2 f'_{n+7} - 288000y_n$ $+ 2491300y_{n+1} - 9788688y_{n+2} + 23239125_{n+3} - 37772000y_{n+4}$ $+ 46777500y_{n+5} - 57430800y_{n+6} + 32771563y_{n+7}),$ $b_7 = -\frac{1}{6350400h^7}(17275860hf_{n+7} - 3263400h^2 f'_{n+7} + 196200y_n$ $- 1768900y_{n+1} + 7228872y_{n+2} - 17805375_{n+3} + 29939000y_{n+4}$ $- 38234700y_{n+5} + 48425400y_{n+6} - 27800497y_{n+7}),$ $b_8 - \frac{1}{25401600h^8}(-462420hf_{n+7} + 88200h^2 f'_{n+7} - 4320y_n$ $+ 40180y_{n+1} - 169344y_{n+2} + 429975_{n+3} - 744800y_{n+4} + 979020y_{n+5}$ $- 1270080y_{n+6} + 739369y_{n+7}),$ $b_9 = -\frac{1}{889056000h^9}(457380hf_{n+7} - 88200h^2 f'_{n+7} + 3600y_n - 34300y_{n+1}$ $+ 148176y_{n+2} - 385875_{n+3} + 686000y_{n+4} - 926100y_{n+5}$ $+ 1234800y_{n+6} - 726301y_{n+7}), y_{n+13/2} = \frac{1}{481689600}(39600y_n - 445900y_{n+1} +$ $2354352y_{n+2} - 7882875y_{n+3} + 19619600y_{n+4} - 44144100y_{n+5} \_176576400y_{n+6}$ $+ 335572523y_{n+7}) + \frac{h}{48168960}(-225855740 f_{n+7} + 12612600hf'_{n+7})$			

4.0 Stability of the methods by plotting the root locus

In this section, we investigate the stability properties of the family of hybrid methods defined in (1.3) for a given value of $k \le 7$. By Gear [7] and Enright [5], a numerical integrator would be stiffly stable provided (a) its region of absolutely stability contains R_1 and R_2 and (b) it is accurate for all $h \in R_2$ when applied to the scalar test equation $y' = \lambda y$, λ a complex constant with $\mathrm{Re}(\lambda) < 0$ where $R_1 = \{z : |\mathrm{Re}(z)| < -D\}$, $R_2 = \{z : -D \le |\mathrm{Re}(z)| \le \delta, \quad -c \le | lm$
$(z)| \le c\}$, and D, δ and c are positive constants.

Applying methods (1.3) in table (1) for a given k to the scalar test problem $y' = \lambda y$, $\mathrm{Re}(\lambda) < 0$ and substituting (1.4) for a corresponding k as the hybrid point from table (2) the stability polynomial in general is found to be

$$\pi(r,z) = r^k - r^{k-1} - z\sum_{j=0}^{k} \beta_j r^j - z\beta_v \left(\sum_{j=0}^{k} \alpha_j r^j + z\beta_{jk} r^k + z^2 \lambda_k r^k \right), \quad z = \lambda h$$

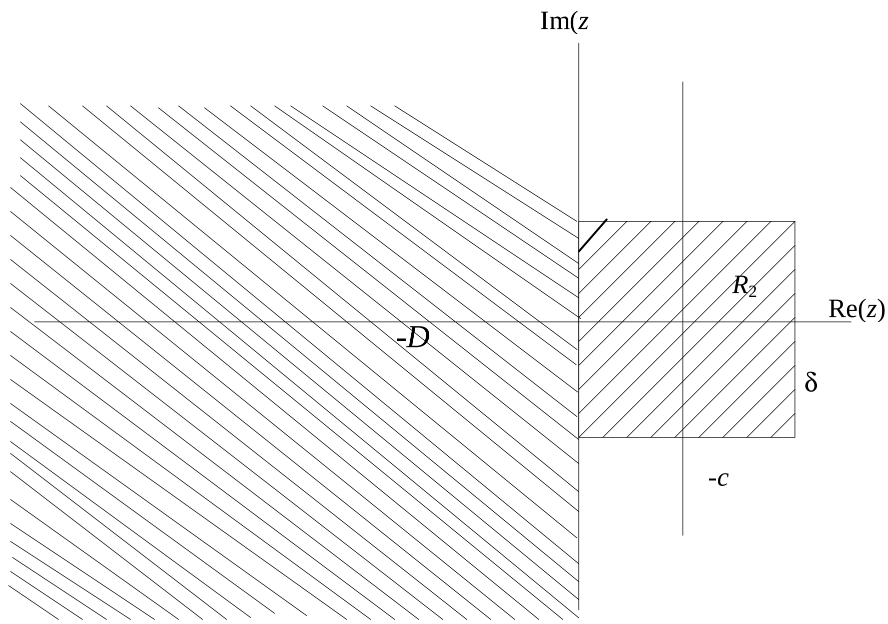

Figure 4.1: Region of a stiffly stable method

for which the root locus are plotted to reveal the stability interval. Methods (1.3) is said to be stable if all roots of the polynomial $\pi(r, z)$ lie in or on the unit circle and with those lying on the unit circle as simple roots. The general form of the stability plot is given below:

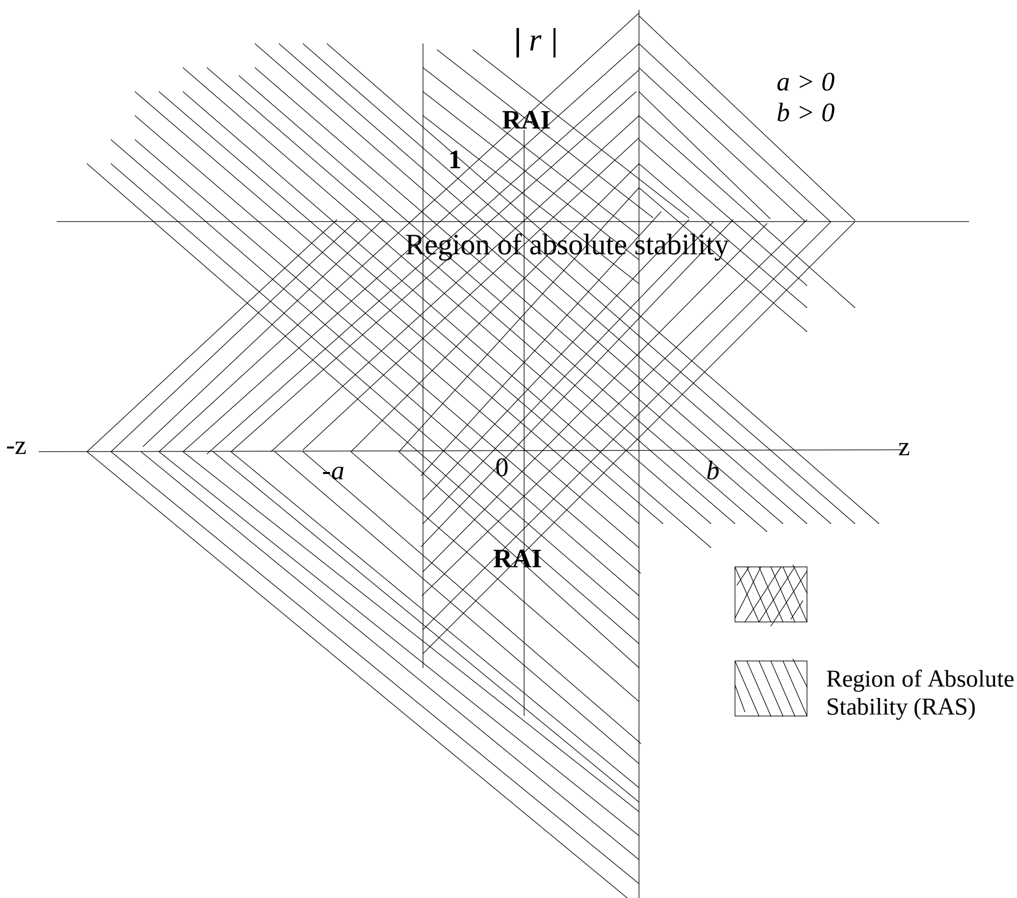

Figure 4.2: Root Locus Plot of the Stability Region of Stiffly stable SDCLMM

Applying the root locus method to $\pi(r, z) = 0$, we observed that the hybrid methods are stiffly stable and the graphs below show the root loci and thus the interval of absolute stability of each method for any given value of $k \leq 7$.

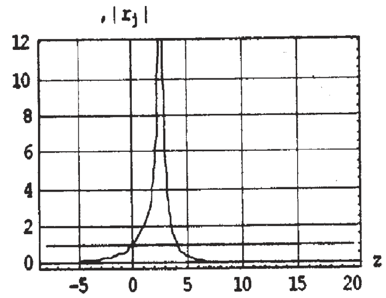

Figure 4.1: Root locus for k = 1

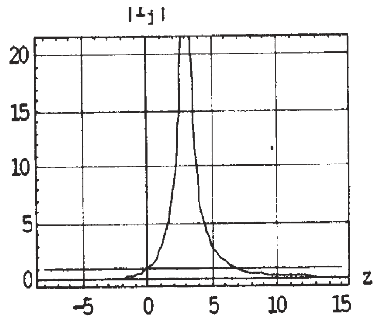

Figure 4.2: Root locus for k = 2

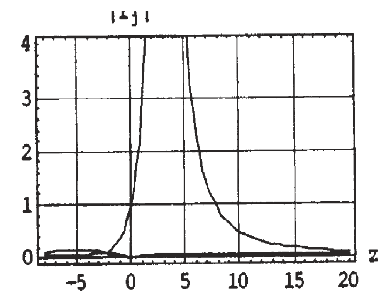

Figure 4.3 Root locus for k = 3

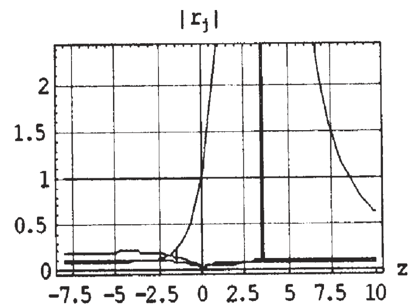

Figure 4.4 Root locus for k = 4

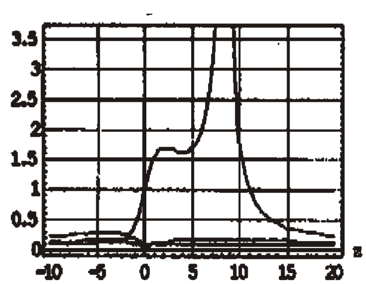

Figure 4.5: Root locus for k =5

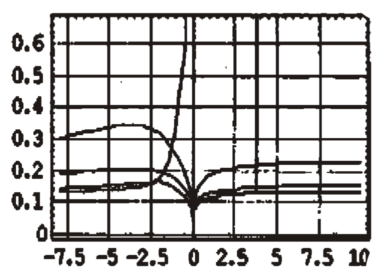

Figure 4.6: Root locus for k = 6

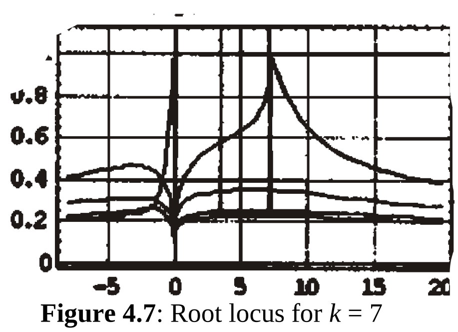

Figure 4.7: Root locus for k = 7

Table 4.1: SDCLMM

k	*SDCLMM*	*Interval of Absolute Stability*
1	SDCLMM 1	$(-\infty,0) \cup (4,\infty)$
2	SDCLMM 2	$(-\infty,0) \cup (6.76,\infty)$
3	SDCLMM 3	$(-\infty,0) \cup (7.74,\infty)$
4	SDCLMM 4	$(-\infty,0) \cup (8.5,\infty)$
5	SDCLMM 5	$(-\infty,0) \cup (11,\infty)$
6	SDCLMM 6	$(-\infty,0) \cup (11,\infty)$
7	SDCLMM 7	$(-\infty,0) \cup (7.2,\infty)$

5.0 Numerical Experiment

The aim now is to demonstrate the application of the methods in sections (2) and (3)

$$y_{n+1} = y_n + \frac{h}{6}(f_n + 4f_{n+1/2} + f_{n+1}) \tag{5.1}$$

$$y_{n+1/2} = +\frac{1}{16}(2y_n + 14y_{n+1}) + \frac{h}{16}(-6f_{n+1} + hf'_{n+1})$$

for k = 1, to solve a linear problem in Enright [5]. The problem is given by:

$$y' = \begin{pmatrix} -0.1 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & -100 & 0 \\ 0 & 0 & 0 & -1000 \end{pmatrix} y,\ y(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

with x in the range [0,1], the exact solution is, $y(x) = (e^{-0.1x}, e^{-10x}, e^{-100x}, e^{-1000x})$ and h = 0.0001. In solving the IVP above the implicitness in the methods has been resolved using the Newton Raphson method, as suggested by Enright [5], Fatunla [6] and Lambert [11], while the inverse Euler's method

$$y_{n+1}^{(0)} = y_n + \frac{hy_n y'_n}{y_n - hy'_n} \tag{5.2}$$

in Fatunla [6], was used to generate the starting values for the iterative schemes. The result of SDCLMM on the problem is found to be very close to the corresponding exact solutions. The numerical result of SDCLMM is compared with that generated by Enright [5], see fig (5.1) and fig. (5.2). The errors in the methods (5.1) for k = 1 are graphically shown in Fig. (5.1) and Fig. (5.2) respectively.

6.0 Conclusion

In this paper, we have considered a class of SDCLMM for solving stiff initial value problems in ordinary differential equations. Our aims is to derive a stiffly stable method with small error constant of high order which also possess stability at the origin and with comparable accuracy of solution to that of Enright [5]. The schemes compare favourably well with Enright [5] as seen from the error graphs in figures (5.1) and (5.2).

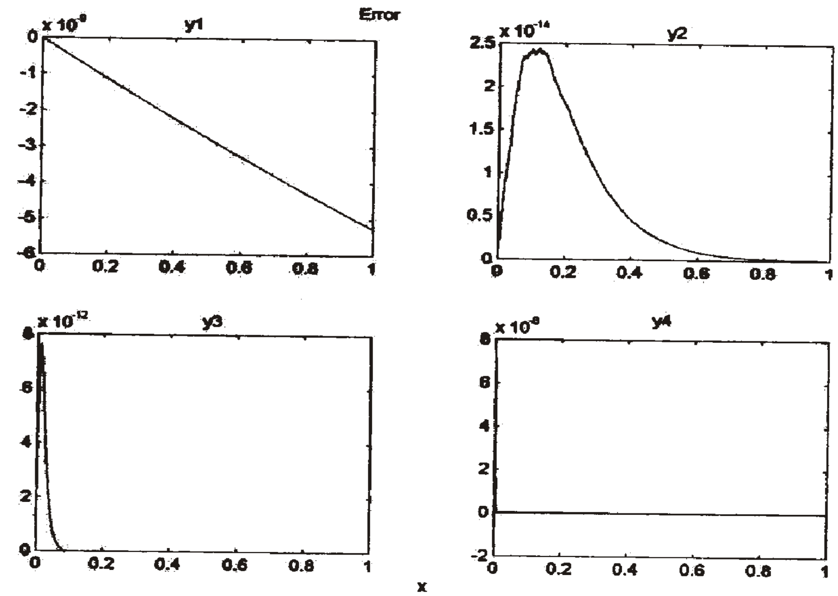

Figure 5.1: SDCLMM

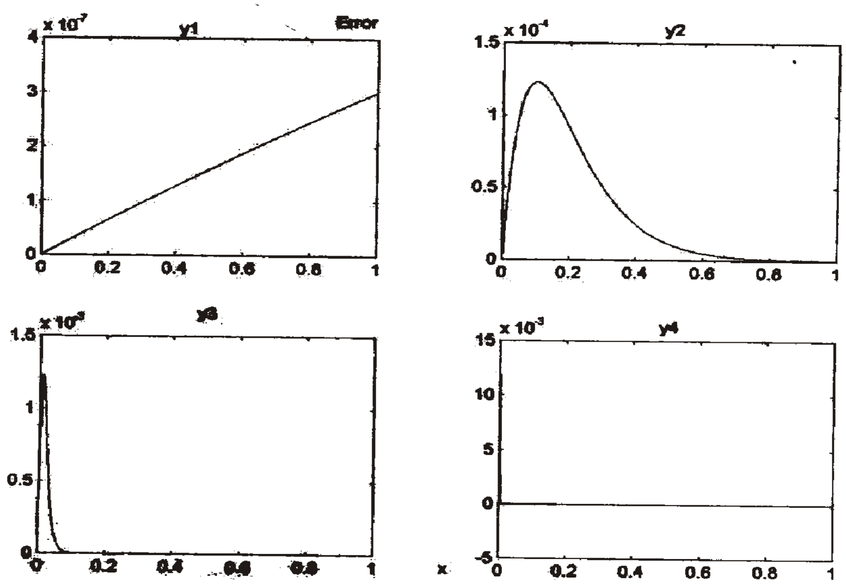

Figure 5.2: Enright method

References

[1] Arevalo, C. Fuherer, C. and Seva, M., (2002), A Collocation Formulation of Multistep methods for Variable Step-size Extensions. Appl. Numer. Math, Vol. 42, pp, 5-16.

[2] Burrage, K. and Tian., T (2001), Stiffly Accurate Runge Kutta Methods for Stiff Stochastic Differential Equations, COmput, Phys. Commun., Vol. 142, pp. 186-190.

[3] Butcher, J.C. and Chen, D. J. L (2001) On the Implementation of ESIRK methods for Stiff IVPs, Numer. Math. Vol. 26, pp, 477-489.

[4] Butcher, J. C. (2002). The A-stability of Methods with Pade and Generalised Stability Functions. Numer. Algorithms; Vol. 31, pp 47-58.

[5] Enright, W. H. (1974), Second Derivative Multistep Methods for Stiff Ordinary Differential Equations. SIAM. J. Numerical Analysis. Vol. 1., No. 2.

[6] Fatunla, S. O. (1988), Numerical Methods for Initial Value Problems in Ordinary DIffereiantial Equation, Academics Press, Inc.

[7] Gear, C. W. (1968), The Auomatic Integration of Stiff Ordinary Differential Equations. Proc. IFIP Congress, North-Holland, Amsterdam, Vol. 1, pp, 187-194.

[8] Hairer, E. and Lubeich, C. H. (2004), Symmetric Multistep Methods Over Long Times. Numer . Math. Vol. 32, pp. 373-379.

[9] Higham, J.D. and Higham J. N. (2000), Matlab Guide. SIAM Philadelphia.

[10] Ikhile, M. N. O. (2004), Coefficients for Studying one-step Rational Schemes for IVPs in ODEs: III. Extrapolation methods. Comp and Maths. With Applic. Vol 47, pp 1463-1475.

[11] Lambert, J.D. (1991), Computational Methods in Ordinary Differential Equations; John Wiley and Sons, New York.

[12] Onumanyi, P. Sirisena, U.W and Awoyemi, D. O. (1996), A new Family of Predictor-Corrector Methods. Spectrum Journal. Vol. 3, No. 1 & 2.

[13] Otunta, F. O. Ikhile, M.N.O. and Okuonghae, R. I. (2007), Second Derivative Continuous Linear Multi Step Methods for the Numerical Integration of Stiff Ordinary Differential Equations. Nigeria Journal of Mathematical Physics (Submitted).

[14] Ontunta, F.O. Ikhile, M.N.O. and Okuonghae, R. I. (2007), Stiffly Stable Second Derivative Continuous Differential Equations. Nigeria Journal of Applied Sciences, University of Benin (Submitted).

[15] Sirisena, U. W., Onumanyi, P. and Chollon, J.P. (2001), Conitnuous Hybrid Methods Through Multistep Collocation. ABACUS, Vol. 28; No. 2; pp. 58-66.