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Quasi-Partial sums of the generalized Bernard integral of certain analytic functions

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Abstract

In this short note we extend a result of Jahangiri and Farahmand* [5] *concerning functions of bounded turning to a more general class of functions

1.0 Introduction

Let C be the complex plane. Denote by A the class of functions:

$$f(z) = z + a_2 z^2 + \cdots \tag{1.1}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$

In [5]Jahangiri and Faramand studied the partial sums of the Libera integral of the class $B(\beta)$, which consist of functions in A satisfying $\operatorname{Re} f'(z) > \beta, 0 \le \beta < 1$. Functions in $B(\beta)$ are called functions of bounded turning. It is known that functions of bounded turning are generally univalent and close-to0convex in the nit disk. In particular they proved that the *mth* partials ums.

$$F_m(z) = z + \sum_{k=2}^{m} \frac{2}{k+1} a_k z^k \tag{1.2}$$

of the Libera intergral

$$F(z) = \int_0^z f(t)dt \tag{1.3}$$

is also of bounded turning. Their result was stated as:

***Theorem* A**.

If $\frac{1}{4} \le \beta < 1$ *and* $f \in B(\beta)$, *then* $F_m \in B\left(\frac{4\beta - 1}{3}\right)$

Earlier and Owa [6] have proved that if $f \in A$ is univalent in E, then the partial sum $F_m(z)$ is starlike in the subdisk $|z| < \frac{3}{8}$, the number $\frac{3}{8}$ being the best possible.

The result of Jahangiri and Farahmand [5] naturally leads to inquisition about a more general class of functions (including $B(\beta)$ as a special case), which was introduced in [7] by Opoola, has been studied extensively in [2]. This is the class $T_n^a(\beta)$ consisting of functions $f \in A$ which satisfy the inequality;

$$\operatorname{Re} \frac{D^n f(z)^a}{\alpha^n z^a} > \beta \tag{1.4}$$

where α > 0 is real, 0 $0 \le \beta < 1,\ D^n (n \in N_0 = \{0, 1, 2, \cdots\})$ is the Salagean derivative operator defined as

$$D^n f(z) = D[D^{n-1} f(z)] = z[D^{n-1} f(z)]' \qquad (1.5)$$

with $D^0 f(z) = f(z)$ and powers in (1.4) meaning principal values only. Obverse that the geometric condition (1.4) slightly modifies the only given originally in [7] (see [2]). Onverse also that the class $B(\beta)$ corresponds to $n = \alpha = 1$.

In a recent work we considered the generalized Bernardi integral operator given by

$$F(z)^a = \frac{\alpha + c}{z^c} \int_0^z t^{c-1} f(t)^a dt, \qquad \alpha + c > 0 \qquad (1.6)$$

and sharpened and extended many earlier results concerning closure, under the integral, of several classes of functions. In the present paper we define a concept of quasi-partial sums and follow a method of Jahagiri an Farahmand [5] to extend their result (Theorem A) to the class $T_n^a(\beta)$.

As we noted in [1], the binomial expansion of (1,1) gives

$$f(z)^a = z^a + \sum_{k=2} a_k(\alpha) z^{\alpha+k-1} \qquad (1.7)$$

where $a_k(\alpha)$ is a polynomial depending on the coefficients of $f(z)$ and the index α. Hence

$$F(z)^\alpha = z^\alpha + \sum_{k+2} \frac{\alpha + c}{\alpha + c + k - 1} a_k(\alpha) z^{\alpha+k-1} \qquad (1.8)$$

and we define the *m*th quasi-partial sums of the integral (1.6) as follows

$$F_m(z)^\alpha = z^\alpha + \sum_{k=2}^{m} \frac{\alpha + c}{\alpha + c + k - 1} a_k(\alpha) z^{\alpha+k-1} \qquad (1.9)$$

In the next section we state the preliminary results.

2.0 Preliminary Results

We will require the following lemmas.

***Lemma* 2.1 [3]**

Let 0 be a real number and 1 a positive integer. If $-1 < \gamma \le A$, *then*

$$\frac{1}{1+\gamma} + \sum_{k=1}^{1} \frac{Cosk\theta}{k+\gamma} \ge 0$$

The constant $A = 4.5678018,\ldots$ is the best possible.

***Lemma* 2.2**

For $z \in E, -1 < \gamma \le A = 4.5678018\ldots,\ \operatorname{Re}\left(\sum_{k=1}^{1} \frac{z^k}{k+\gamma}\right) \ge -\frac{1}{1+\gamma}$

Proof

Let $z = re^{10}$ where $0 \le r < 1,\ 0 < |0| \le \pi$. Then by De Moivre's law and the minimum principle for harmonic functions $\operatorname{Re}\left(\sum_{k=1}^{1} \frac{z^k}{k+\gamma}\right) = \sum_{k=1}^{1} \frac{r^k Cosk\theta}{k+\gamma} > \sum_{k=1}^{1} \frac{Cosk\theta}{k+\gamma}$. Hence by Abel's Lemma [8, pg 6] and Lemma 2.1 above the conclusion follows. Let P denote the class of analytic functions of the form

$$p(z) = 1 + c_1 z + \ldots \tag{2.1}$$

normalized by $p(0) = 1$ and satisfy $\mathrm{Re}\,p(z) > 0$ in E. The next lemma concerns convolution of analytic functions with functions in P. The convolution (or Hadamard product) of two power series $f(z) = \sum_{k=0} a_k b_k z^k$ and $g(z) = \sum_{k=0} b_k z^k$ (written as $f*g$) is defined as $(f * g)(z) = \sum_{k=0} a_k b_k$ z^k. $\blacksquare$

***Lemma* 2.3 [4]**

Let p(z) be analytic in E and satisfy p(0) = 1 and $\mathrm{Re}\, p(z) > \frac{1}{2}$ *in E. For analytic function a(z) in E, the convolution p * q takes values in the convex hull of the image of E under a(z).*

3.0 Main Results

***Theorem* 3.1**

Let f(z) given by (1.1) *be in the class* $T_n^a(\beta)$. *Then*

$$\mathrm{Re}\,\frac{D^n F_m(z)^\alpha}{\alpha^n z^\alpha} > 1 - \frac{2(1-\beta)(a+c)}{(a+c+1)}, \quad \alpha + c \le 4.5678018\ldots \tag{3.1}$$

Furthermore if $\beta \ge \frac{1}{2}\frac{(\alpha+c-1)}{(\alpha+c}$, *then* $F_m(z)$ *belongs to some subclasses of the class* $T_n^a(\beta)$

Proof

From (1.7) and the condition (1.4) we have

$$\mathrm{Re}\left\{1 + \frac{1}{2(1-\beta)} \sum_{k=2} \left(\frac{\alpha+k-1}{\alpha}\right)'' a_k(\alpha) z^{k-1}\right\} > \frac{1}{2} \tag{3.2}$$

Also from (1.9) we have

$$\frac{D'' F_m(z)^\alpha}{\alpha'' z^\alpha} = 1 + \sum_{k=2} \left(\frac{\alpha+k-1}{\alpha}'' \frac{\alpha+c}{\alpha+c+k-1} a_k(\alpha) z^{k-1}\right) = p(z) * q(z) \tag{3.3}$$

where

$$p(z) = 1 + \frac{1}{2(1-\beta)} \sum_{k=2} \left(\frac{\alpha+k-1}{\alpha}''\right) a_k(\alpha) z^{k-1}, \tag{3.4}$$

$$q(z) = 1 + 2(1-\beta) \sum_{k=2}^{m} \frac{\alpha+c}{\alpha+c+k-1} z^{k-1} \tag{3.5}$$

Thus by Lemma 2.3 and the condition (3.1) the geometric quantities $D^n F_m(z)^{\alpha} \big/ d^n z^\alpha$ takes values in the convex hull of $q(E)$. But

$$\mathrm{Re}\, q(z) = 1 + 2(1-\beta)(\alpha+c)\,\mathrm{Re}\left(\sum_{k=1}^{m-1} \frac{z^k}{\alpha+c+k}\right) \tag{3.6}$$

We know from (1.6) that $\alpha + c > 0$. Now suppose $\alpha + c \leq 4.5678018\ldots$, then by taking $l = m - 1$ in Lemma 22, the real part of the series on right of (3.6) is greater than $-(\alpha + c + 1)^{-1}$ so that

$$\operatorname{Re}\frac{D''F_m(z)^{\alpha}}{\alpha'' z^{\alpha}} = \operatorname{Re} q(z) > 1 - \frac{2(1-\beta)(\alpha+c)}{(\alpha+c+1)} \tag{3.7}$$

Now observe that the real number $1 - \frac{2(1-\beta)(\alpha+c)}{(\alpha+c+1)}$ is nonnegative only for $\beta \geq \frac{1}{2}\frac{(\alpha+c-1)}{(\alpha+c)}$.

Thus only in this case it is clear $F_m(z)$ belongs to some subclasses of the class $T_n^a(\beta)$. This completes the proof.

■

Remark

For αα = 1, c= 0, the partial sums

$$F_m(z) = z + \sum_{k=2}^{m} \frac{a_k}{k} z^k \tag{3.8}$$

of the integral

$$F(z) = \int_0^z t^{-1} f(t)dt \tag{3.9}$$

for each $f \in B_n(1)$, belongs to the class $B_n(1)$ in general. More particularly, the partial sum (3.8) of the integral (3.9) of a function of bounded turning in the unit disk is also a function of bounded turning in the unit disk.

4.0 Conclusion

In this paper we defined a new concept of quasi-partial sums of the generalized Bernard integral. We used the new concept to extend an earlier result of Jahangiri and Farahmand [5] concerning functions of bounded turning to a more general class of function.

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