On some functional associated with certain coefficient problems

K. O. Babalola
Department of Mathematics,
University of Ilorin, Nigeria
e-mail: Ummusalamah.kob@unilorin.edu.ng

Abstract

Under certain conditions, we obtain sharp bounds on some functionals defined in the coefficient space of starlike functions. It has been found that the functionals are closely associated with certain coefficient problems, which are of independent interest.

1.0   Introduction

Let *S* denote the family of functions.

$$f(z) = z + a_2 z^2 + \dots \qquad (1.1)$$

which are analytic and univalent in the unit disk $E = \{z : |z| < 1\}$ and map *E* onto some domains *D*. A function $f \in S$ is said to be starlike if the domain *D* is starlike with respect to the origin. The family of starlike functions is denoted by S^*. It is known that a function $f \in S$ is star;ike if and only if it satisfies the geometric condition:

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0. \quad z \in E \qquad 1.2$$

The above geometric condition implies that the quantity $zf'(z)/f(z)$ belongs to the class *P* of analytic functions:

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots \qquad (1.3)$$

which have positive real part in *E*. The family of starlike functions in the unit disk has attracted much attention in the past. The volume of work being publish on this family of functions leaves no one in doubt about the importance attached to it both in the past and the present.

In this paper we determine, under certain conditions, the best possible upper bound on some functionals defined in the coefficient space of starlike functions. These are functionals, which whave arisen from the study of coefficient problems of certain family of univalent functions. The study of functionals similar to those being considered in this paper is not new. For instance, bounds on $|a_n|$ and $|a_3 - \lambda a_2^2|$ can be found in many literatures (for example, [1 – 7]. In particular, the functional $|a_3 - \lambda a_2^2|$ is known as t he Fekete-Szego functional for both real and complex values of the parameter λ. The problem of determining the sharp bound on the Fekete-Szego functional has its origin in a conjecture of Littlewood and Parley (1932) that the true bound on the coefficients of an odd univalent

function is 1 which was disproved in 1933 by Fekete and Szego via the determination of the sharp bound on the functional (see [3]).

The functional has since continued to received attention of researchers in geometric function theory.

In section 3, we consider functionals of the form $|a_4 - \gamma a_2 a_3|$, $|a_4 - \gamma a_2 a_3 - \eta a_2^3|$, $|a_5 - \mu a_2^2 a_3|$ and $|a_5 - \xi a_2 a_4 - \zeta a_3^2|$ where the parameters γ, η, μ, ξ, ζ are all real numbers. These functionals have been found to have applications in certain coefficient problems, which are of independent interest.

2.0   Preliminary Lemmas
We shall need the following well-known inequalities.

Lemma **2.1. [1 – 7]**
    Let $p \in P$. Then $|c_n| \le 2, k = 1,\ 2,\ 3\ldots$ Equality is attained for the Moebius function..

$$L_0(z) = \frac{1+z}{1-z} \tag{2.1}$$

Lemma **2.2 [4, 6]**
    Let $p \in P$. Then

$$\left| c_2 - \frac{c_1^2}{2} \right| \le 2 - \frac{|c_1|}{2} \tag{2.2}$$

This result is sharp. Equality holds for the function

$$p(z) = \frac{1 + \frac{1}{2}(c_1 + \varepsilon c_1)z + \varepsilon z}{1 - \frac{1}{2}(c_1 - \varepsilon c_1)z - \varepsilon z^2} \quad |\varepsilon| = 1 \tag{2.3}$$

Note that the inequality (2.2) can be written as

$$c_2 = \frac{1}{2}c_1^2 + \varepsilon(2 - \frac{1}{2}|c_1|^2), \quad |\varepsilon| \le 1 \tag{2.4}$$

3.0   Main Result
Theorem **3.1**
    Let $f(z)$ given by (1.1) be a starlike function. Then for real numbers $\gamma,\ \eta,\ \mu,\ \xi, \zeta$ such that $1-\gamma,\ 1-2\mu,\ 1-\xi,\ 1-2\zeta$ and $1-2\xi-2\zeta$ are all nonnegative, we have the sharp inequalities.

$$|a_4 - \gamma a_2 a_3| \le 4 - 6\gamma, \quad if \ \ \gamma \le \frac{5}{9}$$

$$|a_4 - \gamma a_2 a_3 - \eta a_2^1| \le 4 - 6\gamma - 8\eta, \quad if \ \ 3\gamma + 4\eta \le \frac{5}{3}$$

$$|a_5 - \mu a_2^2 a_3| \le 5 - 12\mu, \quad if \ \ \mu \le \frac{2}{9}$$

$$|a_5 - \xi a_2 a_4 - \zeta a_3^2| \le 5 - 8\xi - 9\zeta, \quad if \ \ 5\xi + 6\zeta \le 2$$

Proof

Since $f(z)$ is starlike there exists $p \in P$ such that

$$zf'(z) = p(z)f(z) \tag{3.1}$$

Comparing coefficients of both sides of (3.1) using (1.1) and (1.3) we see that

$$a_2 = c_1$$

$$2a_1 = c_2 + c_1^2$$

$$6a_4 = 2c_3 + 3c_2c_1 + c_1^2$$

$$24a_5 = 6c_4 + 8c_1c_1 + 6c_2c_1^2 + 3c_2^2 + c_1^4$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

so that

$$a_4 - \gamma a_2 a_3 = \frac{c_3}{3} + (1-\gamma)\frac{c_1}{2}\left\{c_2 + \frac{2}{3}\frac{(1-3\gamma)}{(1-\gamma)}\frac{c_1^2}{2}\right\} \tag{3.2}$$

$$a_4 - \gamma a_2 a_3 - \eta a_2^3 + (1-\gamma)\frac{c_1}{2}\left\{c_2 \frac{2}{3}\frac{(1-3\gamma-6\eta)}{(1-\gamma)}\frac{c_1^2}{2}\right\} \tag{3.3}$$

$$a_5 - \mu a_2^2 a_3 = \frac{c_4}{4} + \frac{c_3c_1}{3} + \frac{c_2^2}{8} + (1-2\mu)\frac{c_1}{4}\left\{c_2 + \frac{1}{3}\frac{(1-12\mu)}{(1-2\mu)}\frac{c_1^2}{2}\right\} \tag{3.4}$$

$$a_5 - \xi a_2 a_4 - \xi a_3^2 = \frac{c_4}{4} + (1-\xi)\frac{c_3c_1}{3} + (1-2\zeta)\frac{c_2}{8} + (1-2\xi-2\zeta)\frac{c_1^2}{4}\left\{c_2 + \frac{1}{3}\frac{1-4\xi-6\zeta)c_{12}^2}{(1-2\xi-2\zeta)}\right\} \tag{3.5}$$

Recall that real numbers $1-\gamma$, $1-2\mu$, $1-\xi$, $1-2\zeta$ and $1-2\xi-2\zeta$ are nonnegative. We eliminate c_2 in each of the terms in the curly brackets in (3.2) – (3.5) using the equality (2.4). For instance, we have from (3.2).

$$c_2 + \frac{2}{3}\frac{(1-3\gamma)}{(1-\gamma)}\frac{c_1^2}{2} = \frac{(5-9\gamma)}{3(1-\gamma)}\frac{c_1^2}{2} + \varepsilon\left(2 - \frac{|c_1|^2}{2}\right) \tag{3.6}$$

Since $(2 - \frac{1}{2}|c_1|^2) \geq 0$, the absolute value of (3.6) attains its maximum for $|c_1| = 2$ provide $\gamma \leq \frac{5}{9}$ (which is the condition given in the inequality (i) of the theorem). Thus (3.6) yields

$$\left|c_2 + \frac{2}{3}\frac{(1-3\gamma)}{(1-\gamma)}\frac{c_1^2}{2}\right| \leq \frac{2}{3}\frac{(5-9\gamma)}{(1-\gamma)} \tag{3.7}$$

so that, by triangle inequality and Lemma 2.1, (3.2) gives the inequality (*i*). Similar arguments and computations, from (3.3) to (3.5), lead to the inequalities (*ii*) to (*iv*) respectively. For each of the real numbers γ, η, μ, ξ and ζ equality is attained in each case by the Koebe function (up to rotations) given by:

$$k(z) = \frac{z}{(1-z)^2} \tag{3.8}$$

∎

4.0   Conclusion

The study of fucntionals in the theory of analytic and univalent functions is here boosted with the consideration of new ones. The functionals considered in this work are closely associated with certain coefficient functions theory.

References

[1]   P. N. Chichra. *New subclasses of the class of close-to-convex functions.* Proc. Amer. Math. Soc. 62(1) (1977), 37 – 43.

[2]   M. Darus. *The Fekete-Szego theorem for close-to-covex functions of the class $K_{sh}(\alpha,\beta)$* Acta Math. Acad. Paed. Nyiregyhaziensis 18 (2002), 13 – 18.

[3]   M. Dorff and J. Szynal, *Higher order Schwarzian derivatives for convex univalent   functions.* Preprint submitted to Elservier Science (2002).

[4]   P. L. Duren. Coefficients of univalent functions. Bull. Amer. Math. Soc. 83 (1977), 891 – 911.

[5]   P. L. Duren. *Univalent functions.* Springer-Verlag. New York Inc. 1983.

[6]   S. Kanas and A. Lecko. *On the Fekete-Szego problem and the domain of convexity for a certain class of univalent functions.* Sci. Univ. Tech. Res. 73 (10) (1990), 49 –   57.

[7]   R. Singh, *On Bazilevic functions.* Proc. Amer. Math. Soc. 38 (1973), 261 – 271.