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On the existence of continuous selections of solution and reachable sets of quantum stochastic differential inclusions

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Abstract

We prove that the map that associates to the initial value the set of solutions to the Lipschitzian Quantum Stochastic Differential Inclusion (QSDI) admits a selection continuous from the locally convex space of stochastic processes to the adapted and weakly absolutely continuous space of solutions. As a corollary, we show that the reachable sets admit some continuous selections. In the framework of the Hudson - Parthasarathy formulations of quantum stochastic calculus, our results are achieved subject to some compactness conditions on the set of initial values and on some coefficients of the inclusion. The results here compliment similar results in our previous work in* [3] *where continuous selections defined on the set of the matrix elements of initial values were established.

Keywords: Continuous selections, Lipschitzian quantum stochastic differential inclusions, Reachable sets, solution sets

AMS Subject Classification (2000): 81 S 25, 60 H 10

1.0 Introduction

The investigation of the existence of continuous selections and their applications have been an important preoccupation in the analysis of classical differential inclusions defined in finite dimensional Euclidean spaces. Research in this subject has attracted considerable attention in the literature. Some well known results on continuous selections and their applications in the finite dimensional Euclidean settings can be found in [1,2,13,14,15,17,18,19,20]. As in [17, 19, 20], Selection results have been used among other things for the interpolation of a given finite set of trajectories of classical inclusions. However, in the non commutative quantum setting, investigation of the existence of continuous selections and their applications have not received adequate attention in the literature comparable to the classical situation. The analysis of quantum stochastic differential inclusions (QSDI) concerns quantum stochastic processes as solutions that live in certain infinite dimensional locally convex spaces. In addition, there are several locally convex operator topologies that may be defined on the space of such processes arising from several theories of noncommutative stochastic integration. There are several variants of topological conditions depending on the underlying properties of the locally convex spaces of the integrands that may be required of the domain and codomain of the selection maps. In the framework of the Hudson and Parthasarathy [16,18] formulations of quantum stochastic calculus, we established in our previous work [3], some continuous selections of solution sets of QSDI defined on the set of the matrix elements of initial points with values in the set of matrix elements of solutions. However, on this occasion and in the same framework of the Hudson and Parthasarathy formulation of quantum stochastic calculus and using the QSDI formulation due to [10], we establish the existence of a continuous selection map from a compact subset of initial values contained in the locally convex space of stochastic processes into the locally convex space of adapted weakly absolutely continuous quantum stochastic processes. In addition, we show that the reachable set admits a continuous selection. This compliments our result in [3] where the sets of the matrix elements of both the solution and the reachable sets were shown to admit continuous selections and consequently admit some continuous parameterizations.

In what follows, we shall consider ***A*** - valued quantum stochastic processes indexed by the interval $[t_0, T]$ with the space $\tilde{\boldsymbol{A}}$ endowed with weak topology generated by a family of semi-norms as in [3, 10, 11, 12].

The existence of the continuous selections which we study in this paper concerns solution and the reachable sets of quantum stochastic differential inclusions in integral form given by:

$$X(t) \in a + \int_0^t (E(s, X(s)d \wedge_\pi (s) + F(s, X(s)dA_f(s) + F(s, X(s))dA_f(s) \qquad (1.1)$$
$$+ G(s, X(s))dA_g^+(s) + H(s, X(s))ds), \ t \in [t_0, T],$$

where the coefficients *E, F, G,H* lie in the space $L^2_{loc}([t_0, T] \times \overline{A}))_{mvs}$ and $(t_0, \alpha) \in [t_0, T] \times$ $\times \overline{A}$ is a fixed point.

For any pair of $\eta, \xi \in D \underline{\otimes E}$ such that $\eta = c \otimes e(\alpha), \xi = d \otimes e(\beta), \alpha.\beta \in L^2_\gamma(R+)$, as in our previous works in [3,4, 5, 6], we shall in what follows, employ the equivalent form of (1.1) established in [10] given by the non-classical ordinary differential inclusion:

$$\frac{d}{dt}(\eta, X(t)\xi) \in P(t, X(t)(\eta, \xi)$$
$$X(t_0) = a, \ t \in [t_0, T] \qquad (1.2)$$

The multi-valued map *P* appearing in (1.2) is of the form

$$P(t, x)(\eta, \xi) = (\eta, P_{\alpha\beta}(t, x)\xi)$$

where the map $P_{\alpha\beta} : [t_0, T] \times \overline{A} \to 2^{\overline{A}}$ is given by

$$P_{\alpha\beta}(t, x) = \mu_{\alpha\beta}(t)E(t, x) + \nu_\beta(t)F(t, x) + \sigma_\alpha(t)G(t, x) + H(t, x).$$

The complex valued functions $\mu_{\alpha\beta}, \ \nu_\beta, \ \sigma_\alpha [t_0, T] \to C$ are defined by

$$\mu_{\alpha\beta}(t) = (\alpha(t), \pi(t)\beta(t)_\gamma, \ \nu_\beta(t) = \langle f(t), \ \beta(t) \rangle_\gamma, \ \sigma_\alpha(t) = \langle \alpha(t), \ g(t) \rangle_\gamma, \ t \in [t_0, T]$$

for all $(t, x) \in [0, T] \times \overline{A}$ and the coefficient *E, F, G,H* belong to the space $L^2_{loc}([t_0, T] \times \overline{A}))_{mvs}$ of multivalued stochastic processes with closed values. As explained in [10], the map P cannot in general be written in the form: $P(t, x)(\eta, \xi) = (\overline{P}(t, \prec \eta, x\xi \succ)$ for some complex valued multifunction defined on $[t_0, T] \times C$, for t 2 [t0, T], $(t, x) \in [0, T] \times \overline{A}, \eta, \xi \in D \underline{\otimes E}$.

We refer the reader to the works of Ekhaguere in [10, 11, 12] for the details of several spaces and notations employed in this work.

Under the condition of compactness of the values of the map $(t, x) \to P(t, x)(\eta, \xi)$ for arbitrary $\eta, \xi \in D \underline{\otimes E},$, we prove that the map which associates to the initial point $a \in \overline{A}$, the set of solutions $S^{(T)}(a)$ to (1.2) admits continuous selection from the space $\overline{A}$ to the completion $wac(\overline{A})$ of the locally convex space of adapted weakly absolutely continuous stochastic processes indexed by elements of the interval $[t_0, T]$. In particular, we show that the map $(a \to R^{(T)}(a)$ admits a continuous selection, where $R^{(T)}(a)$ is the reachable set at $t = T$ of the *QSDI* (1.1).

The plan for the rest of the paper is as follows: In section 2, we present some fundamental results, notations and assumptions. The main results of the paper are reported in Section 3.

2.0 Preliminary results and assumptions

In what follows, we adopt the notations, formulation and the frameworks as reported in [10, 11, 12]. Detailed definitions of various spaces that appear below can be found in [10]. In what follows, γ is a fixed Hilbert space, D is an inner product space with R as its completion, and $\Gamma(L^2_\gamma(R+)$ is the Boson Fock Space determined by the function space $L^2_\gamma(R+)$. The set E is the subset of the Fock space generated by the exponential vectors. If N is a topological space, then we denote by *clos*(N) (resp. *comp*(N)), the family of all nonempty closed subsets of N (resp. compact members of *clos*(N)).

In our formulations, quantum stochastic processes are $\overline{A}$ - valued maps on $[t_0, T]$. The space $\overline{A}$ is the completion of the linear space

$$\overline{A} = L^+_W(D\underline{\otimes}E,\ R\otimes\Gamma(L^2_\gamma(R+)))$$

endowed with the locally convex operator topology generated by the family of seminorms $\{x \to \|x\|_{\eta\xi} = |\langle\eta, x\xi\rangle|,\ \eta,\xi \in D\underline{\otimes}E\}$. Here A consist of linear operators from $D\underline{\otimes}E$ into $R\otimes\Gamma(L^2_\gamma(R+))$ with the property that the domain of the operator adjoint contains $D\underline{\otimes}E$. We adopt the notation and the definitions of Hausdorff topology on *clos*($\overline{A}$) as explained in [10, 11, 12]. The Hausdorff topology is determined by some family of pseudo-metrics as defined in [10].On the set C of complex numbers, we employ the metric topology on *clos*(C) induced by the Hausdorff metric ρ. Thus for A, B, ∈ *clos*(C), $\rho(A, B)$ is the Hausdorff distance of the sets and for arbitrary pair $\eta,\xi \in D\underline{\otimes}E, N, M \in clos(\overline{A})\rho\eta_\beta(N,M)$ denotes the family of pseudo-metrics as defined in [10, 11, 12].

A quantum stochastic process $\Phi : [t_0, T] \to \overline{A}$ will be said to be weakly continuous on $[t_0, T]$ if for each pair $\eta,\xi \in D\underline{\otimes}E$, the map $t \to \Phi_{\eta\xi}(t)$ is continuous. Here $\Phi_{\eta\xi}(t) := \prec \eta, \Phi(t)\xi \succ$. We shall denote by $C[I,\ \overline{A}]$ the set of all weakly continuous quantum stochastic processes on $[t_0, T]$ and for each $\Phi \in C[I, \overline{A}\}$, we set

$$\|\Phi_{\eta\xi}\|_C := \sup_I |\Phi_{\eta\xi}(t)| = \sup_I \|\Phi(t)\|_{\eta\xi} \tag{2.1}$$

If we denote by $Ad(\overline{A})_{wc}$ the set of all adapted weakly continuous stochastic processes, then we have the following set inclusion $Ad(\overline{A})_{wac} \subseteq Ad(\overline{A})_{wc} \subseteq C[I, \overline{A}]$, since all weakly absolutely continuous stochastic processes are weakly continuous.

As in [10], we denote by $wac(\overline{A})$, the completion of $Ad(\overline{A})_{wac}$ in the topology generated by the family of seminorms

$$|\Phi|_{\eta\xi} = \|\Phi(t_0)\|_{\eta\xi} + \int_{t_0}^{T} \left|\frac{d}{ds} \prec \eta, \Phi(s)\xi \succ\right| ds \tag{2.2}$$

for each $\Phi \in Ad[\overline{A})_{wac}$ and arbitrary $\eta,\xi \in D\underline{\otimes}E$.

In our subsequent presentation, we assume that the multivalued sesquilinear multivalued map (t, x)→ $P(t,x)(\eta,\xi)$ appearing in Equation (1.2) satisfies the following conditions:

$$S(a)P : \Omega \subseteq [t_0 T) \times \overline{A} \to 2^{Sesq(D\underline{\otimes}E}$$

defined on an open subset $\Omega \subseteq [t_0, T]\times \bar{A}$ bounded on Ω by constants $M_{\eta\xi}$ that depend on η, ξ, i.e

$$|P(t,x)(\eta,\xi)| \leq M_{\eta\xi},\ (t,x) \in \Omega, \eta, \xi \in D\underline{\otimes}E .$$

$S(b)$ The map $t \to P(t,x)(\eta,\xi)$ is measurable for fixed $x \in \bar{A}$ and for all $\eta,\xi \in D\underline{\otimes}E$. $S(c)$ the map $(t,x) \to P(t,x)(\eta,\xi)$ is Lipschitzian with Lipschitz function $K_{\eta\xi}(t)$ lying in $L^1_{loc}([t_0, T])$, *i. e.* for $x, y \in \bar{A}$

$$\rho|P(t,x)(\eta,\xi)|,\ P(t,y)(\eta,\xi)) \leq K_{\eta\xi}(t)\|x-y\|_{\eta\xi} .$$

$S(d)$ he set $P(t,x)(\eta,\xi) \subseteq C$ are compact in $C \forall (t,x) \in \Omega, \eta, \xi \in D\underline{\otimes}E$. $S(e)$ There exists a compact set $A \subseteq \bar{A}$ such that $\forall \eta, \xi \in D\underline{\otimes}E$, the set

$$\{(t, a + v(t - t_0) \colon \in A, V \in \bar{A} \text{ such that } \| v \|_{\eta\xi} \leq M_{\eta\xi}, t \in [t_0, T] \subseteq \Omega$$

Moreover, we set

$$Y_{\eta\xi}(t) = \int_{t_0}^{t} K_{\eta\xi}(s)ds . \qquad (2.3)$$

We shall restrict our consideration to a subinterval $I = [t_0, u] \subset [t_0, T]$ such that

$$\Lambda_{\eta\xi} = 3(e^{Y_{\eta\xi}(u) - Y_{\eta\beta}} - 1) \prec 1; \forall \eta, \xi \in D\underline{\otimes}E . \qquad (2.4)$$

On the whole interval $[t_0, T]$, the result can be obtained by applying Theorem 3.1 below on each subinterval satisfying condition (2.4) and by taking composition of maps. In what follows, we set

$$\Gamma_{\eta\xi} = \int_{t_0}^{u} e^{Y_{\eta\xi}(u) - Y_{\eta\xi}(s)} ds .$$

For any pair of $\eta, \xi \in D\underline{\otimes}E$ such that $\eta = c \otimes e(\alpha),\ \xi = d \otimes e(\beta),\ \alpha, \beta \in L^2_{\gamma}(R+)$, we shall in what follows, employ the equivalent form of (1.2).

3.0 Establishment of the selection map

We shall establish the existence of the continuous selection in this section by employing an adaptation of the techniques employed in Cellina [1] suitable for the present analysis of *QSDI* in the framework of the Hudson-Parthasarathy quantum stochastic calculus. As in [1], our main tools in the construction of the selection are some suitable use of Liapunov's Theorem on the range of vector measures (see [14, 15]) and Ekhaguere's existence results of solutions of *QSDI* (1.1) concerning the generalization of Filippov 's extension of Gronwall's inequalities to QSDI (1.1) as reported in [10]. By a solution of (1.1) we mean a quantum stochastic process $\Phi : [t_0, T] \to \bar{A}$ lying in $Ad(\bar{A})_{wac} \cap L^2_{loc}(\bar{A})$ satisfying *QSDI* (1.1). We denote by $S^{(T)}(a)$, the set of solutions of Lipschitzian *QSDI* (1.1). It has been established in [10] that this set is not empty.

Our main result below shows that there exists a continuous map $\Phi : A \to wac\bar{A}$, such that for each $a \in A$, $\tilde{\Phi}(a) \in S^{(T)}(a) \subseteq wac\bar{A}$.

***Theorem* 3.1**

Suppose that the map $(t,x) \to P(t,x)(\eta,\xi)$ satisfy the assumptions $\boldsymbol{S}(a)$ **-** $\boldsymbol{S}(e)$. Then there exists a continuous map $\tilde{\Phi} : A \to wac(\bar{A})$ such that for every a $a \in A$, $\tilde{\Phi}(a)$ is a solution to the *QSDI* (1.2).

Proof

The proof shall be presented in six parts as follows:

Part A

We claim that there exist two sequences of adapted stochastic processes $\Phi^{n}(a),\ \Psi^{n}:[t_{0},u]$ $\to \tilde{A}$ such that

(*i*) $\Psi^{n}\in S^{(u)}(a);\ \Phi^{n}(a))$ is weakly absolutely continuous and adapted such that $\_\Phi^{n}(a)t_{0}=a$. Setting $\Phi^{n}_{\eta\xi}(a)(t):=\prec\eta,(\Phi^{n})(t)\xi\succ$, then,

(*ii*) $$\left\|\Phi^{n}_{\eta\xi}(a)-\Phi^{n}_{\eta\xi}(a)\right\|_{c}=\sup\left|\prec\eta,(\Phi^{n}(a)(t)\xi\succ-\prec\eta,(\Psi^{n}(a)(t)\xi\succ\right|$$
$$\leq M_{\eta\xi}\Gamma_{\eta\xi}\Lambda^{n-1}_{\eta\xi}.$$

(*iii*) For every Є > 0, there exists $\delta(\varepsilon)=\delta(\varepsilon,n,\eta,\xi)\succ 0$ depending on ε,n,η,ξ and a function $R_{\eta\xi}(a,\varepsilon):I\to IR+)$ satisfying $\int_{I}R^{n}_{\eta\xi}(a,\varepsilon)(s)ds\leq 2M_{n\xi}\varepsilon$ such that $|\frac{d}{dt}\prec\eta,(\Phi^{n}(a)(t)\xi\succ-\frac{d}{dt}\prec\eta,(\Phi^{n}(a')(t)\xi\succ|\leq R^{n}_{\eta\xi}(a,\varepsilon)(t)$, whenever $\|a-a'\|_{\eta\xi}\leq\delta(\varepsilon)$.

(*iv*) $$|\frac{d}{dt}\prec\eta,(\Phi^{n}(a)(t)\xi\succ-\frac{d}{dt}\prec\eta,(\Psi^{n}(a)(t)\xi\succ|$$
$$\leq 3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda^{n-2}_{\eta\xi}K_{\eta\xi}(t)e^{Y_{\eta\xi}(t)},n\geq 2$$

(*v*) $$|\Phi^{n}(a)-\Phi^{n-1}(a)|_{\eta\xi}\leq 3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda^{n-1}_{\eta\xi},n\geq 3.$$ ■

Part B

We apply mathematical induction as follows: Set $\Phi^{1}(a)(t)=a$. Then trivially, $\Phi^{1}(a)$ lies in $Ad(\tilde{A})_{wac}$. Also by the boundedness of *P*,

$(\frac{d}{dt}\prec\eta,(\Phi^{1}(a)(t)\xi\succ,\ P(t,\Phi^{1})(t)(\eta,\xi)=d(0,P(t,a)(\eta,\xi)\leq M_{\eta\xi}$. By Ekhaguere (1992) there exists $\Psi^{1}(a)\in S^{(u)}(a)$ such that $\forall t\in[t_{0},u]$,

$$\left\|\Phi^{1}(a)(t)-\Psi^{1}(a)(t)\right\|_{\eta\xi}\leq\int_{t_{0}}^{t}e^{(Y_{\eta\xi}(t)-Y_{n\xi}(s))}M_{\eta\xi}ds\leq M_{\eta\xi}\Gamma_{\eta\xi}\ .$$

The above shows that Φ^{1},Ψ^{1} satisfy items (*i*), (*ii*) in Part A, with *n* = 1. Item (*iii*) also holds by putting $R^{1}_{\eta\xi}(a,\varepsilon)=0$ for *n* = 1. Assume that we have defined $\Phi^{v}(a)$ and $\Psi^{v}(a)$ satisfying items (*i*) – (*iii*), for $v=1,2\cdots,n-1$. We claim that we can define $\Phi^{n}(a)$ and $\Psi^{n}(a)$ satisfying items (*i*) – (*iv*) for $n\geq 2$.

■

Part C

For notational simplification, we will denote Φ^{n-1} by Φ and Ψ^{n-1} by Ψ. The map $\Phi:A\to C[I,\tilde{A}],a\to\Phi(a)$ is uniformly continuous on account of our assumption in *Part A* above. This can be shown as follows:

Let $r > 0$ be a real number satisfying the following $r \le \delta(\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-1})$, where δ is defined in Part A, item (*ii*) above. Then a', a'' lying in the set $B(a, r) = \{ x \in \tilde{A} : \| x - a \|_{\eta\xi} \le r$, $\forall \eta, \xi \in ID \underline{\otimes} IE \}$ implies that $\| a - a' \|_{\eta\xi} \le r \le \delta$ and $\| a - a'' \|_{\eta\xi} \le r \le \delta$. By item (*iii*), Part A,

$$\left| \frac{d}{dt} \prec \eta, (\Phi(a)(t)\xi \succ - \frac{d}{dt} \prec \eta, (\Phi(a'(t)\xi \succ \right| \le R_{\eta\xi}^{n}(a, \varepsilon)(t)$$

and
$$\left| \frac{d}{dt} \prec \eta, (\Phi(a)(t)\xi \succ - \frac{d}{dt} \prec \eta, (\Phi(a''(t)\xi \succ \right| \le R_{\eta\xi}^{n}(a, \varepsilon)(t)$$

so that

$$\left| \frac{d}{dt} \prec \eta, (\Phi(a')(t)\xi \succ - \frac{d}{dt} \prec \eta, (\Phi(a''(t)\xi \succ \right| \le 2R_{\eta\xi}^{n}(a, \varepsilon)(t) \tag{3.2}$$

But by the absolute continuity of the map $t \to (\prec \eta, \Phi(a')(t)\xi \succ - \prec \eta, \Phi(a'')(t)\xi \succ)$, we have

$$| t \prec \eta, (\Phi(a')(t)\xi \succ - \prec \eta, (\Phi(a'')(t)\xi \succ | = \left| \int_I \frac{d}{ds} (\prec \eta, (\Phi(a')(s)\xi \succ - \prec \eta, (\Phi(a''(s))\xi \succ) ds \right| \tag{3.3}$$

Hence from (3.3) and using (3.1)

$$| \prec \eta, \Phi(a')(t)\xi \succ - \prec \eta, \Phi(a'')(t)\xi \succ | \le 2 \int_I R_{\eta\xi}^{n-1}(a, \varepsilon)(s) ds \le 4M_{\eta\xi}\varepsilon .$$

If $r \le \frac{1}{3} M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}$, then $\| a' - a'' \|_{\eta\xi} \le 2r \le \frac{2}{3} M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}$, implies that

$$\| \Phi(a')(t) - \Phi(a'')(t) \|_{\eta\xi} \le \frac{1}{3} M_{\eta\xi}\Gamma_{n\xi}\Lambda_{\eta\xi}^{n-2},$$

where ε is small enough so that $\varepsilon \le \frac{1}{12}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-1} \le \frac{1}{12}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}$. Consequently we have for $a', a'' \in B(a, r)$, $\| \Phi_{\eta\xi}(a') - \Phi_{\eta\xi}(a'')(t) \|_C \le \frac{1}{3} M_{\eta\xi}\Gamma_{n\xi}\Lambda_{\eta\xi}^{n-2}$. Our claim of uniform

continuity of the map $a \to \Phi(a)$ follows. Let $\{B[a_i, r], i = 1, 2, \cdots, m\}$ be a finite open cover of the compact set A, $a_i \in \forall_i$ and $\Pi_i : A \to IR+$, a partition of unity subordinate to the cover. Here $B[a, r] = \{ x \in \tilde{A} : \| x - a \|_{\eta\xi} \prec r, \forall_{\eta}, \xi \in ID \underline{\otimes} IE \}$ and $\sum_{i=1}^{m} \Pi_i(a) = 1$, $\Pi(a) >$ $0, \forall a \in A$. For the existence of partition of unity, see [2].

Next, we define

$$\sigma(j, a) = \sum_{1 \le i \le j} \Pi_i(a) \text{ and } \Psi_i(t) = \Psi(a_i)(t) .$$

Let $\delta \succ 0$ be such that $\frac{u - t_0}{\delta} = m'$, an integer and $\delta \prec \frac{1}{12}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}$. The subintervals $J(j) =$ $[t_0 + (j-1)\delta, t_0 + j\delta$, $j = 1, 2, \cdots, m'$ form a partition of the interval $I = [t_0, u)$.

Corresponding to an arbitrary pair of elements $\eta, \xi \in ID \underline{\otimes} IE$, we consider the family of complex valued maps on $[t_0, u)$ defined by

$$D_{\eta\xi, i, j}(t) = \frac{d}{dt} \prec \eta, \Psi_i(t)\xi \succ I_{J(j)}(t), i = 1, 2, \cdots m; j = 1, 2 \cdots m', \tag{3.4}$$

where $I_{J(j)}$ is the characteristic function on the set $J(j)$. Let $\{A(\alpha)\}$ be a nested family of measurable subsets of the interval $[t_0, u)$, $A(0) = \emptyset$, $A(1) = [t_0, u)$ such that

$$\int_{A(\alpha)} D_{\eta\xi,i,j}(t)dt = \alpha \int_{t_0}^{u} D_{\eta\xi,i,j}(t)dt,\ \mu(A(\alpha)) = \alpha(u - t_0) \tag{3.5}$$

Such a family exists by a Corollary to Liapunov's theorem (see [1, 14]). Since $\Psi_i \in S^{(u)}(a_i)$ then as shown in [10], there exists processes $V_i : I \to \tilde{A}$ lying in $L^1_{loc}(\tilde{A})$ such that $\Psi_i(t) = a_i + \int_{t_0}^{t} V_i(s)ds$ and $\frac{d}{dt} \prec \eta, \Psi_i(t)\xi \succ = \prec \eta, V_i(t)\xi \succ$. It follows from (3.4) that

$$D_{\eta\xi,i,j}(t) = \prec \eta, V_i(t) I_{J(j)}(t)\xi \succ, i = 1,2,\cdots m;\ j = 1,2\cdots m'.$$

Hence by (3.5) and putting $V_{ij}(t) = V_i(t) I_{J(j)}(t),\ i = 1,2,\cdots m;\ j = 1,2\cdots m'$, we have

$$\int_{A(\alpha)} V_{i,j}(t)dt = \alpha \int_{t_0}^{u} V_{i,j}(t)dt, \tag{3.5b}$$

Next we define the stochastic process $\Phi^n(a) : [t_0, u] \to \tilde{A}$ by

$$\Phi^n(a)(t) = a + \sum_i \int_{t_0}^{t} V_i(s) I_{A(\sigma(i,a))\setminus A(\sigma(i-1,a))}(s)ds \tag{3.6}$$

with its matrix element given by

$$\prec \eta, (\Phi^n(a)(t)\xi \succ = \prec \eta, a\xi \succ + \sum_i \int_{t_0}^{t} \prec \eta, (V_i(s))\xi \succ I_{A(\sigma(i,a))\setminus A(\sigma(i-1,a))}(s)ds$$

We remark that the process $\Phi^n(a)$ given by (3.6) lies in $wac(\tilde{A})$ since each $V_i \in L^1_{loc}(\tilde{A})$ and in addition, $\Phi^n(a)$ is an adapted and weakly absolutely continuous process.

To show that $\Phi^n(a)$ satisfies item (*iii*) of Part A, we note that as in the proof of the only Theorem in [1], $\frac{d}{dt} \prec \eta,\ \Phi^n(a)\xi \succ$ and $\frac{d}{dt} \prec \eta,\ \Phi^n(a')\xi \succ$ differ only on the subset and $E' \subset [t_0, u]$ given by

$E' = \bigcup_{i=1}^{m} \{(A(\sigma(i,a)) \setminus A(\sigma(i-1,a)))\Delta A(\sigma(i,a')) \setminus A(\sigma(i-1,a')))\}$ and that

$$E' \subset \bigcup_{i=1}^{m} \{(A(\sigma(i,a))\Delta A(\sigma(i,a')))\} \tag{3.7}$$

As in [1], we fix any $\varepsilon > 0$ and let $\Theta_{\eta\xi} = \Theta_{\eta\xi}(\varepsilon)$ be the common modulus of continuity of the maps $a \to \sigma(i, a)$ in the seminorm $\|\cdot\|_{\eta\xi}$, *i.e*

$$\Theta_{\eta\xi}(\varepsilon) := \sup_{a \in A} \left[\sup_{a' : \|a - a'\|_{\eta\xi} \le \varepsilon} |\sigma(i,a) - \sigma(i,a')| \right] \tag{3.8}$$

Then whenever $\| a - a' \|_{\eta\xi} \prec \Theta_{\eta\xi} \frac{\varepsilon}{2m}$, the superset in (3.7) is contained in the set

$$E''(a,\varepsilon) = \bigcup_{i=1}^{m} \{(A(\sigma(i,a) + \frac{\varepsilon}{2m}) \setminus A(\sigma(i,a) - \frac{\varepsilon}{2m})\} \tag{3.9}$$

and the total measure of E"(a, ε) is bounded by ε or $\int_I I_{E''(a,\varepsilon)} \prec \varepsilon$. Consequently, we have

$$|\frac{d}{dt} \prec \eta, \Phi^n(a)(t)\xi \succ - \frac{d}{dt} \prec \eta, (\Phi^n(a')(t)\xi \succ| \leq 2M_{\eta\xi} I_{E''(a,\varepsilon)}(t) \tag{3.10}$$

so that item (iii) in Part A follows with

$$\delta(\varepsilon) = \Theta_{\eta\xi}(\frac{\varepsilon}{2m}) \text{ and } R^n_{\eta\xi}(a,\varepsilon)(t) = 2M_{\eta\xi} I_{E''(a,t)}(t)$$ ■

Part D

We shall estimate here the pseudo-distance of $\Phi^n(a)$ from the set of solution $S^{(u)}(a)$. To this end, let $t \in [t_0 + r\delta, t_0 + (r+1)\delta)$). At the point $t = t_0 + r\delta$, the integral in (3.6) can be written as

$$\begin{aligned}\sum_i \int_{t_0}^{t_0 + r\delta} V_i(s) I_{A(\sigma(i,a)) \setminus A(\sigma(i-1),a)} ds &= \sum_i \sum_{l \leq r} \int V_i(s) I_{A(\sigma(i,a)) \setminus A(\sigma(i-1,a))} I_{J(l)} ds \\ &= \sum_i \sum_{l \leq r} \int V_i(s) I_{A(\sigma(i,a)) \setminus A(\sigma(i-1,a))}(s) ds \\ &= \sum_i \sum_{l \leq r} \int_{A(\sigma(i,a)) \setminus A(\sigma(i-1,a))} V_{i,l}(s) ds \\ &= \sum_{l \leq r} \sum_i \Pi_i(a) \int V_{i,l}(s) ds, \\ &= \sum_i \sum_{l \leq r} \Pi_i(a) \{\Psi_i(t_0 + l\delta)\Psi_i(t_0 + (l-1)\delta)\} \\ &= \sum_i \Pi_i(a) \{\Psi_i(t_0 + r\delta) - \Psi_i(t_0)\}\end{aligned}$$

This follows from (3.5)b and the definition of $\sigma(\cdot,\cdot)$. Hence, we have

$$\Phi^n(a)(t_0 + r\delta) - a = \sum_i \Pi_i(a)(\Psi_i(t_0 + r\delta) - a_i$$

For any $j \in \{1, 2, \ldots, m\}$ and arbitrary $\eta\xi \in ID \underline{\otimes} IE$, we can write

$$\begin{aligned}\| \Phi^n(a)(t) - \|_{\eta\xi} \leq \| \Phi^n(a)(t_0 + r\delta) - \Psi_j(t_0 + r\delta \|_{\eta\xi} + \| \Phi^n(a)(t_0 + r\delta) \\ - \Phi^n(a)(t) \|_{\eta\xi} + \| \Psi_i(t) - {}_j(t_0 + r\delta \|_{\eta\xi}\end{aligned} \tag{3.11}$$

Since

$$|\frac{d}{dt} \prec \eta, \Phi^n(a)(t)\xi \succ| \leq M^n_{\eta\xi} \text{ and } |\frac{d}{dt} \prec \eta, \Psi_j(t)\xi \succ| \leq M^n_{\eta\xi},$$

by our choice of δ, the sum of the last two terms in (3.11) is bounded by

$$\frac{1}{3} M_{\eta\xi} \Gamma_{n\xi} \Lambda^{n-2}_{\eta\xi}.$$

Hence, from (3.11)

$$\| \Phi^n(a)(t) - \Psi_i(t) \|_{\eta\xi} \leq \| a - \sum_i \Pi_i(a) a_i \|_{\eta\xi} + \| \sum_i \Pi_i(a)(\Psi_i(t_0 + r\delta) - \Psi_j(t_0 + r\delta \|_{\eta\xi}$$

$$+\frac{1}{3}M_{\eta\xi}\Gamma_{n\xi}\Lambda_{\eta\xi}^{n-2} \tag{3.12}$$

By our choice of r in Part C, whenever $\Pi_i(a) > 0$, then $\| a - a_i \| \leq \frac{1}{3}M_{\eta\xi}\Gamma_{n\xi}\Lambda_{\eta\xi}^{n-2}$. This estimate also holds for the first term at the right hand side of (3.12). Furthermore,

$$\begin{aligned}\| \Psi_i(t_0 + r\delta) - \Psi_j(t_0 + r\delta \|_{\eta\xi} \leq \| \Psi_i(t_0 + r\delta) - \Phi(a_i)(t_0 + r\delta) \|_{\eta\xi} + \| \Phi(a_i)(t_0 + r\delta) \\ - \Phi(a_j)(t_0 + r\delta) \|_{\eta\xi} + \| (a_j)(t_0 + r\delta) - \Psi_j(t_0 + r\delta \|_{\eta\xi}\end{aligned} \tag{3.13}$$

When both $\Pi_i(a) > 0$ and $\Pi_j(a) > 0$ and by the choice of r, the second term on the right of (3.13) satisfies

$$\| \Phi^n(a_i)(t_0 + r\delta) - \Phi(a_j)(t_0 + r\delta) \|_{\eta\xi} \leq \frac{1}{3}M_{\eta\xi}\Gamma_{n\xi}\Lambda_{\eta\xi}^{n-2} \tag{3.14}$$

so that by item (ii) in Part A and the recursive assumption, we finally have

$$\| \Phi^n(a)(t) - \Psi_j(t) \|_{\eta\xi} \leq 3M_{\eta\xi}\Gamma_{n\xi}\Lambda_{\eta\xi}^{n-2} \tag{3.15}$$

Equation (3.15) holds for every j such that $\Pi_j > 0$. By the definition of $\Phi^n(a)(t)$ given by (3.6), at any point t except on a set of measure zero in I, $\frac{d}{dt}\prec \eta, \Phi^n(a)(t)\xi \succ = \frac{d}{dt}\prec \eta, \Psi_j(t)\xi \succ$ for some j such that $\Pi_j(a) > 0$. Since $\Psi_j \in S^{(u)}(a_j)$, then $\frac{d}{dt}\prec \eta, \Psi_j(t)\xi \succ \in P(t, \Psi_j(t))(\eta, \xi)$ and therefore we have

$$\begin{aligned} d\left(\frac{d}{dt}\prec \eta, \Phi^n(a)(t)\xi \succ, P(t, \Phi^n(a)(t))(\eta, \xi)\right) &\leq \rho(P(t, \Psi_j(t))(\eta, \xi),\ P(t, \Phi^n(a)(t))(\eta, \xi) \\ &\leq K_{\eta\xi}(t) \| \Psi_j(t) - \Phi^n(a)(t) \|_{\eta\xi} \leq 3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\varepsilon ta\xi}^{n-2}K_{\eta\xi}(t)\end{aligned} \tag{3.16}$$

on account of (3.15) and the fact that the map $(t, x) \to P(t, x)(\eta, \xi)$ is Lipschitzian. Estimate (3.16) is independent of j and therefore hold on $I = [t_0, u]$. Again by the existence result of [10], there exists a stochastic process $\Psi^n(a) \in S^{(u)}(a)$ such that

$$\| \Psi^n(a)(t) - \Phi^n(a)(t) \|_{\eta\xi} \leq 3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\varepsilon ta\xi}^{n-2}(e^{Y_{\eta\xi}(t)} - 1) \leq M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-1} \tag{3.17}$$

and $| \frac{d}{dt}\prec \eta, \Psi^n(a)(t)\xi \succ - \frac{d}{dt}\prec \eta, \Phi^n(a)(t)\xi \succ | \leq 3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}K_{\eta\xi}(t)e^{Y_{\eta\xi}(t)}$ (3.18)

Inequalities (3.17) and (3.18) prove items (ii) and (iv) in Part A for all $n \geq 2$. ■

Part E

It is now left for us to show that if items (i) – (iv) hold up to n – 1, then item (v) holds for n. We use the same notations as before to fix any t and let j be such that $\frac{d}{dt}\prec \eta, \Phi^n(a)(t)\xi$ $\succ = \frac{d}{dt}\prec \eta, \Psi_j(a)(t)\xi \succ$ so that $\Pi_j(a) > 0$. Then we have

$$\begin{aligned} &| \frac{d}{dt}\prec \eta, \Phi^n(a)(t)\xi \succ - \frac{d}{dt}\prec \eta, \Phi^{n-1}(a)(t)\xi \succ | = | \frac{d}{dt}\prec \eta, \Psi_j(t)\xi \succ - \frac{d}{dt}\prec \eta, \Phi(a)(t)\xi \succ | \\ &\leq | \frac{d}{dt}\prec \eta, \Psi_j(t)\xi \succ - \frac{d}{dt}\prec \eta, \Phi((a_j)(t))\xi \succ | + | \frac{d}{dt}\prec \eta, \Phi(a_j)(t)\xi \succ - \frac{d}{dt}\prec \eta, \Phi(a)(t)\xi \succ | \end{aligned} \tag{3.19}$$

By item (iv), the first term in (3.18) is bounded by

$$3M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}K_{\eta\xi}(t)e^{Y_{\eta\xi}(t)}$$

while by the choice of r, and applying item (*iii*), the second term in (3.18) is bounded by the functions

$$R_{\eta\xi}^{n-1}(a,\Gamma_{n\xi}\Lambda_{\eta\xi}^{n-1}:I\to IR+$$

satisfying the conditions of item (*iii*). These bounds do not depend on *j* and so hold on the whole of interval *I*. Since

$$\int_I R_{\eta\xi}^{n-1}(a,\varepsilon)(t)dt\prec 2M_{\eta\xi}\ ,$$

we have

$$|\Phi^n(a)-\Phi^{n-1}(a)|_{\eta\xi}=\int_I|\frac{d}{dt}\prec\eta,(\Phi^n(a)(t)-\Phi^{n-1}(a)(t)\xi\succ|\,dt$$

$$\le 3\int_I M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-2}K_{\eta\xi}e^{Y_{\eta\xi}(t)}\,dt+\int_I R_{\eta\xi}^{n-1}(a,\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-1}(t)dt$$

$$\le M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-1}+2M_{\eta\xi}\Gamma_{\eta\xi}\Lambda_{\eta\xi}^{n-1}$$

proving item (*v*). ■

Part F

By item (*iii*), each map Φ^n:$A\to wac(\tilde{A})$ is uniformly continuous. Since $\Lambda_{\eta\xi}<1$ for arbitrary pair $\eta,\xi\in ID\underline{\otimes}IE$, item (*v*) shows that the sequence $\{\Phi^n(a)\}$ is Cauchy. Since $wac(\tilde{A})$ is complete, the sequence converges to a continuous map Φ: $A\to wac(\tilde{A})$. By construction, $\left\{\frac{d}{dt}\prec\eta,\Phi^n(a)(t)\xi\succ\right\}$ converges in $L^1[I]$ to $\frac{d}{dt}\prec\eta,\Phi(a)(t)\xi\succ$. Hence, a subsequence converges to $\frac{d}{dt}\prec\eta,\tilde{\Phi}(a)(t)\xi\succ$ pointwise almost every where.

By item (*iv*),

$$d\left(\frac{d}{dt}\prec\eta,\Phi^n(a)(t)\xi\succ,P(t,\Phi^n(a)(t))(\eta,\xi)\right)\to 0\ \text{as}\ n\to\infty$$

Since the images $P(t,x)(\eta,\xi)$ are compact and therefore closed and the map $(t,x)\to P(t,x)(\eta,\xi)$ is continuous, then we have:

$$\frac{d}{dt}\prec\eta,\tilde{\Phi}(a)(t)\xi\succ\in P(t,\tilde{\Phi}(a)(t))(\eta,\xi)$$

showing that

$$\tilde{\Phi}(a)\in S^{(u)}(a)\subseteq wac(\tilde{A}).$$

The next result is a direct consequence of Theorem 3.1 concerning the reachable sets of *QSDI* (1.1) defined by:

$$R^{(T)}(a)=\{\Psi(a)(T):\Psi(a)\in S^{(T)}(a)\}\subseteq\tilde{A}$$ ■

C*orollary* 3.2

The multivalued map $R^{(T)}:A\to 2^{\tilde{A}}$ admits a continuous selection where $R^{(T)}(a)$ is the reachable set at time $t=T$ for each $a\in A$.

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References

[1] Arrigo Cellina: On the set of Solutions to Lipschitzian Differential inclusions, Differential and Integral Equations, 1(4), (1988), 495 - 500.

[2] Aubin, J. P; Cellina, A. Differential Inclusions, Springer - Verlag, Berlin, 1984.

[3] Ayoola, E. O. Continuous selections of solution sets of Lipschitzian quantum stochastic differential inclusions, International Journal of Theoretical Physics, 43(10),(2004), 2041 - 2059.

[4] Ayoola, E. O. Error estimates for discretized quantum stochastic differential inclusions, Stochastic Analysis and Applications, 21(6) (2003),1215 - 1230.

[5] Ayoola, E. O. Exponential formula for the reachable sets of quantum stochastic differential inclusions, Stochastic Analysis and Applications 21(3),(2003) 515 - 543.

[6] Ayoola, E. O. Construction of approximate attainability sets for Lipschitzian quantum stochastic di_erential inclusions, Stochastic Analysis and Applications, 19(3), (2001) 461 - 471.

[7] Broucke, M; Arapostathis,A: Continuous interpolation of solutions of Lipsschitz inclusions. J.Math. Analysis and Appl., (258), (2001), 565 -572.

[8] Dontchev, A. L; Farkhi, E. M: Error estimates for discretized differential inclusions.Computing, 41, (1989) 349 - 358.

[9] Dontchev, A; Lempio, F.: Difference methods for differential inclusions: A survey. SIAM Reviews, 34 (2), (1994) 263 - 294.

[10] Ekhaguere, G. O. S. Lipschitzian quantum stochastic differential inclusions, International Journal of Theoretical Physics Vol.31(11),(1992), 2003-2034.

[11] Ekhaguere, G. O. S. Quantum stochastic differential inclusions of hypermaximal monotone type, International Journal of Theoretical Physics, 34(3), (1995), 323-353.

[12] Ekhaguere, G.O.S. Quantum stochastic evolutions, International Journal of Theoretical Physics 35(9) (1996), 1909-1946.

[13] Filippov, A. F. Classical solutions of differential equations with multivalued right hand side, SIAM Journal on Control(5),(1967), 609 -621.

[14] Fryszkowski, A.: Continuous selections for a class of nonconvex multivalued maps, Studia Math. (76), (1983), 163 - 174.

[15] Hermes, H; LaSalle, J.P: Functional Analysis and Time Optimal Control, Academic Press, (1969), New York.

[16] Hudson, R. L; Parthasarathy, K. R. Quantum Ito's formulae and stochastic evolutions, Communications in Mathematical Physics, 93,(1984), 301-324

[17] Michal Kisielewicz: Differential Inclusions and Optimal Control, Mathematics and its Applications (East European Series), Volume 44, Kluwer Academic Publishers, Dordrecht, Boston, London, 1991.

[18] Parthasarathy, K. R. An introduction to quantum stochastic calculus Monographs in Mathematics, Vol. 85, Birkhause Verlag, 1992.

[19] Repovs, D; Semenov, P.V.: Continuous selections of multivalued mappings. Mathematics and its Applications. Kluwer, Dordrecht, 1998.

[20] Smirnov, G. V.: Introduction to the Theory of Differential Inclusions. Graduate Studies in Mathematics, AMS, Providence, RI Volume 41, 2002.