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Computational model for speed of efflux in liquids

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Abstract

We have looked at the efflux of a viscous liquid from an orifice. Assuming the steady flow of a Newtonian fluid, a model for the energy loss due to viscous shearing stress is derived, and a first-order non-linear ordinary differential equation of second degree is obtained for the speed of efflux. Numerically, the equation is quasi-stiff, due to the small value of kinematic viscosity of common liquids. We resolve the equation numerically using a modified Rosenbrock formula for the speed of efflux at different depths of the orifice, below the free surface of the liquid. Generally, the results show that the speed of efflux for a liquid with a large kinematic viscosity is lower than that for a liquid with a small kinematic viscosity at any particular depth. At a low hydrostatic pressure, the speed of efflux of a viscous liquid is less than that of an inviscid fluid. Thus there is a significant energy loss if the kinematic viscosity of a liquid is high. Also, the results suggest that liquids with a large kinematic viscosity are more likely to support steady flow if subject to a high pressure gradient.

Keywords: speed of efflux; viscous loss; non-linear ordinary differential equation.

1.0 Introduction

This work is part of an effort to introduce students to methods of computational physics using illustrations that are conceptually simple, but mathematically non-trivial. Easy access to modern computing resources enable us examine, within a typical undergraduate curriculum, problems which hitherto we ignore because the resulting equations are often not amenable to simple analytic treatment. Here, we consider the efflux of a viscous liquid from an orifice. If a small hole is made on the wall of a large open reservoir, which is kept full with a liquid, the liquid issues from the orifice as a jet, under the influence of gravity. Imagine that the speed of efflux of this jet is measured, by say measuring the volume discharge in a given time interval, for various liquids; it is most likely that the values of speed of efflux obtained will not be the same for liquids with different viscosities. We seek a relationship linking the speed of efflux with the viscosity of the liquid. Such an expression may be obtained by considering the conservation of energy for a fluid particle moving with the fluid.

Usually, to obtain the conservation equation for the energy of a fluid particle we form the inner (i.e. scalar) product of the velocity (***v***) with the equation of motion (a form of the conservation of momentum equation consistent with the particular fluid of interest). This leads to what is essentially an energy balance equation. For the steady, incompressible flow of an inviscid fluid, the integral of the inner product along a streamline in a time independent gravity field leads to the Bernoulli equation [1]

$$\frac{1}{2}v^2 + \frac{p}{\rho} + gz = C, \qquad (1.1)$$

where p is pressure, ρ is density, g is gravitational acceleration and z is the position of the fluid particle above a fixed horizontal reference. In writing (1.1) it is assumed that the z- direction is anti-parallel to the gravity field. In general, the constant C varies from one streamline to another, but is the same for all points on a streamline.

For a moving fluid particle, the terms on the left-hand-side of the energy balance equation (1.1) represent, respectively, the kinetic energy, pressure energy, and potential energy (per unit mass) and their sum is conserved along a streamline in the steady incompressible flow of an inviscid fluid. Thus, in Bernoulli equation, the terms which involve energy loss or 'heat' are specifically excluded.

2.0 Effect of viscosity

The energy balance equation in the steady incompressible flow of a viscous liquid will involve both mechanical and heat energy components, because of the fact that mechanical energy can be changed into heat. Thus associated with the flow is the irreversible energy dissipation arising from the change of mechanical energy into heat by the action of friction. And so, for different points along a streamline in a viscous liquid, the sum of the kinetic energy, pressure energy and potential energy (per unit mass) is conserved only if the amount of heat energy arising from viscous friction is included. Consequently, the energy balance equation in a viscous liquid requires an expression for the mechanical energy converted into heat by viscous friction.

For any two points designated by the symbols *1* and *2* along a streamline, the energy balance equation implies that

$$\frac{1}{2}v_1^2 + \frac{p_1}{\rho} + gz_1 - C_1 = \frac{1}{2}v_2^2 + \frac{p_2}{\rho} + gz_2 - C_2. \qquad (2.1)$$

For these points, the law of conservation of mass leads to the equation of continuity which when expressed as a balance equation for the volume discharge is

$$A_1 v_1 = A_2 v_2, \qquad (2.2)$$

where A_1 and A_2 are cross-sectional areas at the respective points. Let *1* designate a point on the free surface of the liquid, *2* designate a point on the plane of the orifice, and the origin of coordinates coincide with the free surface. Then the pressure $p_1 = p_2$ is atmospheric pressure, $z_1 = 0$, $z_2 = -z$, v_1 is the speed of the free surface, and $v_2 = v_x$ is the speed of efflux of the liquid from the orifice. Thus equation (2.1) reduces to

$$\frac{1}{2}\left[\frac{A_1^2 - A_2^2}{A_1^2}\right]v_x^2 = gz - (C_1 - C_2). \qquad (2.3)$$

The term ($C_1 - C_2$) corresponds to the amount of mechanical energy which is converted into heat per unit mass. In an inviscid fluid this term is zero and (2.3) reduces to the Torricelli theorem:

$$v_X = \sqrt{2gz\frac{A_1^2}{A_1^2 - A_2^2}}. \qquad (2.4)$$

3.0 Model of viscous loss

The energy balance equation in a viscous liquid requires an expression for the term (C_1 – C_2), representing the amount of mechanical energy converted irreversibly into heat by viscous friction. To determine the energy loss due to viscosity per unit mass, it is necessary to make an assumption concerning the nature of the liquid. Assume that the liquid is a Newtonian fluid. In the simple shearing flow of an incompressible Newtonian fluid whose velocity field is given by [2]

$$v_x = v_x(z), \quad v_y = 0, \quad v_z = 0, \qquad (3.1)$$

the stress tensor reduces to

$$\overline{\overline{T}} = \begin{bmatrix} -p & 0 & \eta \frac{\partial v_x}{\partial z} \\ 0 & -p & 0 \\ \eta \frac{\partial v_x}{\partial z} & 0 & -p \end{bmatrix}, \tag{3.2}$$

where η is the shear viscosity of the fluid. Given the stress tensor $\overline{\overline{T}}$, the total force due to stress across any surface S is, the vector sum of the forces on its elements [3]:

$$\boldsymbol{F} = \int \overline{\overline{T}}.\boldsymbol{dS} . \tag{3.3}$$

For a moving fluid particle, the rate at which work is done by the stresses on the medium is

$$\boldsymbol{F.v} = \int \left(\overline{\overline{T}}.\boldsymbol{v}\right).\boldsymbol{dS} . \tag{3.4}$$

If *S* is a closed surface surrounding the volume *V*, we define the outward unit normal **n** to the surface such that

$$\boldsymbol{F.v} = -\int \left(\overline{\overline{T}}.\boldsymbol{v}\right).\boldsymbol{n}dS . \tag{3.5}$$

Using the Gauss divergence theorem, we then have $$\boldsymbol{F.v} = -\int \nabla.\left(\overline{\overline{T}}.\boldsymbol{v}\right)dV . \tag{3.6}$$

Thus the rate at which work is done by stresses on the medium per unit volume is

$$\frac{dW}{dt} = -\nabla.\left(\overline{\overline{T}}.\boldsymbol{v}\right) . \tag{3.7}$$

Now, consider the generalized (Euler) equation of motion $$\rho \frac{d\boldsymbol{v}}{dt} + \nabla.\overline{\overline{T}} = \boldsymbol{f} , \tag{3.8}$$

where ***f*** is the body force per unit volume. For a moving fluid particle of mass δm = ρδV, the inner product of ***v*** and equation (3.8) leads to $$\frac{d}{dt}\left(\frac{1}{2}\rho v^2 \delta V\right) = \boldsymbol{v}.\left(\boldsymbol{f} - \nabla.\overline{\overline{T}}\right)\delta V . \tag{3.9}$$

Thus, ignoring the body force, the rate of production of kinetic energy per unit volume due to stresses is

$$Q = -\boldsymbol{v}.\left(\nabla.\overline{\overline{T}}\right) . \tag{3.10}$$

We know that the work done on a system is equal to the sum of the change in kinetic energy of the system and the energy loss due to friction. Thus we can write

$$\nabla.\left(\overline{\overline{T}}.\boldsymbol{v}\right) = \boldsymbol{v}.\left(\nabla.\overline{\overline{T}}\right) + \left(\overline{\overline{T}}.\nabla\right).\boldsymbol{v} , \tag{3.11}$$

where the last term in equation (3.11) represents the rate at which energy is dissipated by viscous shearing stress per unit volume in a moving fluid. In three dimensional rectangular coordinates and with the velocity profile and stress tensor given by (3.1) and (3.2), respectively,

we have

$$\left(\overline{\overline{T}}.\nabla\right).\boldsymbol{v} = \eta\left(\frac{dv_x}{dz}\right)^2 . \tag{3.12}$$

Thus, for an incompressible Newtonian fluid with the velocity profile (3.1), the amount of mechanical energy converted irreversibly into heat by viscous friction per unit mass is

$$\left(C_1 - C_2\right) = \frac{\eta}{\rho}\left(\frac{dv_x}{dz}\right)^2 . \tag{3.13}$$

Upon substituting equation (3.13) in (2.3), the speed of efflux of our incompressible fluid satisfies the ordinary differential equation

$$\frac{\eta}{\rho}\left(\frac{dv_x}{dz}\right)^2 + \frac{1}{2}\left(\frac{A_1^2 - A_2^2}{A_1^2}\right)v_x^2 - gz = 0, \quad v_x(0) = 0 \ . \tag{3.14}$$

The ratio η/ρ is the kinematic viscosity of the fluid. Note that according to the derivation, z in equation (3.14) is simply the depth of the orifice below the free surface of the liquid and thus a positive quantity.

4.0 Numerical procedure

Mathematically, our task is to solve an initial-value non-linear first-order ordinary differential equation numerically. For physical reasons, the solution to the problem is expected to remain bounded. When a numerical solution is (or tends to be) unbounded, it is an indication of the non-suitability of the numerical method that is used. Also, the value of the speed of efflux given by Torricelli theorem should be greater than the value obtained from (3.14) at any given depth, since viscosity is ignored in Torricelli theorem.

According to Press et al [4] "for many scientific users, fourth-order Runge-Kutta is not just the first word in ODE integrators, but the last word as well". Thus, we tried a fourth-order Runge-Kutta method on this problem. Incidentally, the outcome is not satisfactory. In general, Runge-Kutta methods assume that the numerical solution to an ordinary differential equation is represented using

$$V_{i+1} = V_i + \varphi(z_i, V_i, \Delta z)\Delta z\,, \tag{4.1}$$

where V_i is the present function value, V_{i+1} is the future function value, Δz is a small increment in the independent variable and $\varphi(z_i, V_i, \Delta z)$ is an increment function. Various Runge-Kutta methods determine the increment function using different schemes, but geometrically the increment function is a generalized representation of the slope.

We can re-write (3.14) to obtain a kind of slope function, namely,

$$\frac{dv_x}{dz} = \sqrt{\frac{\rho}{\eta}\left[gz - \frac{1}{2}\frac{A_1^2 - A_2^2}{A_1^2}v_x^2\right]}\ . \tag{4.2}$$

Common liquids are such that the kinematic viscosity is numerically a small quantity, thus the slope calculated using (4.2) is numerically large since it is proportional to the inverse of the kinematic viscosity. This gives the differential equation a kind of stiff attribute. Thus the numerical form of the slope function in this situation seems not to favour the use of the classical fourth-order Runge-Kutta method, since the increment function is evaluated using the slope function.

The numerical solution is implemented in MATLAB. Of all the differential equation solvers in the MATLAB ODE suite, the ode23s is more efficient for our problem. ode23s is an ordinary differential equation solver for stiff problems based on a modified Rosenbrock formula of second-order. Basically, a Rosenbrock formula is an implicit Runge-Kutta scheme [5]; it is numerically stable when applied to stiff ordinary differential equations.

5.0 Numerical experiments and results

We model the efflux of a viscous liquid and calculate the speed of efflux of the liquid from an orifice at various depths. In one instance we compare the speed of efflux for some common liquids. Next, we consider the speed of efflux for water at different temperatures. Though temperature does not appear explicitly in the model, it is incorporated indirectly through the kinematic viscosity, which varies with temperature. Recall that Torricelli theorem predicts the same speed of efflux (varying only with depth) in all these situations.

The values of liquid density and viscosity taken from Kaye and Laby [6] are given in table 1, and the acceleration due to gravity (*g*) is assumed to be 9.80665 ms^{-2}. In some instances the values are an interpolation. All values are in S.I. units. The ratio of the area of the orifice to the area of the free surface, A_2/A_1, is 0.01. For each liquid the computation was attempted for values of z beginning at the liquid surface down to a depth of 10.0 m, though it is only for castor oil that the computation went through to z = 10.0 m.

In Fig. 1 we present the results when the model is applied to castor oil, observe the speed deviation from the value predicted by Torricelli theorem for an inviscid fluid. Intuitively, we expect the speed of efflux of a viscous liquid to be less than that of an inviscid fluid. Generally, the results agree with this during the early part of the computation. However, for all samples, beyond a certain depth the speed of efflux of the liquid becomes greater than that of an inviscid fluid. The depth at which this occurs varies from one liquid to another, but is a maximum in castor oil and a minimum in mercury (Fig. 2). Beyond this depth the speed of efflux obtained from the model increases monotonically, until the ode-solver is unable to continue with the computation. Figure 3 is the result which compares the speed of efflux of water at temperatures between 10 ºC and 60 ºC. Observe that at any given depth the speed of efflux is less at lower temperatures. Essentially, there is an indication of the variation of speed of efflux with temperature, occasioned by the change in values of kinematic viscosity.

6.0 Discussion and conclusion

The model has been tested on liquids which may not be classified as highly viscous. In this domain of low to moderate viscosity, the model predicts a small but noticeable influence of (kinematic) viscosity on the speed of efflux. This is especially evident on comparing the results of the model when applied to castor oil and olive oil. These are liquids with the same density but different viscosities. Observe that, in general, the speed of efflux is lower in those liquids with a high kinematic viscosity. Thus in the flow of a viscous liquid the energy loss is significant if the kinematic viscosity is high. Also, beyond a certain depth, z_c, of the orifice below the free surface of the liquid, the speed of efflux obtained from the model exceeds that for an inviscid fluid. If this depth is rendered as a hydrostatic pressure, $\rho g z_c$, then it represents the pressure difference causing the efflux at that depth. According to the results this hydrostatic pressure is least in mercury, which has the smallest kinematic viscosity. This suggests that there is a critical pressure gradient for each liquid beyond which the model may not be applicable. The equation on which our derivation is based assumes steady flow, and a critical pressure gradient beyond which the model 'fails' may imply a change in the flow regime, such that the flow regime is different for pressure gradient values larger than this critical value than for lower values. It seems that for a small pressure gradient the flow is steady, while at a much larger pressure gradient there is a deviation from steady flow. Thus we anticipate an early deviation from steady flow as the pressure gradient is increased in liquids having a low kinematic viscosity than in those with a high kinematic viscosity. The manifestation of a critical pressure gradient, here, may be an indication of a change to unsteady or asymmetric flow, when the liquid is subject to a high pressure gradient.

Essentially, we may conclude from these calculations that liquids with a high kinematic viscosity are much more likely to support steady flow when subject to a high pressure gradient. Also, because low kinematic viscosity liquids tend to lose this ability at a relatively small pressure gradient, their motion may not be adequately represented by an equation of motion which assumes steady flow.

Liquid	**Density (Kg M^{-3})**	**Viscosity(10^{-3} N S M^{-2})**
Acetone	800.0	0.295
Castor oil	900.0	451.0
Mercury	13521.4	1.499
Olive oil	900.0	52.0
Water	995.21	0.7982

Table 1(a): Density and viscosity of various liquids at 30 ºC.

Temperature (ºC)	**Density (Kg M^{-3})**	**Viscosity (10^{-3} N S M^{-2})**
10	999.02	1.3037
20	998.20	1.0019
30	995.21	0.7982
40	992.22	0.6540
50	987.71	0.5477
60	983.20	0.4674

Table 1(b): Density and viscosity of water at various temperatures.

Figure 1: Speed of efflux for castor oil and the inviscid fluid, (a) at small depths and (b) at large depths.

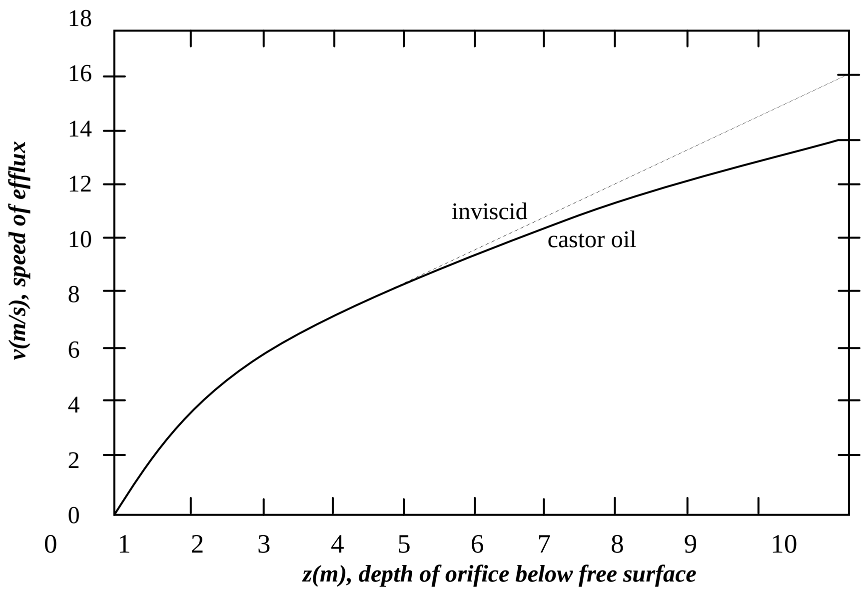


Figure 1(a)

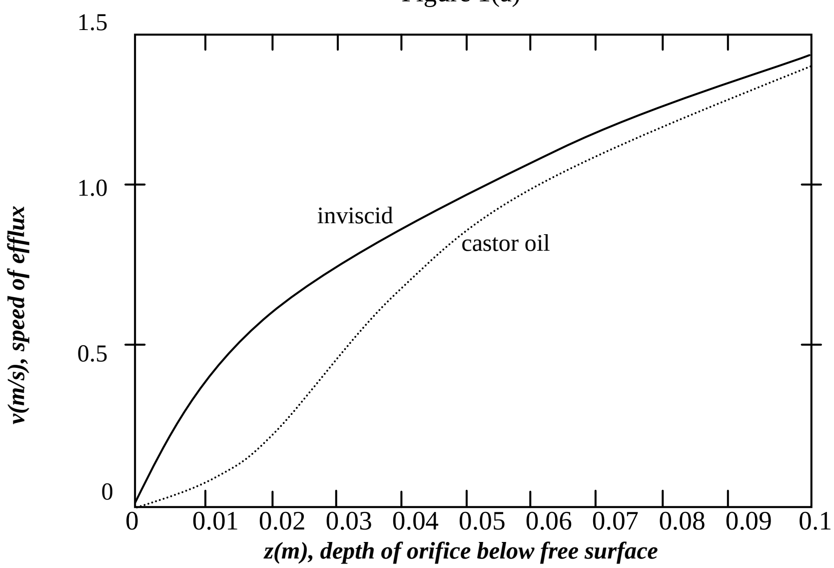
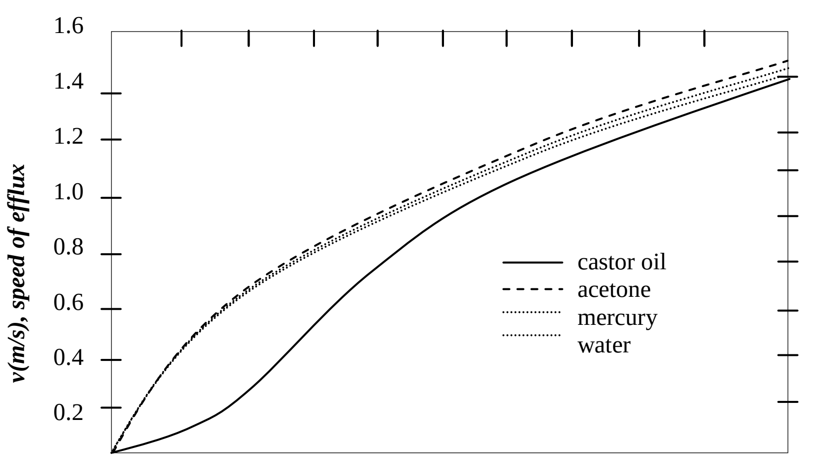


Figure 1(b)



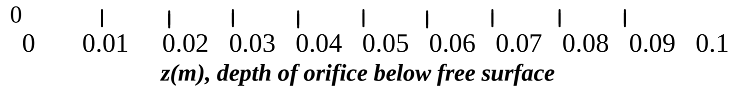


Figure 2(a): Speed of efflux for various liquids at small depths.

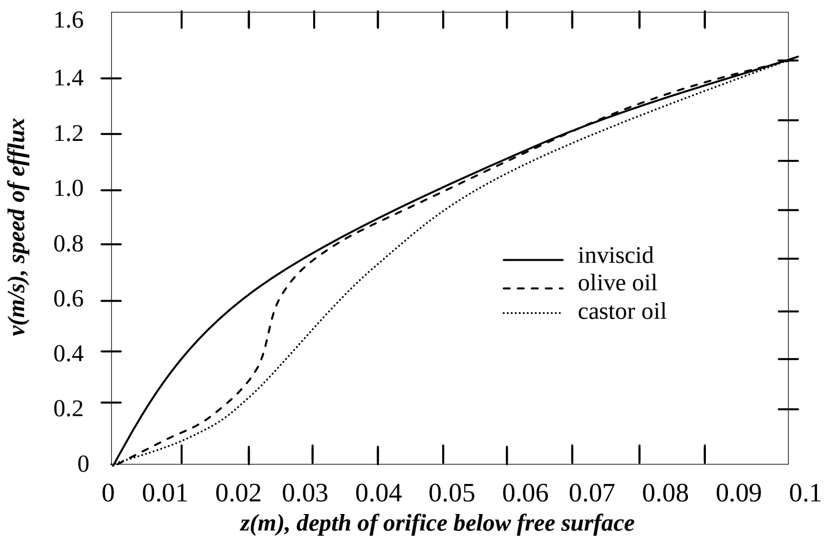


Figure 2(b): Speed of efflux for liquids with same density.

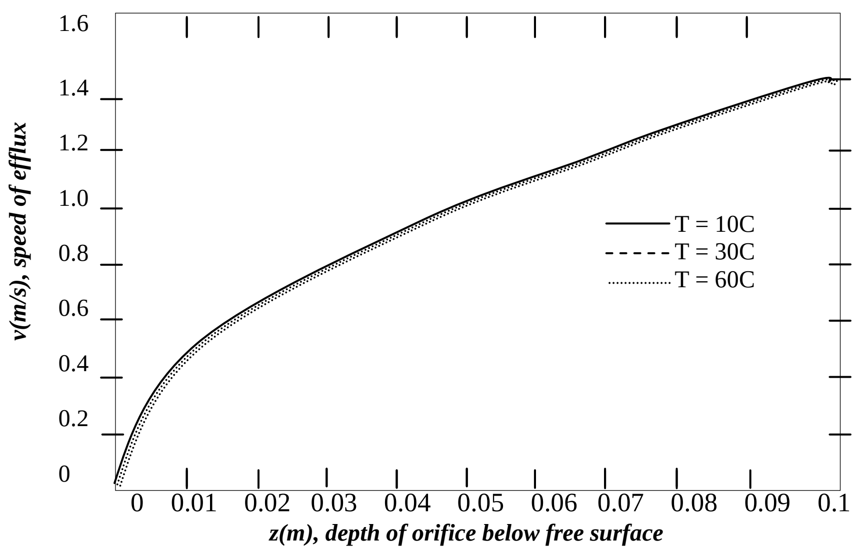


Figure 3: Speed of efflux for water at various temperatures

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